Breakdown of self-similar evolution in homogeneous perfect fluid collapse

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The stability analysis of self-similar solutions is an important approach to confirm whether they act as an attractor in general non-self-similar gravitational collapse. Assuming that the collapsing matter is a perfect fluid with the equation of state $P = \alpha\rho$, we study spherically symmetric non-self-similar perturbations in homogeneous self-similar collapse described by the flat Friedmann solution. In the low pressure approximation $\alpha \ll 1$, we analytically derive an infinite set of the normal modes and their growth (or decay) rate. The existence of one unstable normal mode is found to conclude that the self-similar behavior in homogeneous collapse of a sufficiently low pressure perfect fluid must terminate and a certain inhomogeneous density profile can develop with the lapse of time.

I. INTRODUCTION

Spherically symmetric self-similar gravitational collapse of a perfect fluid with pressure P given by the equation of state $P = \alpha\rho$ is one of the most extensively-studied phenomena in general relativity. Many efforts have been made to solve the Einstein's equations governing its dynamics, which are reduced to a set of ordinary differential equations with respect to the single variable $z \equiv r/t$. The flat Friedmann solution is well known as the unique analytically-found exact solution regular at the center and has played an important role in finding a family of solutions regular at the center [1,2]. The homogeneous collapse described by this solution has been considered as the most basic process to spacelike singularity formation in the self-similar dynamics, while essential features of inhomogeneous collapse have been understood mainly through the detailed analysis of the general relativistic Larson-Penston solution [3]. In addition, it is also noteworthy that the perfect fluid critical collapse corresponding to the threshold of black hole formation has been confirmed to be described by one of the self-similar solutions [4–6].

Several works have been devoted to the stability analysis of such self-similar solutions for spherically symmetric non-self-similar perturbations and have given important implications to more general non-self-similar gravitational collapse. In particular, it is remarkable that the flat Friedmann solution and the general relativistic Larson-Penston solution were numerically confirmed to be able to act as an attractor in general spherically symmetric gravitational collapse for α lying in the range $0 < \alpha \lesssim 0.036$ [7]. In addition, the critical phenomena are illustrated in terms of the time evolution of the single unstable normal mode which was found by the numerical analysis of the perturbations in the critical collapse [8,9]. (See [10] for a recent review on the role of self-similar solutions as an attractor and the critical phenomena.) Although it may

be interesting to study the stability problem more extensively, these numerical results should be confirmed through an analytical treatment of the perturbations.

Recently, we have developed an analytical scheme to treat the stability problem by constructing the single wave equation governing non-self-similar spherically symmetric perturbations [11], which is reduced to the ordinary differential equation if we assume the perturbations to have the time dependence given by $\exp(i\omega \log|t|)$. In this paper, using this analytical scheme, we study the stability problem for the flat Friedmann solution in the low pressure limit, i.e., $0 < \alpha \ll 1$. Fortunately, in the expansion with respect to the small parameter α, we can explicitly solve the master ordinary differential equation for the normal modes and consequently find the single unstable normal mode, which was not found in the numerical analysis [7].

We begin with a brief description of the perturbation theory for the flat Friedmann solution in Sec. II. In Sec. III, a discrete set of the normal modes and their growth (or decay) rate are derived in the low pressure limit, and the self-similar behavior turns out to be unstable in homogeneous collapse of a sufficiently low pressure perfect fluid. In Sec. IV, we see density inhomogeneities generated by the normal modes and explain the affection of the background transonic flow upon the growth (or decay) of the normal modes. In the final section, we summarize this paper and give a suggestion for the result of the numerical study [7]. In addition, we discuss an implication of the breakdown of the self-similar evolution in relation to critical phenomena.

II. PERTURBATION THEORY FOR SELF-SIMILAR HOMOGENEOUS PERFECT FLUID COLLAPSE

In this section, we briefly illustrate our analytical scheme to treat spherically symmetric non-self-similar perturbations in homogeneous self-similar perfect fluid collapse described by the flat Friedmann solution (see [11] for the details). The line element considered throughout this paper is given by

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$$ds^2 = -e^{2\nu(t,r)}dt^2 + e^{2\lambda(t,r)}dr^2$$
$$+ R^2(t,r)(d\theta^2 + \sin^2\theta d\varphi^2) \qquad (2.1)$$

with the comoving coordinates t and r. In addition, the energy momentum tensor of a perfect fluid is expressed as

$$T^{ab} = (\rho + P)u^a u^b + P g^{ab}, \qquad (2.2)$$

where ρ, P and the vector u^a are energy density, pressure and fluid four velocity, respectively. As was mentioned in Sec. I, its equation of state is assumed to be

$$P = \alpha\rho \qquad (2.3)$$

with a constant α lying in the range $0 < \alpha \le 1$. To discuss the self-similar behavior later, we use a new variable z defined by $z \equiv r/t$, instead of r. In addition, instead of the four unknown functions ν, λ, R and ρ, we introduce the following dimensionless functions:

$$S(t,r) \equiv \frac{R}{r}, \qquad \eta(t,r) \equiv 8\pi r^2 \rho,$$
$$\qquad (2.4)$$
$$M(t,r) \equiv \frac{2m}{r}, \qquad V(t,r) \equiv z e^{\lambda - \nu},$$

where the function $m(t,r)$ is the Misner-Sharp mass. The function V is interpreted as the velocity of a $z = \text{const}$ surface relative to the fluid element.

From the Einstein's field equations for the system (2.1), (2.2), and (2.3), we can obtain the four partial differential equations governing the functions ν, λ, S and η. By the two equations, the metrics ν and λ are explicitly given by the functions S and η. The relations between the functions S and η given by the remaining two equations become the simpler first order partial differential equations if we use the function M, which is explicitly given by the functions S and η. Therefore we hereafter focus our concern on the functions S, η and M.

Now we consider spherically symmetric non-self-similar perturbations in the flat Friedmann background by expressing the solutions for the Einstein's equations as

$$S(t,z) = S_B(z)\{1 + \epsilon S_1(t,z) + O(\epsilon^2)\},$$
$$\eta(t,z) = \eta_B(z)\{1 + \epsilon \eta_1(t,z) + O(\epsilon^2)\}, \qquad (2.5)$$
$$M(t,z) = M_B(z)\{1 + \epsilon M_1(t,z) + O(\epsilon^2)\}$$

with a small parameter ϵ. The functions S_B, η_B and M_B for the flat Friedmann solution are given by

$$S_B(z) \propto (-z)^{-p}, \qquad \eta_B(z) \propto z^2, \qquad M_B(z) \propto (-z)^{2-3p} \qquad (2.6)$$

with the constant p defined as

$$p \equiv \frac{2}{3(1+\alpha)}. \qquad (2.7)$$

Note that the line element (2.1) for the flat Friedmann solution can be reduced to a form more familiar in cos-

mology through the coordinate transformation $r \to \hat{r} \propto r^{1-p}$ [1].

As seen from the behaviors such that $S_B \to 0$ and $\eta_B \to \infty$ in the limit $z \to \infty$, a big crunch singularity appears at $t = 0$ in the flat Friedmann background. Therefore we hereafter consider time evolution of the perturbations during $t < 0$ (i.e., $z < 0$). As was mentioned in Sec. I, in such a region there are two characteristic surfaces at which a constraint is imposed on the behavior of the perturbations. One is the regular center located at $z = 0^-$, and the other is the sonic point located at $z = z_s$ defined as a point at which the velocity of a $z = \text{const}$ surface relative to the fluid is equal to the sound speed, i.e., $V_B = -\sqrt{\alpha}$, where the function V_B for the background flat Friedmann solution is found to be

$$V_B(z) \propto (-z)^{1-p}. \qquad (2.8)$$

Using the perturbation equations for S_1, η_1 and M_1, we can obtain the following single wave equation:

$$\Psi_{,uu} - \Psi_{,\zeta\zeta} + W(\zeta)\Psi_{,u} + F(\zeta)\Psi_{,\zeta} + U(\zeta)\Psi = 0 \quad (2.9)$$

for the function Ψ defined as

$$\Psi(t,z) = S_1(t,z) - f(z)M_1(t,z), \qquad (2.10)$$

where the functions W, F, U and f are given in Appendix A. The new variables u and ζ are related to the variables t and z by

$$u = \log(-t) + I(z), \qquad I(z) = \frac{1}{2(1-p)}\log(1 - x^2(z)), \qquad (2.11)$$

$$\zeta = \frac{1}{2(1-p)}\log\frac{1 - x(z)}{1 + x(z)}, \qquad (2.12)$$

where the function x is defined as

$$x = -\frac{V_B}{\sqrt{\alpha}}. \qquad (2.13)$$

The remaining perturbation equations yield the relations

$$M_1(t,z) = B_1(z)\dot{\Psi}(t,z) + B_2(z)\Psi'(t,z) + B_3(z)\Psi(t,z), \qquad (2.14)$$

$$\eta_1(t,z) = M_1(t,z) + \frac{1}{3(1-p)}M_1'(t,z) - 3S_1(t,z)$$
$$- \frac{1}{1-p}S_1'(t,z), \qquad (2.15)$$

where the dot and the prime represent the partial derivative with respect to $\log|t|$ and $\log|z|$, respectively, and the functions B_1, B_2 and B_3 are also given in Appendix A. The perturbations S_1, η_1 and M_1 are determined by the solution Ψ for the wave Eq. (2.9) via Eqs. (2.14), (2.10), and (2.15).

The wave Eq. (2.9) allows us to consider the modes ϕ defined as

$$\Psi(t, z) = \phi(x, \omega)e^{i\omega(u+\zeta)} \tag{2.16}$$

with the spectral parameter ω. It is mathematically convenient to use the variable x, instead of ζ, which can cover the whole region between the regular center and the sonic point in the finite range $0 \leq x \leq 1$, irrespective of the parameter value α. Then the equation for ϕ is found to be

$$\phi_{,xx} + \frac{2i\omega - 2(1-p)x - F}{(1-p)(1-x^2)}\phi_{,x} - \frac{i\omega(F+W) + U}{(1-p)^2(1-x^2)^2}\phi = 0. \tag{2.17}$$

We will obtain the normal modes as the solutions for Eq. (2.17) satisfying the boundary conditions such that ϕ is analytic both at the regular center $x = 0$ and at the sonic point $x = 1$. (The same boundary conditions were required in the numerical analysis done by [7].) The leading behavior of ϕ near $x = 1$ is given by

$$\phi \simeq C_1(\omega) + C_2(\omega)(1-x)^k, \tag{2.18}$$

where $k = \{(1 + i\omega)/(1 - p)\} + 1$. The ratio C_2/C_1 will be uniquely determined by the requirement of the analyticity of ϕ at $x = 0$, and discrete eigenvalues $\omega = \omega_n$ giving the normal modes $\phi(x, \omega_n) \equiv \phi_n(x)$ will be derived by the equation

$$C_2(\omega) = 0. \tag{2.19}$$

Here we would like to note that there exists an exact solution $\phi = \phi_g$ for Eq. (2.17) written by

$$\phi_g \propto \frac{x(1+x)^{3(1-\alpha)/(3\alpha+1)}}{2x + 1 + 3\alpha} \tag{2.20}$$

if the spectral parameter ω is equal to ω_g defined as

$$\omega_g = \frac{1-\alpha}{1+\alpha}i. \tag{2.21}$$

This solution ϕ_g is one of the normal modes ϕ_n but corresponds to a gauge mode due to an infinitesimal transformation of t. In fact, all the perturbations η_1, S_1 and M_1 obtained from ϕ_g are found to be independent of z. Although the gauge mode is obviously unphysical, the presence of such an exact solution will be mathematically useful for checking the validity of the analysis of Eq. (2.17).

III. NORMAL MODES IN THE LOW PRESSURE LIMIT

Although the master Eq. (2.17) for the perturbations in the flat Friedmann background is given by a simpler form (if compared with the form for any other self-similar backgrounds), it is still a difficult task to solve the eigenvalue problem, and in [11] the absence of the unstable normal modes was clearly proven only in a limited range of ω.

Such a difficulty may be overcome if we consider the low pressure limit $\alpha \to 0$, keeping the variable x finite in the range $0 \leq x \leq 1$ and expanding the solution $\phi(x, \omega, \alpha)$ analytic at $x = 0$ as follows,

$$\phi(x, \omega, \alpha) = \sum_{i=0}^{\infty} \alpha^i \phi^{(i)}(x, \omega). \tag{3.1}$$

For the lowest-order solution $\phi^{(0)}$ we obtain the equation

$$\mathcal{L}\,\phi^{(0)}(x, \omega) = 0, \tag{3.2}$$

where the ordinary differential operator $\mathcal{L}$ is given by

$$\mathcal{L} = \frac{d^2}{dx^2} + \frac{3}{1-x^2}\left\{2i\omega + \frac{2x^3 + 3x^2 + 10x + 3}{3x(1+2x)}\right\}\frac{d}{dx}$$
$$- \frac{3}{x(1+2x)(1+x)(1-x^2)}\left\{2i\omega(2x^2 - 2x - 1)\right.$$
$$\left. - \frac{2x^4 + 4x^3 - x^2 - 2x - 1}{x}\right\}. \tag{3.3}$$

We would like to emphasize that this limit does not mean to consider an exactly pressureless fluid (i.e., a dust fluid) because the requirement of the analyticity of ϕ at the sonic point $x = 1$ is not missed. The crucial point in this approach is that we can explicitly derive general solutions for Eq. (3.2). In particular, the solution $\phi^{(0)}$ satisfying the boundary condition at $x = 0$ is written as

$$\phi^{(0)}(x, \omega) = Z_1(x, \omega) - Z_2(x, \omega), \tag{3.4}$$

where the functions Z_1 and Z_2 are the two independent solutions for Eq. (3.2) and given by

$$Z_1(x, \omega)$$
$$= \frac{(1+x)^3\{-6x^2\omega^2 - 4x(x^2 - 3x + 1)i\omega + (1-x)^4\}}{x^3(1+2x)}, \tag{3.5}$$

$$Z_2(x, \omega) = \frac{(1-x)^{4+3i\omega}(1+x)^{1-3i\omega}\{x^2 + 2(1+i\omega)x + 1\}}{x^3(1+2x)}. \tag{3.6}$$

It is clear that this solution $\phi^{(0)}$ becomes analytic also at the sonic point $x = 1$ if the spectral parameter ω is equal to $\omega_n^{(0)}$ given by

$$\omega_n^{(0)} = \frac{4-n}{3}i \tag{3.7}$$

with n defined as non-negative integers. However, for $n = 0, 2$ and 4, the function $\phi^{(0)}$ turns out to vanish. Hence, the values of $\omega_n^{(0)}$ are given only for $n = 1, 3, 5, 6, \cdots$. It can be easily found that the value of $\omega_1^{(0)}$ and the function $\phi^{(0)}(x, \omega_1^{(0)})$ are identical with the value of ω_g and the gauge mode ϕ_g in the limit $\alpha \to 0$. While this assures that $\omega_n^{(0)}$ represents the eigenvalues approximately written

in the limit $\alpha \to 0$, it may be unclear how the condition (2.19) to obtain the eigenvalues ω_n is used in this approach.

To discuss this point, we consider the next order solution $\phi^{(1)}(x, \omega)$ in Eq. (3.1), using Eq. (3.4). The inhomogeneous ordinary differential equation for the function $\phi^{(1)}$ is written as

$$\mathcal{L}\, \phi^{(1)}(x, \omega) = J(x, \omega), \qquad (3.8)$$

where the function J is given in Appendix A. Here we note that the imaginary part of $\omega_n^{(0)}$ is smaller than $4/3$. Because we consider the solution $\phi^{(1)}$ analytic at $x = 0$ for ω nearly

equal to $\omega_n^{(0)}$, from Eq. (3.8) we obtain

$$\phi^{(1)}(x, \omega) = Z_2(x, \omega) \int_0^x \frac{\phi^{(0)}(y, \omega)J(y, \omega)}{w(y, \omega)} dy$$
$$+ \phi^{(0)}(x, \omega) \int_x^1 \frac{Z_2(y, \omega)J(y, \omega)}{w(y, \omega)} dy$$
$$+ a(\omega)\phi^{(0)}(x, \omega), \qquad (3.9)$$

where $a(\omega)$ is an arbitrary constant and the Wronskian w of $\phi^{(0)}$ and Z_2 is given by

$$w(x, \omega) = \phi^{(0)}Z_{2,x} - \phi^{(0)}_{,x}Z_2 = -\frac{8(1 - x)^{3(1+i\omega)}(1 + x)^{3(1-i\omega)}\omega(\omega - 2i)(3\omega - 2i)(3\omega - 4i)}{x^3(1 + 2x)^2}. \qquad (3.10)$$

The eigenvalues $\omega_n(\alpha)$ giving the normal modes ϕ_n up to the first order of α will have the form

$$\omega_n(\alpha) = \omega_n^{(0)} + \omega_n^{(1)}\alpha + O(\alpha^2). \qquad (3.11)$$

Because of this expansion of $\omega_n(\alpha)$, Eq. (3.1) for the normal modes $\phi_n (\equiv \phi(x, \omega_n(\alpha), \alpha))$ can be rewritten to the form

$$\phi_n(x, \alpha) = \phi_n^{(0)}(x) + \left(\frac{\partial \phi^{(0)}}{\partial \omega}(x, \omega_n^{(0)})\omega_n^{(1)} + \phi_n^{(1)}(x)\right)\alpha$$
$$+ O(\alpha^2), \qquad (3.12)$$

where $\phi_n^{(i)}(x) \equiv \phi^{(i)}(x, \omega_n^{(0)})$. In this expansion scheme, the function ϕ_n may be nonanalytic at $x = 1$ owing to terms containing the logarithmic factor $\log(1 - x)$ in $\partial \phi^{(0)}/\partial \omega(x, \omega_n^{(0)})$ and $\phi_n^{(1)}$. It is straightforward to derive such nonanalytic terms, and we have

$$\frac{\partial \phi^{(0)}}{\partial \omega}(x, \omega_n^{(0)}) = K_n(x) - 3iZ_2(x, \omega_n^{(0)})\log(1 - x),$$
$$(3.13)$$

$$\phi_n^{(1)}(x) = L_n(x) - b_n Z_2(x, \omega_n^{(0)})\log(1 - x), \qquad (3.14)$$

where the functions K_n and L_n are analytic at $x = 1$ and the coefficient b_n is obtained from the term proportional to $(1 - x)^{-1}$ in the expansion of the function $Z_1 J/w$ for $\omega = \omega_n^{(0)}$ around $x = 1$. (Note that the function $Z_1 J/w$ appears in the integrand of the first integral in the right hand side of Eq. (3.9) because of Eq. (3.4).) It is clear that the non-analyticity of ϕ_n at $x = 1$ can be removed if $\omega_n^{(1)}$ is chosen as follows,

$$\omega_n^{(1)} = \frac{b_n i}{3}. \qquad (3.15)$$

To estimate the coefficient b_n, we rewrite the function $Z_1 J/w$ for $\omega = \omega_n^{(0)}$ into the form

$$\frac{Z_1(x, \omega_n^{(0)})J(x, \omega_n^{(0)})}{w(x, \omega_n^{(0)})} = B(x, \omega_n^{(0)})\{(1 - x)^{3-n}J_1(x, \omega_n^{(0)})$$
$$+ (1 - x)^{-n}J_2(x, \omega_n^{(0)})$$
$$+ (1 - x)^{-1}J_3(x, \omega_n^{(0)})\}, \qquad (3.16)$$

where the functions J_1, J_2 and J_3 are given in Appendix A and the function B is defined as

$$B(x, \omega) \equiv \frac{Z_1(x, \omega)}{w(x, \omega)(1 - x)^{-3(1+i\omega)}}. \qquad (3.17)$$

Note that the functions B, J_1, J_2 and J_3 for $\omega = \omega_n^{(0)}$ become finite at $x = 1$. From the form given by Eq. (3.16), we can easily find

$$b_n = B(1, \omega_n^{(0)})J_3(1, \omega_n^{(0)}) + \bar{b}_n, \qquad (3.18)$$

where $\bar{b}_n$ is given by

$$\bar{b}_1 = B(1, \omega_1^{(0)})J_2(1, \omega_1^{(0)}), \qquad (3.19)$$

$$\bar{b}_3 = \frac{1}{2}(B(x, \omega_3^{(0)})J_2(x, \omega_3^{(0)}))_{,xx}\Big|_{x=1}, \qquad (3.20)$$

and for $n \geq 5$

$$\bar{b}_n = \frac{(-1)^{n-4}}{(n-4)!}\frac{d^{n-4}}{dx^{n-4}}(B(x, \omega_n^{(0)})J_1(x, \omega_n^{(0)}))\Big|_{x=1}$$
$$+ \frac{(-1)^{n-1}}{(n-1)!}\frac{d^{n-1}}{dx^{n-1}}(B(x, \omega_n^{(0)})J_2(x, \omega_n^{(0)}))\Big|_{x=1}. \quad (3.21)$$

We can easily calculate the first term in the right hand side of Eq. (3.18) to be

$$B(1, \omega_n^{(0)})J_3(1, \omega_n^{(0)}) = -2(n - 1). \qquad (3.22)$$

In addition we find that the value of $\bar{b}_n$ is given by the unified form

$$\bar{b}_n = 6(-1)^n \qquad (3.23)$$

for $n = 1, 3, 5, 6, \cdots$. Thus we arrive at the result

$$\omega_n(\alpha) = \frac{4-n}{3} i + \left\{ -\frac{2}{3}(n-1) + 2(-1)^n \right\} \alpha i + O(\alpha^2)$$

(3.24)

for the eigenvalue problem Eq. (2.19) giving the normal modes ϕ_n written as

$$\phi_n(x, \alpha) = \phi_n^{(0)}(x) + \left[K_n(x) \left\{ -\frac{2}{3}(n-1) + 2(-1)^n \right\} i + L_n(x) \right] \alpha + O(\alpha^2),$$

(3.25)

where the functions K_n and L_n are explicitly given through Eqs. (3.13) and (3.14). It should be noted that the value of ω_1 given by the above equation is identical with the value of ω_g up to the first order of α, as was expected. Although the normal modes ϕ_n appear to have an ambiguity due to the existence of the constant a in Eq. (3.9), it will be uniquely determined if we require the analyticity at $x = 1$ up to the second order of α. In fact, we can confirm that Eq. (3.25) for $n = 1$ is identical with the gauge mode ϕ_g only for $a = 7-12 \log 2$.

The most important result is that the imaginary part of $\omega_3^{(0)}$ given by Eq. (3.7) is positive, namely, there exists one unstable normal mode, at least for sufficiently small values of α. The proof concerning the absence of unstable normal modes shown in [11] is not applicable to the normal mode ϕ_3 with the small growth rate $\mathrm{Im}(\omega_3^{(0)} + \alpha \omega_3^{(1)})$ obtained here. Hence the flat Friedmann solution does not act as an attractor in homogeneous collapse of a sufficiently low pressure perfect fluid. However, the first order correction $\omega_3^{(1)}$ has a negative imaginary part. This means that the growth rate of the unstable normal mode becomes smaller as the value of α increases, and the self-similar behavior might be stable in homogeneous collapse of a higher pressure perfect fluid such as the radiation fluid ($\alpha = 1/3$) and the stiff fluid ($\alpha = 1$). This is an interesting problem to be further studied in future works. Moreover, it is also interesting that there exists an infinite set of the stable normal modes (i.e., ϕ_n for $n \geq 5$). In the next section, we will derive the density perturbation η_1 corresponding to the normal modes ϕ_n to see what configuration of perturbed inhomogeneous fields can develop or decay with the lapse of time.

IV. DENSITY PERTURBATIONS

Let us denote the perturbed density η_1 corresponding to the normal modes ϕ_n by $\eta_{1(n)}$, which is expanded with respect to α as follows,

$$\eta_{1(n)}(t, x, \alpha) = \eta_{1(n)}^{(0)}(x) \exp\{i\omega_n^{(0)} \log(-t)\} + O(\alpha).$$

(4.1)

Here we focus our attention on the leading term $\eta_{1(n)}^{(0)}$

depending on x. Through Eqs. (2.10), (2.11), (2.12), (2.14), (2.15), (2.16), and (3.4), in the limit $\alpha \to 0$ with the eigenvalues $\omega = \omega_n^{(0)}$, we obtain

$$\eta_{1(n)}^{(0)}(x) = \frac{2+n}{6x} [(1-x)^{n-2}\{3x^2 + 3(n-2)x + (n-1)(n-3)\} - (1+x)^{n-2}\{3x^2 - 3(n-2)x + (n-1)(n-3)\}].$$

(4.2)

In Fig. 1, we show the configuration of the density perturbation $\eta_{1(n)}^{(0)}(x)$ normalized by its value at $x = 0$, namely,

$$\eta_{1(n)}^{(0)}(0) = -\frac{(n-4)(n-2)n(n+2)}{3},$$

(4.3)

for $n = 1, 3, 5, 6, 7, 8, 9$ and 10 in the range $0 \leq x \leq 1$. It is shown in this figure that the normal mode $\eta_{1(1)}^{(0)}$ corresponds to the gauge mode $\eta_g = \mathrm{const}$ at a given time as was mentioned in Sec. II. In addition, it should be emphasized that the normal mode $\eta_{1(6)}^{(0)}$, which is also constant at any x, is a physical normal mode because the corresponding perturbation M_1 depends on x in the range $0 \leq x \leq 1$.

From Fig. 1, we note that the amplitude of all the stable normal modes (i.e., the normal modes for $n \geq 5$) at the sonic point $x = 1$ remains nonzero and the amplitude of the normal modes for $n \geq 7$ rather increases towards the sonic point $x = 1$ from the center $x = 0$. The ratio $\eta_{1(n)}^{(0)}(1)/\eta_{1(n)}^{(0)}(0)$ increases as n becomes larger and the decay rate $-\mathrm{Im}(\omega_n)$ given by Eq. (3.24) has the same tendency. This implies that the density perturbation generated near the sonic point is rapidly carried away to the supersonic region $x > 1$ by the background transonic flow and the growth of such a density perturbation in the subsonic region is prevented. It is remarkable that only for $n = 3$, the value of $\eta_{1(n)}^{(0)}$ vanishes at $x = 1$. This seems to be the most favorable configuration of η_1 to allow the growth of the density perturbation due to the effect of its own self-
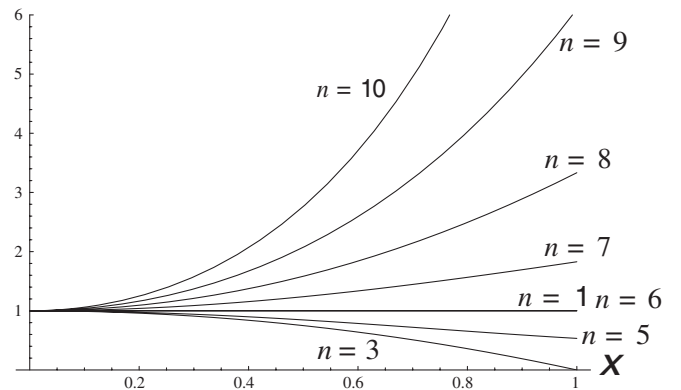

FIG. 1. Configuration of the density perturbation given by $\eta_{1(n)}^{(0)}(x)/\eta_{1(n)}^{(0)}(0)$.

gravitation. Further, Eq. (3.24) clearly shows that the imaginary part of ω_n decreases as α increases. The dispersive effect due to the pressure against the self-gravitation can enhance the above-mentioned decay process of the perturbations.

In this section, we have focused our concern on the configuration of the normal modes in the subsonic region $0 \leq x \leq 1$. This is mainly because the eigenvalue problem was set under the boundary conditions at $x = 0$ and $x = 1$ and from the viewpoint of causality, any disturbances in the supersonic region cannot affect the process in the subsonic region. Further, as another reason, we would like to point out that the approximation $\alpha \simeq 0$ to derive Eq. (3.24) becomes mathematically unreliable in the region far away from the sonic point, i.e., $x \gg 1$. If one tries to understand more global features of the normal modes by analyzing Eq. (2.17) in the limit $\alpha \rightarrow 0$, the terms included in Eq. (2.17) which can be negligible in the subsonic region but become important for $x \gg 1$ should be taken into consideration.

V. SUMMARY AND DISCUSSION

In this paper, we have studied the stability problem for the self-similar behavior in homogeneous collapse of a perfect fluid with the equation of state $P = \alpha\rho$, using the perturbation theory developed in [11]. We have derived the single ordinary differential Eq. (2.17) governing spherically symmetric non-self-similar perturbations with the time dependence $\exp\{i\omega \log(-t)\}$ in the flat Friedmann background and set up the eigenvalue problem to determine the value of the spectral parameter ω. In the low pressure approximation $\alpha \rightarrow 0$, we have succeeded in deriving explicitly an infinite set of the eigenvalues and the normal modes given by Eqs. (3.24) and (3.25). Because one of such normal modes is an unstable normal mode, we have concluded that non-self-similar inhomogeneous disturbances can develop in homogeneous collapse of a sufficiently low pressure perfect fluid.

As was mentioned in Sec. I, the unstable normal mode obtained in this paper was not found in the numerical analysis [7], which has rather claimed that for α lying in the range $0 < \alpha \lesssim 0.036$, the geometrical structure and the fluid motion at late stages in general non-self-similar collapse of a perfect fluid (which is initially homogeneous) can be well described by the flat Friedmann solution. However, recalling that the unstable normal mode obtained in this paper has the small growth rate (i.e., $\text{Im}(\omega_3)$) less than $1/3$, we suggest that even if the geometrical structure and the fluid motion once become similar to those of the flat Friedmann solution, non-self-similar disturbances become significant at much later stages which were missed in the numerical simulation [7]. Although the general relativistic version of the Larson-Penston solution was also suggested to act as an attractor in general inhomogeneous collapse by the results of the numerical simulation and the normal mode analysis in [7], we claim that its stability should be also confirmed in our analytical scheme.

Because of the above-mentioned result of the numerical simulation [7], the result obtained in this paper allows us to interpret the flat Friedmann solution as an intermediate attractor in general non-self-similar perfect fluid collapse which starts from a nearly homogeneous density profile, at least for sufficiently small α. If the general relativistic Larson-Penston solution is confirmed to be stable, a transition from the flat Friedmann stage to the general relativistic Larson-Penston stage may occur in gravitational collapse. (Such a transition was also mentioned in [7].) The so-called critical solution corresponding to the threshold between the black hole formation and the complete dispersion of the fluid is the well-studied self-similar perfect fluid solution (see e.g., [6]) acting as an intermediate attractor in general inhomogeneous collapse. It is interesting to note that there is a common feature between the flat Friedmann solution and the critical solution, namely, the single unstable normal mode exists for these solutions. It was found from the idea of the renormalization group in [8,9] that such a feature is essential to the scaling-law and the universality observed in the critical phenomena. Therefore what critical phenomena are relevant to the flat Friedmann solution will be an interesting problem to be investigated in future works.

APPENDIX: FUNCTIONS IN THE PERTURBATION EQUATIONS

It is an easy task to calculate the functions involved in Eqs. (2.9) and (2.14) by using the formulae given in [11]. The results are summarized as follows,

$$W = \frac{(1 + 3\alpha)x^3 + (7 + 9\alpha)x^2 + (1 + 3\alpha)(7 + 9\alpha)x + (1 + 3\alpha)^2}{3(1 + \alpha)x(1 + 3\alpha + x)}, \tag{A1}$$

$$F = -\frac{6(1 - \alpha)x^4 - (1 + 3\alpha)(9\alpha - 11)x^3 + 3(1 + 3\alpha)(5 + 7\alpha)x^2 + (1 + 3\alpha)^2(13 + 3\alpha)x + 3(1 + 3\alpha^3)}{3(1 + \alpha)x(1 + 3\alpha + x)(1 + 3\alpha + 2x)}, \tag{A2}$$

$$U = \frac{(1 + 3\alpha)(1 - x^2)\{2(\alpha - 1)x^4 + 4(\alpha - 1)(1 + 3\alpha)x^3 - (1 + 3\alpha)(5\alpha - 1)x^2 + 2(1 + 3\alpha)^2x + (1 + 3\alpha)^3\}}{3(1 + \alpha)^2x^2(1 + 3\alpha + x)(1 + 3\alpha + 2x)}, \tag{A3}$$

$$f = \frac{2\alpha x - 3\alpha - 1}{3\alpha(1 + 3\alpha + 2x)}, \qquad \text{(A4)}$$

$$B_1 = \frac{9\alpha(1 + 3\alpha + 2x)^2}{8(1 + 3\alpha)(1 + 3\alpha + x)}, \qquad \text{(A5)}$$

$$B_2 = -\frac{9\alpha(1 + x)(1 + 3\alpha + 2x)^2}{8(1 + 3\alpha)x(1 + 3\alpha + x)}, \qquad \text{(A6)}$$

$$B_3 = \frac{9\alpha(1 + 3\alpha + 2x)\{2(\alpha - 1)x^2 + (1 + 3\alpha)^2\}}{8(1 + \alpha)(1 + 3\alpha)x(1 + 3\alpha + x)}. \qquad \text{(A7)}$$

If the functions involved in Eq. (2.17) are expanded with respect to α as follows,

$$F + 2(1 - p)x = Q_0(x) + Q_1(x)\alpha + O(\alpha^2), \qquad \text{(A8)}$$

$$F + W = P_0(x) + P_1(x)\alpha + O(\alpha^2), \qquad \text{(A9)}$$

$$U = U_0(x) + U_1(x)\alpha + O(\alpha^2), \qquad \text{(A10)}$$

the inhomogeneous term J in Eq. (3.8) is given by

$$J(x, \omega) = -\frac{1}{1 - x^2}\{-6(2i\omega - Q_0) - 3Q_1\}\phi_{,x}^{(0)}$$
$$+ \frac{1}{(1 - x^2)^2}\{-36(i\omega P_0 + U_0)$$
$$+ 9(i\omega P_1 + U_1)\}\phi^{(0)}, \qquad \text{(A11)}$$

which leads us to the final result

$$J(x, \omega) = \frac{6}{(1 - x)(1 + x)^{2+3i\omega}x^4(1 + 2x)^4} \times [(1 + x)^{3+3i\omega}\{(x - 1)^3(64x^6 + 122x^5 + 119x^4 + 74x^3 + 34x^2 + 16x + 3)$$
$$- 2(104x^8 - 102x^7 - 45x^6 + 4x^5 - 109x^4 - 43x^3 + 2x^2 + 7x + 2)i\omega - 2x(160x^6 - 122x^5 - 217x^4 + 67x^3$$
$$+ 98x^2 + 28x + 4)\omega^2 + 36x^4(1 + 2x)(3 + 4x)i\omega^3\} + (1 - x)^{3+3i\omega}\{(1 + x)^3(64x^6 + 122x^5 + 119x^4 + 74x^3$$
$$+ 34x^2 + 16x + 3) + 2(88x^8 + 420x^7 + 880x^6 + 1047x^5 + 821x^4 + 447x^3 + 155x^2 + 28x + 2)i\omega$$
$$- 2x(1 + 2x)(56x^5 + 238x^4 + 336x^3 + 225x^2 + 73x + 8)\omega^2 - 24x^2(1 + x)(1 + 2x)^3 i\omega^3\}]. \qquad \text{(A12)}$$

In addition, the functions J_1, J_2 and J_3 in Eq. (3.16) are given by

$$J_1(x, \omega) = -\frac{6(1 + x)}{x^4(1 + 2x)^4}(64x^6 + 122x^5 + 119x^4 + 74x^3 + 34x^2 + 16x + 3), \qquad \text{(A13)}$$

$$J_2(x, \omega) = \frac{6i\omega(1 + x)}{x^4(1 + 2x)^4}\{-2(104x^8 - 102x^7 - 45x^6 + 4x^5 - 109x^4 - 43x^3 + 2x^2 + 7x + 2)$$
$$+ 2x(160x^6 - 122x^5 - 217x^4 + 67x^3 + 98x^2 + 28x + 4)i\omega + 36x^4(1 + 2x)(3 + 4x)\omega^2\}, \qquad \text{(A14)}$$

$$J_3(x, \omega) = \frac{6}{(1 + x)^{2+3i\omega}x^4(1 + 2x)^4} \times \{(1 + x)^3(64x^6 + 122x^5 + 119x^4 + 74x^3 + 34x^2 + 16x + 3) + 2(88x^8 + 420x^7$$
$$+ 880x^6 + 1047x^5 + 821x^4 + 447x^3 + 155x^2 + 28x + 2)i\omega - 2x(1 + 2x)(56x^5 + 238x^4 + 336x^3 + 225x^2$$
$$+ 73x + 8)\omega^2 - 24x^2(1 + x)(1 + 2x)^3 i\omega^3\}. \qquad \text{(A15)}$$

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