

String-inspired Gauss-Bonnet gravity reconstructed from the universe expansion history and yielding the transition from matter dominance to dark energy

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We consider scalar-Gauss-Bonnet and modified Gauss-Bonnet gravities and reconstruct these theories from the universe expansion history. In particular, we are able to construct versions of those theories (with and without ordinary matter), in which the matter-dominated era makes a transition to the cosmic acceleration epoch. In several of the cases under consideration, matter dominance and the deceleration-acceleration transition occur in the presence of matter only. The late-time acceleration epoch is described asymptotically by de Sitter space but may also correspond to an exact Λ CDM cosmology, having in both cases an effective equation of state parameter w close to -1 . The one-loop effective action of modified Gauss-Bonnet gravity on the de Sitter background is evaluated and it is used to derive stability criteria for the ensuing de Sitter universe.

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I. INTRODUCTION

Modified gravity is a promising theory that has become a very attractive gravitational alternative for dark energy (for a recent review, see Ref. [1–3]). It is, to begin with, a powerful scheme. Indeed, depending on the specific model considered, modified gravity is able to realize any of the proposed scenarios that have been delimited by the observational constraints leading to cosmic acceleration: effective phantom models, cosmological constant theories or quintessence. Also, it can easily account for the different epochs in the evolution of the universe that start to emerge clearly from the observational data (for a recent review of these data and their comparison with dark energy models, see Ref. [4]). The qualitative understanding of gravitational dark energy (see Ref. [3] for a review) is quite simple; some gravitational terms different from the usual general relativity ones may dominate at the very early or very late universe epochs. Thus, general relativity seems to be only approximately valid, both at very early as well as at very late times. To have the possibility to explain—in a unified way as modified gravity effects—fundamental cosmological phenomena such as early-time inflation and late-time acceleration, is very appealing. Moreover, modi-

fied gravity has the possibility to solve the coincidence problem, too, and can also clarify the role of dark matter in the formation and evolution of the universe.

Among the different classes of modified gravities, a very interesting one is the family of the string-inspired gravities. String-inspired scalar-Gauss-Bonnet gravity has been suggested in Ref. [5] as a possibility for gravitational dark energy. The idea that these theories may lie close to effective models which may come from fundamental string theories is in itself very appealing. Some time ago, Gauss-Bonnet (GB) gravity was applied to the possible resolution of the initial singularity problem [6]. Another version of string-inspired gravity, namely, the modified GB or $F(G)$ theory [7] can also play the role of gravitational dark energy. The investigation of different regimes of cosmic acceleration in such string-inspired gravity models has been carried out in Refs. [5,7–14].

A very strong theoretical restriction on modified gravity is caused by the natural emergence of the known classical universe expansion history. In other words, as in the case of the usual Λ CDM cosmology, it must faithfully reproduce, to start with, the sequence of the well-established cosmological phases: radiation/matter dominance, deceleration-acceleration transition, and the cosmic speed-up. Recently, a reconstruction scheme for the case of $f(R)$ gravity has been developed [15–17] where such a cosmological sequence is seen to occur quite naturally for some models (for a review on reconstruction from the universe expansion history, see Ref. [17,18]). Having in mind the fundamental importance of the correct description of the past and the current universes, in this work we develop a reconstruction program for string-inspired gravity from the universe expansion history. Using the method devel-

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oped in Refs. [13,19], it will be here demonstrated that scalar-Gauss-Bonnet or $F(G)$ gravity can be reconstructed for any given Friedman-Robertson-Walker (FRW) cosmology. Moreover, the role of string-inspired gravitational terms may be quite important, even in the matter-dominated epoch. Examples of exact and/or approximate Λ CDM cosmology in the above modified gravity are discussed too. The stability of such form of FRW cosmology, which naturally develops an asymptotically de Sitter future is investigated. As de Sitter universe naturally occurs at the early or late times in such models, special attention is paid to the de Sitter universe. Using the results of the calculation of the one-loop effective action of $F(G)$ gravity in the de Sitter space, the semiclassical stability of de Sitter space is investigated.

The paper is organized as follows. In the next section we develop the reconstruction scenario for string-inspired, scalar-Gauss-Bonnet gravity. Elaborating on the approach of Refs. [13,19], it is shown that the cosmological sequence of matter dominance, deceleration-acceleration transition and cosmic acceleration may occur in the specific version of that theory with some given potentials. It is also shown that an exact Λ CDM cosmology can indeed be reconstructed from such theory. Section III is devoted to the reconstruction of $F(G)$ gravity from the universe expansion history. For the same two classes of FRW universes (an approximate and an exact one), Λ CDM cosmology is derived in versions of the theory where this cosmology naturally occurs. By introducing perturbations around such cosmological solutions, the stability of those two FRW universes is investigated at the classical level. It is shown that the very early universe may be unstable, which opens the very interesting possibility of a natural exit from inflation, while the late stage, e.g. the asymptotically de Sitter universe, can be stable. Sec. IV is devoted to the study of the same reconstruction scenario for $F(G)$ gravity in the presence of several types of usual matter. Two specific versions of $F(G)$ gravity with matter are constructed where the matter dominance and deceleration-acceleration transition occur only *with matter*. After that, the accelerating universe becomes asymptotically (or even exactly) the de Sitter one.

In Sec. V we discuss the following problem: does it exist any scenario which may still improve the above situation, with the emergence of the matter-dominated era for scalar-Gauss-Bonnet or $F(G)$ gravity? It is shown that such scenario may be realized: by adding the compensating dark energy which is relevant in the matter dominance epoch mainly, the cosmological sequence of matter dominance, deceleration-acceleration transition, and cosmic acceleration, do occur in the versions of such theories which, otherwise, do not contain the matter dominance. Some discussion and outlook are presented in the last section.

Appendix A is devoted to the calculation of the one-loop effective action of $F(G)$ gravity on a de Sitter background.

Some technical remarks about the appearance of the multiplicative anomaly in those calculations are done. In Appendix B, this one-loop effective action is applied to study the semiclassical stability of de Sitter space in $F(G)$ gravity. This is an alternative method to investigate stability, which is advantageous as compared with the very involved one that uses cosmological perturbations. Numerical calculations provide examples of modified Gauss-Bonnet gravity which have a stable de Sitter vacuum solution.

II. RECONSTRUCTION OF THE SCALAR-GAUSS-BONNET THEORY FROM THE UNIVERSE EXPANSION HISTORY

In the present section, we will show how the string-inspired, scalar-Gauss-Bonnet gravity theory, proposed as a dark energy model in Ref. [5], can indeed be reconstructed, for any requested cosmology. The starting action is

$$S = \int d^4x \sqrt{-g} \left[\frac{R}{2\kappa^2} - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) - \xi_1(\phi)G \right]. \quad (1)$$

Here G is the Gauss-Bonnet invariant and the scalar field ϕ is canonical in (1). We now assume that the FRW universe with scale factor $a(t)$: $ds^2 = -dt^2 + a(t)^2 \sum_{i=1, \dots, 4} (dx^i)^2$ and the scalar field ϕ only depend on t : $\phi = \phi(t)$. These assumptions are based on the observational data indicating that the spatial part of the universe is flat and the universe could be regarded to be homogeneous on large scales. In the early universe, the small inhomogeneity could have played an important role. For example, the large scale structure of the present universe could be generated from the very small initial fluctuation and the gravitational instability for the perturbation. In this paper, however, we are interested in the behavior of the universe on the scales larger than the “large” scale structure and we now assume the homogeneity of the universe. Hence, we now drop the spatial coordinate (x^i) dependence of a and ϕ . In the following, the Hubble rate H is defined by \dot{a}/a .

Then the FRW equations look like [5,13]:

$$0 = -\frac{3}{\kappa^2} H^2 + \frac{1}{2} \dot{\phi}^2 + V(\phi) + 24H^3 \frac{d\xi_1(\phi(t))}{dt}, \quad (2)$$

$$0 = \frac{1}{\kappa^2} (2\dot{H} + 3H^2) + \frac{1}{2} \dot{\phi}^2 - V(\phi) - 8H^2 \frac{d^2 \xi_1(\phi(t))}{dt^2} - 16H\dot{H} \frac{d\xi_1(\phi(t))}{dt} - 16H^3 \frac{d\xi_1(\phi(t))}{dt}. \quad (3)$$

and the scalar field equation is

$$0 = \ddot{\phi} + 3H\dot{\phi} + V'(\phi) + \xi_1'(\phi)G. \quad (4)$$

Now $G = 24(\dot{H}H^2 + H^4)$, and combining (2) and (3), one gets

$$\begin{aligned}
 0 &= \frac{2}{\kappa^2} \dot{H} + \dot{\phi}^2 - 8H^2 \frac{d^2 \xi_1(\phi(t))}{dt^2} - 16H\dot{H} \frac{d\xi_1(\phi(t))}{dt} \\
 &\quad + 8H^3 \frac{d\xi_1(\phi(t))}{dt} \\
 &= \frac{2}{\kappa^2} \dot{H} + \dot{\phi}^2 - 8a \frac{d}{dt} \left(\frac{H^2}{a} \frac{d\xi_1(\phi(t))}{dt} \right). \quad (5)
 \end{aligned}$$

Equation (5) can be solved with respect to $\xi_1(\phi(t))$, as

$$\begin{aligned}
 \xi_1(\phi(t)) &= \frac{1}{8} \int^t dt_1 \frac{a(t_1)}{H(t_1)^2} W(t_1), \quad (6) \\
 W(t) &\equiv \int^t \frac{dt_1}{a(t_1)} \left(\frac{2}{\kappa^2} \dot{H}(t_1) + \dot{\phi}(t_1)^2 \right).
 \end{aligned}$$

Combining (2) and (6), the scalar potential $V(\phi(t))$ is

$$V(\phi(t)) = \frac{3}{\kappa^2} H(t)^2 - \frac{1}{2} \dot{\phi}(t)^2 - 3a(t)H(t)W(t). \quad (7)$$

We now identify t with $f(\phi)$ and H with $g'(t)$, where f and g are some functions. Such identification has a close analogy with the method suggested for the reconstruction of the scalar-tensor theory in Ref. [19]. Let us consider the model where $V(\phi)$ and $\xi_1(\phi)$ can be expressed in terms of two functions f and g as

$$\begin{aligned}
 V(\phi) &= \frac{3}{\kappa^2} g'(f(\phi))^2 - \frac{1}{2f'(\phi)^2} - 3g'(f(\phi))e^{g(f(\phi))}U(\phi), \\
 \xi_1(\phi) &= \frac{1}{8} \int^\phi d\phi_1 \frac{f'(\phi_1)e^{g(f(\phi_1))}}{g'(f(\phi_1))^2} U(\phi_1), \\
 U(\phi) &\equiv \int^\phi d\phi_1 f'(\phi_1) e^{-g(f(\phi_1))} \left(\frac{2}{\kappa^2} g''(f(\phi_1)) + \frac{1}{f'(\phi_1)^2} \right). \quad (8)
 \end{aligned}$$

By choosing $V(\phi)$ and $\xi_1(\phi)$ as (8), we find the following solution for Eqs. (2) and (3) (compare with Ref. [19]):

$$\begin{aligned}
 \phi &= f^{-1}(t) \quad (t = f(\phi)), \\
 a &= a_0 e^{g(t)}, \quad (H = g'(t)). \quad (9)
 \end{aligned}$$

One can straightforwardly check that the solution (9) satisfies the field Eq. (4) [13].

One should now suggest some realistic ansatz for scale factor and scalar. Our choice below is motivated by the possibility to realize the matter-dominated stage as well as cosmic acceleration for such scale factor. The choice of scalar is dictated by the consistency with scale factor and simplicity condition. Nevertheless, still it remains some arbitrariness in our choice so several examples will be discussed.

Consider now as an example the metric

$$e^{g(t)} = \left(\frac{t}{t_0} \right)^{g_1} e^{g_0 t}, \quad \phi = f^{-1}(t) = \phi_0 \ln \frac{t}{t_0}, \quad (10)$$

where t_0 , g_0 , g_1 , and ϕ_0 are constants. One further chooses

$$\phi_0^2 = \frac{2g_1}{\kappa^2}. \quad (11)$$

Then U is a constant $U = U_0$, and

$$\begin{aligned}
 V(\phi) &= \frac{3}{\kappa^2} \left(g_0 + \frac{g_1}{t_0} e^{-\phi/\phi_0} \right)^2 - \frac{g_1}{\kappa^2 t_0^2} e^{-2\phi/\phi_0} \\
 &\quad - 3U_0 \left(g_0 + \frac{g_1}{t_0} e^{-\phi/\phi_0} \right) e^{g_1 \phi/\phi_0} e^{g_0 t_0} e^{\phi/\phi_0}, \\
 \xi_1(\phi) &= \frac{U_0}{8} \int^{t_0 e^{\phi/\phi_0}} dt_1 \left(g_0 + \frac{g_1}{t_1} \right)^{-2} \left(\frac{t}{t_0} \right) e^{g_0 t}. \quad (12)
 \end{aligned}$$

Equation (10) leads to

$$H = g_0 + \frac{g_1}{t}. \quad (13)$$

Hence, when t is small, the second term in Eq. (13) dominates and the scale factor behaves as $a \sim t^{g_1}$. Therefore, if $g_1 = 2/3$, a matter-dominated period, where a scalar may be identified with matter, could be realized. On the other hand, when t is large, the first term in Eq. (13) dominates and the Hubble rate H becomes constant. Therefore, the universe is asymptotically de Sitter space, which is an accelerating universe. The three-year Wilkinson Microwave Anisotropy Probe (WMAP) data are analyzed in Ref. [20], which shows that the combined analysis of WMAP with the supernova legacy survey (SNLS) constrains the dark energy equation of state w_{DE} pushing it clearly towards the cosmological constant value. The marginalized best fit values of the equation of state parameter at 68% confidence level are given by $-1.14 \leq w_{\text{DE}} \leq -0.93$. In case one takes as a prior that the universe is flat, the combined data gives $-1.06 \leq w_{\text{DE}} \leq -0.90$. As in our model the universe goes asymptotically to de Sitter space, we find $w_{\text{DE}} \rightarrow -1$. Therefore, it can easily accommodate these values of w_{DE} . For example, if $g_0 \approx 40$, $w_{\text{DE}} = -0.98$.

In the limit $U_0 \rightarrow 0$, the Gauss-Bonnet term in Eq. (1) vanishes and the action (1) reduces into that of the usual scalar-tensor theory with potential

$$V(\phi) = \frac{3}{\kappa^2} \left(g_0 + \frac{g_1}{t_0} e^{-\phi/\phi_0} \right)^2 - \frac{g_1}{\kappa^2 t_0^2} e^{-2\phi/\phi_0}, \quad (14)$$

which reproduces the result in Ref. [21].

Let us now consider a second example. In Einstein gravity with a cosmological constant and matter characterized by the equation of state (EOS) parameter w , the FRW equation has the following form:

$$\frac{3}{\kappa^2} H^2 = \rho_0 a^{-3(1+w)} + \frac{3}{\kappa^2 l^2}. \quad (15)$$

Here l is the length parameter coming from the cosmological constant. The solution of (15) is given by

$$a = a_0 e^{g(t)}, \quad (16)$$

$$g(t) = \frac{2}{3(1+w)} \ln \left(\alpha \sinh \left(\frac{3(1+w)}{2l} (t - t_0) \right) \right).$$

Here t_0 is a constant of the integration and

$$\alpha^2 \equiv \frac{1}{3} \kappa^2 l^2 \rho_0 a_0^{-3(1+w)}. \quad (17)$$

We now reconstruct the scalar-Gauss-Bonnet gravity

$$V(\phi) = \frac{3}{\kappa^2 l^2} \cosh^2 \left(\frac{\kappa \sqrt{3(1+w)}}{2} \phi \right) - \frac{3(1+w)}{2l^2 \kappa^2} \sinh^2 \left(\frac{\kappa \sqrt{3(1+w)}}{2} \phi \right) - \frac{3U_0}{l} \cosh \left(\frac{\kappa \sqrt{3(1+w)}}{2} \phi \right) \left\{ -\frac{1}{\alpha} \sinh \left(\frac{\kappa \sqrt{3(1+w)}}{2} \phi \right) \right\}^{-2/(3(1+w))}, \quad (19)$$

$$\xi_1(\phi) = -\frac{\alpha U_0 l^3 \kappa}{8\sqrt{3(1+w)}} \int^\phi d\phi_1 \cosh^{-2} \left(\frac{\kappa \sqrt{3(1+w)}}{2} \phi_1 \right) \left\{ -\frac{1}{\alpha} \sinh \left(\frac{\kappa \sqrt{3(1+w)}}{2} \phi_1 \right) \right\}^{-2/(3(1+w))-1},$$

which again reproduces the result in Ref. [21] in the limit of $U_0 \rightarrow 0$. Thus, the scalar-Gauss-Bonnet gravity with the scalar potentials under considerations reproduces the exact Λ CDM cosmology.

Equation (16) shows that when $t \sim t_0$, the scale factor behaves as $a \sim (t - t_0)^{2/(3(1+w))}$. Therefore, if $w = 0$ the matter-dominated period can be reproduced. On the other hand, when $t \rightarrow \infty$, a behaves as $a \sim e^{t/l}$, which tells us that the universe goes asymptotically to de Sitter space, with $w_{\text{DE}} \rightarrow -1$. Therefore, it could be consistent with WMAP and also with the combined data.

The Gauss-Bonnet term is usually induced in low-energy string theory. Hence, the string theories could determine the form of the functions $V(\phi)$ and $\xi_1(\phi)$. In fact, various types of string compactification give the exponential potentials $V(\phi)$ and the exponential function for $\xi_1(\phi)$. Then the functions $V(\phi)$ and $\xi_1(\phi)$ given in this section, if they are rather simple, could be given by such compactified string theories. At present, we do not know the theory which could give rather complicated functions $V(\phi)$ and $\xi_1(\phi)$ as in Eq. (19). We cannot, however, exclude such models, which might appear in future. Such functions might be related with nonperturbative string effects. In our formulation, the universe expansion history dictates the possible forms of $V(\phi)$ and $\xi_1(\phi)$, which may be obtained from (some) string compactifications and the nonperturbative effects.

III. $F(G)$ GRAVITY RECONSTRUCTION FROM THE UNIVERSE EXPANSION HISTORY AND ITS STABILITY

We can extend the formulation in the previous section to $F(G)$ gravity [7], whose action is given by

$$S = \int d^4x \sqrt{-g} \left[\frac{R}{2\kappa^2} + F(G) \right]. \quad (20)$$

model reproducing (16). If a function $g(t)$ is given by (16) and $f(\phi)$ is given by

$$f(\phi) = t_0 - \frac{2l}{3(1+w)} \ln \tanh \left(-\frac{\kappa \sqrt{3(1+w)}}{4} \phi \right), \quad (18)$$

$U(\phi)$ in Eq. (8) becomes a constant again, $U = U_0$. Then, $V(\phi)$ and $\xi(\phi)$ are found to be

The above action can be rewritten by introducing the auxiliary scalar field ϕ as [13]

$$S = \int d^4x \sqrt{-g} \left[\frac{R}{2\kappa^2} - V(\phi) - \xi_1(\phi)G \right]. \quad (21)$$

By variation over ϕ , one obtains

$$0 = V'(\phi) + \xi_1'(\phi)G, \quad (22)$$

which could be solved with respect to ϕ as

$$\phi = \phi(G). \quad (23)$$

By substituting the expression (23) into the action (21), we obtain the action of $F(G)$ gravity, with

$$F(G) = -V(\phi(G)) + \xi_1(\phi(G))G. \quad (24)$$

Note that the action (21) can also be obtained by dropping the kinetic term of ϕ from the action (1).

Assuming a spatially-flat FRW universe and the scalar field ϕ to depend only on t , we obtain the field equations corresponding to Eqs. (2) and (3):

$$0 = -\frac{3}{\kappa^2} H^2 + V(\phi) + 24H^3 \frac{d\xi_1(\phi(t))}{dt}, \quad (25)$$

$$0 = \frac{1}{\kappa^2} (2\dot{H} + 3H^2) - V(\phi) - 8H^2 \frac{d^2 \xi_1(\phi(t))}{dt^2} - 16H\dot{H} \frac{d\xi_1(\phi(t))}{dt} - 16H^3 \frac{d\xi_1(\phi(t))}{dt}. \quad (26)$$

Combining the above equations, one gets

$$\begin{aligned}
 0 &= \frac{2}{\kappa^2} \dot{H} - 8H^2 \frac{d^2 \xi_1(\phi(t))}{dt^2} - 16H\dot{H} \frac{d\xi_1(\phi(t))}{dt} \\
 &\quad + 8H^3 \frac{d\xi_1(\phi(t))}{dt} \\
 &= \frac{2}{\kappa^2} \dot{H} - 8a \frac{d}{dt} \left(\frac{H^2}{a} \frac{d\xi_1(\phi(t))}{dt} \right), \tag{27}
 \end{aligned}$$

which can be solved with respect to $\xi_1(\phi(t))$ as

$$\begin{aligned}
 \xi_1(\phi(t)) &= \frac{1}{8} \int^t dt_1 \frac{a(t_1)}{H(t_1)^2} W(t_1), \\
 W(t) &\equiv \frac{2}{\kappa^2} \int^t \frac{dt_1}{a(t_1)} \dot{H}(t_1). \tag{28}
 \end{aligned}$$

Combining (25) and (28), the expression for $V(\phi(t))$ follows:

$$V(\phi(t)) = \frac{3}{\kappa^2} H(t)^2 - 3a(t)H(t)W(t). \tag{29}$$

As there is a freedom of redefinition of the scalar field ϕ (which corresponds to the choice $f(\phi) = \phi$ in the last section), we may identify t with ϕ . Hence, we consider the model where $V(\phi)$ and $\xi_1(\phi)$ can be expressed in terms of a single function g as

$$\begin{aligned}
 V(\phi) &= \frac{3}{\kappa^2} g'(\phi)^2 - 3g'(\phi)e^{g(\phi)}U(\phi), \\
 \xi_1(\phi) &= \frac{1}{8} \int^\phi d\phi_1 \frac{e^{g(\phi_1)}}{g'(\phi_1)^2} U(\phi_1), \tag{30} \\
 U(\phi) &\equiv \frac{2}{\kappa^2} \int^\phi d\phi_1 e^{-g(\phi_1)} g''(\phi_1).
 \end{aligned}$$

By choosing $V(\phi)$ and $\xi_1(\phi)$ as (30), one can easily find the following solution for Eqs. (25) and (26):

$$a = a_0 e^{g(t)}, \quad (H = g'(t)). \tag{31}$$

After that, one can reconstruct $F(G)$ gravity in a way very similar to the scalar-Gauss-Bonnet theory in the last section.

Although the above formulation is very similar to that in the scalar-Gauss-Bonnet theory, there could be some difference: in scalar-GB gravity, since the scalar field has a kinetic term, it can propagate and there can be the possi-

bility to generate an extra force besides the Newtonian one. On the other hand, $F(G)$ gravity has no kinetic term for the scalar field and an extra force cannot be generated. In fact, one can consider the perturbation around the de Sitter background (compare with Sec. V), by writing the metric as $g_{\mu\nu} = g_{(0)\mu\nu} + h_{\mu\nu}$. Here, the Riemann tensor in the de Sitter background is given by

$$R_{(0)\mu\nu\rho\sigma} = H_0^2 (g_{(0)\mu\rho}g_{(0)\nu\sigma} - g_{(0)\mu\sigma}g_{(0)\nu\rho}). \tag{32}$$

The flat background corresponds to the limit $H_0 \rightarrow 0$. For simplicity, if we choose the gauge conditions $g_{(0)}^{\mu\nu} h_{\mu\nu} = \nabla_{(0)}^\mu h_{\mu\nu} = 0$, we find from the equation of motion without the energy-momentum tensor

$$0 = \frac{1}{4\kappa^2} (\nabla^2 h_{\mu\nu} - 2H_0^2 h_{\mu\nu}). \tag{33}$$

Since the contribution from the Gauss-Bonnet term does not appear except in the length parameter $1/H_0$ of the de Sitter space, the only propagating mode should be the graviton in the $F(G)$ gravity.

Let us now investigate the stability of cosmological solutions in $F(G)$ gravity. The stability study of such theory on a de Sitter background will be presented in seventh section where constraints to $F(G)$ from stability condition will be defined. For $V(\phi)$ and $\xi_1(\phi)$ given by (30), Eqs. (22) and (25) yield

$$\begin{aligned}
 \dot{H} &= -H^2 + \frac{g'(\phi)^2}{H^2} (g''(\phi) + g'(\phi)^2), \tag{34} \\
 \dot{\phi} &= \frac{g'(\phi)^3}{H^3} + \frac{g'(\phi)^2}{\kappa^2 H^3} e^{-g(\phi)} (H^2 - g'(\phi)).
 \end{aligned}$$

We now consider the perturbation of solution (31), that is,

$$H = g'(t), \quad \phi = t, \tag{35}$$

as

$$H = g'(t) + \delta H, \quad \phi = t + \delta\phi. \tag{36}$$

Here it is assumed that δH and $\delta\phi$ only depend on time t . From (34), we obtain

$$\frac{d}{dt} \begin{pmatrix} \delta H \\ \delta\phi \end{pmatrix} = M \begin{pmatrix} \delta H \\ \delta\phi \end{pmatrix}, \quad M = \begin{pmatrix} -\frac{2g''(t)}{g'(t)} - 4g'(t) & g'''(t) + 4g'(t)g''(t) + \frac{2g''(t)^2}{g'(t)} \\ -\frac{3}{g'(t)} + \frac{2e^{-g(t)}}{\kappa^2} & -\left(-\frac{3}{g'(t)} + \frac{2e^{-g(t)}}{\kappa^2}\right)g''(t) \end{pmatrix}. \tag{37}$$

If the two real parts of both eigenvalues are either negative or vanish, the system turns out to be stable under the perturbation. If λ is the eigenvalue, the corresponding eigenvalue equations are written as

$$\begin{aligned}
 0 &= \lambda^2 - \left(-\frac{g''(t)}{g'(t)} - 4g'(t) - \frac{2g''(t)e^{-g(t)}}{\kappa^2} \right) \lambda \\
 &\quad + \left(\frac{3}{g'(t)} - \frac{2e^{-g(t)}}{\kappa^2} \right) g'''(t), \tag{38}
 \end{aligned}$$

whose determinant is given by

$$\begin{aligned}
 D &= \left(-\frac{g''(t)}{g'(t)} - 4g'(t) - \frac{2g''(t)e^{-g(t)}}{\kappa^2} \right)^2 \\
 &\quad - 4 \left(\frac{3}{g'(t)} - \frac{2e^{-g(t)}}{\kappa^2} \right) g'''(t). \tag{39}
 \end{aligned}$$

The solutions of (38) are

$$\lambda = \lambda_{\pm} \equiv \frac{1}{2} \left(-\frac{g''(t)}{g'(t)} - 4g'(t) - \frac{2g''(t)e^{-g(t)}}{\kappa^2} \pm \sqrt{D} \right). \quad (40)$$

When $D < 0$, the eigenvalues are complex and, in order that the real parts are negative (as required for stability), we have

$$A(t) \equiv -\frac{g''(t)}{g'(t)} - 4g'(t) - \frac{2g''(t)e^{-g(t)}}{\kappa^2} < 0. \quad (41)$$

When $D > 0$, both eigenvalues are negative and the system is stable if and only if

$$\begin{aligned} A(t) &= -\frac{g''(t)}{g'(t)} - 4g'(t) - \frac{2g''(t)e^{-g(t)}}{\kappa^2} < 0, \\ B(t) &\equiv \left(\frac{3}{g'(t)} - \frac{2e^{-g(t)}}{\kappa^2} \right) g'''(t) > 0. \end{aligned} \quad (42)$$

One may investigate the stability of the cosmological solution (13):

$$g(t) = \int dtH = g_0 t + g_1 \ln \frac{t}{t_0}. \quad (43)$$

Here t_0 is a constant of the integration. Let us consider the case that $g_1 = 2/3$, which corresponds to dust. In the early universe, where $t \rightarrow 0$, we find

$$A(t) \sim \frac{4t_0^{2/3}}{3\kappa^2} t^{-8/3} > 0. \quad (44)$$

Hence, the solution is unstable, which is the typical property of the early-time universe. On the other hand, in the late universe, where $t \rightarrow \infty$, one finds

$$A(t) \sim -4g_0 < 0, \quad B(t) \sim \frac{6}{g_0 t^2} > 0. \quad (45)$$

Thus, the solution is stable and the future universe does not enter into any big rip-like era [22].

We also investigate the stability of the Λ CDM cosmology corresponding to (16). It is convenient to consider the case that $w = 0$, which corresponds to dust. Hence, in the early universe, where $t \rightarrow t_0$, we find

$$A(t) \sim \frac{4}{3\kappa^2} \left(\frac{2l}{3\alpha} \right)^{2/3} (t - t_0)^{-8/3} > 0. \quad (46)$$

Therefore, the solution is unstable. On the other hand, in the late universe, it follows that

$$A \sim -\frac{4}{l} < 0, \quad B(t) \sim \frac{54}{l^2} e^{-3(t-t_0)/l} > 0. \quad (47)$$

Thus, the solution is stable again.

In general, one can consider the case when $g(t)$ behaves asymptotically (that is, in either the early or the late universe) as

$$g(t) = h_0 \ln \frac{t}{t_0}. \quad (48)$$

The case $h_0 = 2/3$ corresponds to dust dominated universe. For simplicity, we only consider the case that $h_0 > 0$ (nonphantom case). Then, we find

$$\begin{aligned} A(t) &= \frac{1 - 4h_0}{t} + \frac{2h_0 t_0^{h_0}}{\kappa^2} t^{-h_0-2}, \\ B(t) &= \left(\frac{3}{h_0} t - \frac{2}{\kappa^2} \left(\frac{t}{t_0} \right)^{-h_0} \right) \frac{2h_0}{t^3}. \end{aligned} \quad (49)$$

When $0 < h_0 < 1/4$, it always follows that $A(t) > 0$; therefore, the solution is unstable. In the early universe, where $t \rightarrow 0$, $A(t)$ behaves as

$$A(t) \sim \frac{2h_0}{\kappa^2} \left(\frac{t}{t_0} \right)^{-h_0} \frac{1}{t^2} > 0. \quad (50)$$

As a consequence the solution is always unstable in the early universe. The early universe instability indicates that the simple solution (43) is unstable and the inflationary epoch should end as is widely expected.

On the other hand, in the late universe, where $t \rightarrow \infty$, we find

$$A(t) \sim \frac{1 - 4h_0}{t}, \quad B(t) \sim \frac{6}{t^2} > 0. \quad (51)$$

Then as long as $h_0 > 1/4$, the solution should be stable.

We have thus presented several examples of the reconstruction program of modified Gauss-Bonnet gravity from the universe expansion history which include the epochs of early-time inflation, the matter-dominated era, the deceleration-acceleration transition, and the acceleration epoch. The stability conditions of such Λ CDM-like cosmology have been investigated, showing that the late universe can indeed be stable, without exhibiting any type of big rip behavior (for their classification, see Ref. [23]). In a similar way, one can consider dark energy cosmology in other versions of the modified Gauss-Bonnet theory [7,12,24] and relate them with the radiation and matter-dominated epochs.

IV. RECONSTRUCTION OF MODIFIED GAUSS-BONNET GRAVITY WITH MATTER

It is not difficult to extend the above formulation in scalar-Gauss-Bonnet gravity and $F(G)$ gravity to include several matter terms with EOS parameters $w_i \equiv p_i/\rho_i$. Here ρ_i and p_i are the energy density and pressure of the i -th matter term. We now include several kinds of matter, like radiation, baryons, cold dark matter, etc. Then, instead of Eqs. (2) and (3) or Eqs. (25) and (26), the corresponding FRW equations are given as

$$0 = -\frac{3}{\kappa^2}H^2 + \frac{\eta}{2}\dot{\phi}^2 + V(\phi) + \sum_i \rho_i + 24H^3 \frac{d\xi_1(\phi(t))}{dt}, \quad (52)$$

$$0 = \frac{1}{\kappa^2}(2\dot{H} + 3H^2) + \frac{\eta}{2}\dot{\phi}^2 - V(\phi) + \sum_i p_i - 8H^2 \frac{d^2\xi_1(\phi(t))}{dt^2} - 16H\dot{H} \frac{d\xi_1(\phi(t))}{dt} - 16H^3 \frac{d\xi_1(\phi(t))}{dt}. \quad (53)$$

Here $\eta = 1$ corresponds to scalar-Gauss-Bonnet gravity and $\eta = 0$ to $F(G)$ gravity in the form (21). The energy conservation law

$$\dot{\rho}_i + 3H(\rho_i + p_i) = 0, \quad (54)$$

gives

$$\rho_i = \rho_{i0}a^{-3(1+w_i)}, \quad (55)$$

with a constant ρ_{i0} . Instead of (8), one should consider the model with

$$V(\phi) = \frac{3}{\kappa^2}g'(f(\phi))^2 - \frac{\eta}{2f'(f(\phi))^2} - 3g'(f(\phi))e^{g(f(\phi))}U_m(\phi),$$

$$\xi_1(\phi) = \frac{1}{8} \int^\phi d\phi_1 \frac{f'(\phi_1)e^{g(f(\phi_1))}U_m(\phi_1)}{g'(f(\phi_1))^2},$$

$$U_m(\phi) \equiv \int^\phi d\phi_1 f'(\phi_1)e^{-g(f(\phi_1))} \times \left(\frac{2}{\kappa^2}g''(f(\phi_1)) + \frac{\eta}{2f'(\phi_1)^2} + \sum_i (1+w_i)\rho_{i0}a_0 e^{-3(1+w_i)g(f(\phi_1))} \right). \quad (56)$$

In this way we reobtain the solution (9) also in the case when matter is included. This expression is different from (8) due to the last terms in U_m , that is, the terms proportional to ρ_{i0} . This term appears since we include matter. In (8), even without matter, the transition from the matter-dominated phase to acceleration can occur. Therefore in order for the transition to occur, we do not always need the matters themselves. In the real universe, of course, there are matters as in the model in this section. If we consider the model where $V(\phi)$ and $\xi_1(\phi)$ are given in Eq. (56) but matters are *not* included, the matter-dominated phase would not appear. Then in the model (56) *with* matter, the matter-dominated phase to the acceleration phase occurs not only due to pure Gauss-Bonnet term effects but due to combined effect of Gauss-Bonnet term and matter presence.

In case of $F(G)$ gravity with $\eta = 0$, we can choose $f(\phi) = \phi$. Then, expressions (56) can be simplified:

$$V(\phi) = \frac{3}{\kappa^2}g'(\phi)^2 - 3g'(\phi)e^{g(\phi)}\tilde{U}_m(\phi),$$

$$\xi_1(\phi) = \frac{1}{8} \int^\phi d\phi_1 \frac{e^{g(\phi_1)}\tilde{U}_m(\phi_1)}{g'(\phi_1)^2}, \quad (57)$$

$$\tilde{U}_m(\phi) \equiv \int^\phi d\phi_1 e^{-g(\phi_1)} \times \left(\frac{2}{\kappa^2}g''(\phi_1) + \sum_i (1+w_i)\rho_{0i}a_0 e^{-3(1+w_i)g(\phi_1)} \right).$$

It is instructive to consider an example of $F(G)$ gravity with only dust as matter, which could be baryons and dark matter with $w = 0$, and g given by

$$g(\phi) = \frac{2}{3} \ln(\sinh(C\phi)), \quad C \equiv \frac{2}{3} \sqrt{\frac{\rho_{0d}a_0\kappa^2}{3}}. \quad (58)$$

This $g(\phi)$ corresponds to Eq. (16) with only dust: $w = 0$. Here $g(\phi)$ is written in a little bit different way as will soon be seen. Equation (58) shows that U is a constant $U = U_0$, and

$$V(\phi) = \frac{2C^2}{\kappa^2} \coth^2(C\phi) - 3CU_0 \coth(C\phi) \sinh^{2/3}(C\phi),$$

$$\xi_1 = \frac{U_0}{8} \int^\phi d\phi_1 \sinh^{-4/3}(C\phi) \cosh^2(C\phi). \quad (59)$$

Equation (58) indicates that, when $\phi = t$ is small, $g(\phi)$ behaves as $g(\phi) \sim (2/3) \ln \phi$ and, therefore, the Hubble rate behaves as $H(t) = g'(t) \sim (2/3)/t$, which surely reproduces the matter-dominated phase. On the other hand, when $\phi = t$ is large, $g(\phi)$ behaves as $g \sim (2/3)(C\phi)$, that is, $H \sim 2C/3$ and the universe asymptotically goes to de Sitter space. Therefore, the model given by (59) *with matter* shows the transition from the matter-dominated phase to the acceleration universe, which is asymptotically de Sitter space. In fact, by comparing (58) with (16), one can identify

$$\alpha = 1, \quad C = \frac{3(1+w)}{2l}. \quad (60)$$

We now check if the transition from the matter-dominated phase to the acceleration phase could occur or not in this model *without matter*. It can be shown that, if one assumes the existence of the matter-dominated phase, a contradiction appears. Note that in this model without matter, there is no reason to identify ϕ with t . Equation (22) gives

$$0 = -\frac{4C^2}{\kappa^2} \frac{\cosh(C\phi)}{\sinh^3(C\phi)} + B^2U_0 \sinh^{-4/3}(C\phi) \times (3 - 2\cosh^2(C\phi)) + \frac{U_0}{8} \sinh^{-4/3}(C\phi) \cosh^2(C\phi)G. \quad (61)$$

In the matter-dominated phase, the curvature R and, there-

fore, the Gauss-Bonnet invariant should be large. Then Eq. (61) tells us that ϕ should be small in the matter-dominated phase, and we find

$$G \sim \frac{32C^{1/3}}{\kappa^2 U_0} \phi^{-5/3}. \quad (62)$$

On the other hand, in the matter-dominated phase, the Hubble rate $H(t)$ behaves as $H \sim (2/3)/t$ and, therefore,

$$G = 24(H^2 \dot{H} + H^4) \sim -\frac{64}{27t^4}. \quad (63)$$

Comparing (63) with (62), it follows that $\phi \sim t^{12/5}$. Hence,

$$V(\phi) \sim \frac{2C^2}{\kappa^2 \phi^2} \sim t^{-24/5}, \quad \xi_1 \sim -\frac{3U_0}{8B^{4/3}} \phi^{-1/3} \sim t^{-4/5}. \quad (64)$$

This behavior is in conflict with Eq. (25). The first Eq. (25) may be written as the usual FRW equation with an inhomogeneous EOS ideal fluid [25] (for a particular example of such an inhomogeneous EOS ideal fluid interpreted as time-dependent bulk viscosity, see Ref. [26])

$$\frac{3}{\kappa^2} H^2 = \rho_G, \quad \rho_G \equiv V(\phi) + 24H^3 \frac{d\xi_1(\phi(t))}{dt}. \quad (65)$$

Eqs. (63) and (64) show that ρ_G behaves as $t^{-24/5}$, but H^2 behaves as t^{-2} . Therefore, there is a discrepancy between the power of t on both sides of Eq. (65). Thus, the matter-dominated phase can in no way be realized *without* matter, and consequently the model (59) does *not* generate the transition from the matter-dominated phase to the acceleration phase *without* matter.

As the EOS parameter of the dark energy in the present universe is almost -1 , the universe could approach the de Sitter space asymptotically. Let us consider the form of the action when the universe does become de Sitter space. We now assume

$$g(t) = H_0 t, \quad f(\phi) = f_0 \phi. \quad (66)$$

Here H_0 and f_0 are constant. Since the Hubble parameter H is given by $H = \dot{g} = H_0$, the universe is in fact de Sitter space. For $F(G)$ gravity, one may put $f_0 = 1$. It follows that

$$\begin{aligned} U_m(\phi) &= -\frac{\eta}{2H_0 f_0^2} e^{-H_0 f_0 \phi} + \sum_i \frac{(1+w_i)\rho_{i0} a_0}{(2+3w_i)H_0} e^{(2+3w_i)H_0 f_0 \phi} + U_0, \\ V(\phi) &= \frac{3H_0^2}{\kappa^2} + \frac{\eta}{f_0^2} - \sum_i \frac{3(1+w_i)\rho_{i0} a_0}{2+3w_i} e^{3(1+w_i)H_0 f_0 \phi} - 3H_0 U_0 e^{H_0 f_0 \phi}, \\ \xi_1(\phi) &= -\frac{\eta \phi}{16H_0^3 f_0} + \sum_i \frac{(1+w_i)\rho_{i0} a_0}{24(1+w_i)(2+3w_i)H_0^4} e^{3(1+w_i)H_0 f_0 \phi} + \frac{U_0}{8H_0^3} e^{H_0 f_0 \phi} + \xi_0. \end{aligned} \quad (67)$$

Here U_0 and ξ_0 are integration constants. Effectively $\xi_0 = 0$, since a constant times the GB term is a total derivative. Even in the presence of matter, the de Sitter universe could still be realized by adding an exponential potential and the GB coupling.

It is interesting to consider the effect of matter to the corresponding cosmology (16). This could be done by adding the part compensating the contribution from matter to $U(\phi)$, which is chosen to be a constant in the model (16), as

$$U_m(\phi) = U_0 + \sum_i U_i(\phi), \quad U_i(\phi) \equiv (1+w_i)\rho_{i0} a_0 \int^{f(\phi)} dt \left[\alpha \sinh\left(\frac{3(1+w)}{2l}(t-t_0)\right) \right]^{-(4+3w_i)/(3(1+w))}. \quad (68)$$

Then $V(\phi)$ and $\xi_1(\phi)$ (19) are modified as

$$\begin{aligned} V(\phi) &= \frac{3}{\kappa^2 l^2} \cosh^2\left(\frac{\kappa\sqrt{3(1+w)}}{2}\phi\right) - \frac{3(1+w)}{2l^2 \kappa^2} \sinh^2\left(\frac{\kappa\sqrt{3(1+w)}}{2}\phi\right) \\ &\quad - \frac{3}{l} \left(U_0 + \sum_i U_i(\phi) \right) \cosh\left(\frac{\kappa\sqrt{3(1+w)}}{2}\phi\right) \left\{ -\frac{1}{\alpha} \sinh\left(\frac{\kappa\sqrt{3(1+w)}}{2}\phi\right) \right\}^{-2/(3(1+w))}, \\ \xi_1(\phi) &= -\frac{\alpha l^3 \kappa}{8\sqrt{3(1+w)}} \int^\phi d\phi_1 \left(U_0 + \sum_i U_i(\phi_1) \right) \cosh^{-2}\left(\frac{\kappa\sqrt{3(1+w)}}{2}\phi_1\right) \left\{ -\frac{1}{\alpha} \sinh\left(\frac{\kappa\sqrt{3(1+w)}}{2}\phi_1\right) \right\}^{-2/(3(1+w))-1}, \end{aligned} \quad (69)$$

Therefore, there exists a scalar-GB gravity which shows the transition from the matter-dominated phase to the acceleration phase, even in the presence of matter. This theory could be regarded as an extension of (58) with only dust (baryon and/or dark matter). Thus, in the model (68) without matter, the matter-dominated phase cannot be realized and therefore the transition from the matter-dominated phase to the acceleration era does not occur.

Let us investigate the asymptotic behavior of $U_i(\phi)$ in Eq. (68). When $t = f(\phi) \rightarrow t_0$, one gets

$$U_i(\phi) \rightarrow \frac{2l}{\alpha\{-1 + 3(w - w_i)\}} \times \left\{ \frac{3\alpha(1 + w)}{2l} (f(\phi) - t_0) \right\}^{\{-1 + 3(w - w_i)\}/3(1 + w)}. \quad (70)$$

When the model containing the matter-dominated phase is considered, one can find $w = 0$. Since usually $w_i \geq 0$, the power $[-1 + 3(w - w_i)]/3(1 + w)$ could be always negative. Hence, $U_i(\phi)$ could be singular when $t = f(\phi) \rightarrow t_0$. This is not something extraordinary. For example, the scale factor of the FRW universe with matter, whose EOS parameter is a constant w , behaves as $a \sim t^{-(2/3(1+w))}$, which is singular at $t = 0$.

On the other hand, when $t = f(\phi) \rightarrow \infty$, we find

$$U_i(\phi) \rightarrow U_{iF} - \left(\frac{\alpha}{2} \right)^{-(4+3w_i)/3(1+w)} \frac{2l}{4 + 3w_i} \times \exp\left(-\frac{4 + 3w_i}{2l} (f(\phi) - t_0) \right). \quad (71)$$

Here U_{iF} is an integration constant. When $t = f(\phi) \rightarrow \infty$, the second term vanishes very rapidly. We may choose $U_{iF} = 0$. Thus, in the case $U_0 \neq 0$, the corrections generated by adding matter could be neglected. These corrections become important only when $t = f(\phi) \rightarrow t_0$, which may correspond to the early universe.

Our study shows the existence of a big class of modified Gauss-Bonnet gravity models with matter where the cosmological sequence of matter dominance, deceleration-acceleration transition and cosmic acceleration occur *only* in the presence of matter.

V. A COMPENSATING DARK ENERGY

Some versions of scalar-Gauss-Bonnet or $f(G)$ gravity (with specific potentials) do not have a matter-dominated stage as a solution, not even in the presence of matter. Similarly, some of them have a stable matter-dominated era as a solution, hence, no transition to an acceleration epoch occurs. It is remarkable that even in this case, a realistic cosmology may emerge at the price of introducing a compensating dark energy. Such scenario has been proposed in Ref. [16], based on the example of modified gravity of Ref. [27]. Let us introduce a compensating

dark energy, which could be an ideal fluid and may help to realize the matter-dominated and deceleration-acceleration transition phases in modified GB gravity. One may question the necessity of such ideal fluid, saying, that already the Einstein gravity with negative pressure ideal fluid may describe the acceleration of the universe. The below scenario is more sophisticated: it is supposed that the acceleration is produced by scalar-GB interaction and compensating dark energy is negligible in the late epoch. (Hence, without GB term no cosmic acceleration occurs). The role of compensating dark energy is relevant only in matter-dominated era: it helps in the realization of matter dominance.

When the energy density ρ_c and p_c of the compensating dark energy are added, the FRW equations become

$$0 = -\frac{3}{\kappa^2} H^2 + \frac{\eta}{2} \dot{\phi}^2 + V(\phi) + 24H^3 \frac{d\xi_1(\phi(t))}{dt} + \rho_c + p_m, \quad (72)$$

$$0 = \frac{1}{\kappa^2} (2\dot{H} + 3H^2) + \frac{\eta}{2} \dot{\phi}^2 - V(\phi) - 8H^2 \frac{d^2\xi_1(\phi(t))}{dt^2} - 16H\dot{H} \frac{d\xi_1(\phi(t))}{dt} - 16H^3 \frac{d\xi_1(\phi(t))}{dt} + p_c + p_m. \quad (73)$$

Here, ρ_m and p_m are contributions from matter, which may also include dark matter. In the matter-dominated phase, the contribution from the scalar-GB terms could correspondingly be canceled by that coming from the compensating dark energy. In other words, in the matter-dominated era the following conditions hold

$$\rho_c \sim -\left[\frac{\eta}{2} \dot{\phi}^2 + V(\phi) + 24H^3 \frac{d\xi_1(\phi(t))}{dt} \right], \quad (74)$$

$$p_c \sim -\left[\frac{\eta}{2} \dot{\phi}^2 - V(\phi) - 8H^2 \frac{d^2\xi_1(\phi(t))}{dt^2} - 16H\dot{H} \frac{d\xi_1(\phi(t))}{dt} - 16H^3 \frac{d\xi_1(\phi(t))}{dt} \right]. \quad (75)$$

If matter is dust, as baryons or cold dark matter, the Gauss-Bonnet invariant G behaves as Eq. (63).

For simplicity, we will first consider $F(G)$ gravity with $\eta = 0$. When $H = h/t$, G is given by $G = 24h^3(h - 1)/t^4$. Then G changes its sign when $h = 1$. In the decelerating universe, $h < -1$, but in the accelerating one the sign of h changes: $h > -1$. When the decelerating universe turns to the accelerating phase, G changes its sign. In this case, solving (22), if we do not choose $V(\phi)$ and ξ_1 properly, there is no solution. In the case $H = h/t$, Eqs. (30) yield

$$V(\phi) = \frac{3h^2(h - 1)}{(h + 1)\kappa^2\phi^2}, \quad \xi_1(\phi) = \frac{\phi^2}{8h(h + 1)\kappa^2}. \quad (76)$$

We now consider a new model by replacing h in (76) with $h(\phi)$, adiabatically depending on ϕ , so that one can neglect all derivatives $h'(\phi)$, $h''(\phi)$, \dots :

$$V(\phi) = \frac{3h(\phi)^2(h(\phi) - 1)}{(h(\phi) + 1)\kappa^2\phi^2}, \quad (77)$$

$$\xi_1(\phi) = \frac{\phi^2}{8h(\phi)(h(\phi) + 1)\kappa^2}.$$

Using (22) and neglecting $h'(\phi)$, $h''(\phi)$, \dots , we find, as expected, that

$$\phi^2 = \frac{24h(\phi)^3(1 - h(\phi))}{G}. \quad (78)$$

As the adiabatic approximation is used, our identification $\phi = t$ can again be introduced. For the matter-dominated era, by using (74) and (75), we find

$$\rho_c \sim -\frac{3h(t)^2}{\phi^2}, \quad p_c \sim \frac{h(3h(t) - 2)}{\kappa^2\phi^2}, \quad (79)$$

which gives the EOS parameter

$$w_c = \frac{p_c}{\rho_c} \sim \frac{2 - 3h}{3h}. \quad (80)$$

Hence, when $h \rightarrow 2/3$, which corresponds to dust, $w_c \rightarrow 0$ as expected. We should note that the energy density of the compensating dark energy ρ_c is now negative, which might be possible if there is a negative cosmological constant (for instance, the one produced by anti-de Sitter space), which effectively shifts the energy ρ_c by negative constant.

We now consider the scalar-Gauss-Bonnet theory (with $\eta = 1$) of Ref. [5], where

$$V = V_0 e^{-(2\phi/\phi_0)}, \quad \xi_1(\phi) = \xi_0 e^{2\phi/\phi_0}. \quad (81)$$

Without matter, the theory with those potentials (81) can be solved [5]

$$H = \frac{h_0}{t}, \quad \phi = \phi_0 \ln \frac{t}{t_1}, \quad (\text{when } h_0 > 0),$$

$$H = -\frac{h_0}{t_s - t}, \quad \phi = \phi_0 \ln \frac{t_s - t}{t_1}, \quad (\text{when } h_0 < 0). \quad (82)$$

Here h_0 and t_1 are related with V_0 and ξ_0 , as

$$V_0 t_1^2 = -\frac{1}{\kappa^2(1 + h_0)} \left[3h_0^2(1 - h_0) + \frac{\phi_0^2 \kappa^2 (1 - 5h_0)}{2} \right],$$

$$\frac{48\xi_0 h_0^2}{t_1^2} = \frac{6}{\kappa^2(1 + h_0)} \left(h_0 - \frac{\phi_0^2 \kappa^2}{2} \right). \quad (83)$$

Since the above h_0 is a constant, the EOS parameter w is

also a constant. Therefore, in this model without matter and compensating dark energy, the transition from the matter-dominated phase to the acceleration phase could *not* be generated.

In the following, it is enough to consider the case with $h_0 > 0$ only. Let us now add the contribution from the compensating dark energy and usual matter, and assume $H(t) = h(t)/t$ with a slowly varying function $h(t)$. The time dependence of ϕ can be determined by solving the scalar field Eq. (4). Neglecting $h'(\phi)$, $h''(\phi)$, \dots in the adiabatic regime, we get

$$\phi = \phi_0 \ln \frac{t}{t_1(t)}. \quad (84)$$

Here $t_1(t)$ is found from the following equation:

$$0 = (1 - 3h(t))\phi_0^2 + 2V_0 t_1(t)^2 - \frac{48\xi_0 h(t)^3}{t_1^2} (h(t) - 1). \quad (85)$$

Hence, in the matter-dominated phase, Eqs. (74) and (75) are

$$\rho_c \sim -\left\{ \frac{\phi_0^2}{2} + V_0 + 48\xi_0 h(t)^3 \right\} \frac{1}{t^2}, \quad (86)$$

$$p_c \sim -\left\{ \frac{\phi_0^2}{2} - V_0 + 16\xi_0 h(t)^2(1 - 2h(t)) \right\} \frac{1}{t^2}, \quad (87)$$

which give the effective EOS parameter

$$w_c = \frac{\frac{\phi_0^2}{2} - V_0 + 16\xi_0 h(t)^2(1 - 2h(t))}{\frac{\phi_0^2}{2} + V_0 + 48\xi_0 h(t)^3}. \quad (88)$$

Thus, scalar-Gauss-Bonnet gravity with potentials (81) and without matter and/or compensating dark energy does not contain the transition from the matter-dominated phase to the acceleration phase. Nevertheless, adding the matter and the compensating dark energy (86), the matter-dominated phase could be realized even for the model (81), similarly to other classes of modified gravity [16]. As expressed in Eq. (74), the compensating dark energy cancels the contributions from the scalar-Gauss-Bonnet term in the early universe and, therefore, the system effectively reduces to Einstein gravity coupled with matter, where the matter-dominated universe can in fact be generated. In the late universe, the contributions from matter and the compensating dark energy become *small* and the accelerated expanding universe naturally takes over. A similar scenario can be devised to improve the emergence of the matter dominance stage in other versions of modified gravity and scalar-tensor theories (for a recent study of accelerated cosmologies in scalar-tensor theories, see Ref. [28] and the references therein).

VI. DISCUSSION

In summary, we have tried to show in this paper that string-inspired scalar-Gauss-Bonnet gravity and modified Gauss-Bonnet gravity are indeed interesting alternatives for dark energy. These theories are in fact closely related and, what is also important, may have a stringy origin. Here, the reconstruction program from the universe expansion history for those theories has been carried out successfully. Several explicit examples of the same (with some specific potentials) have been presented where the cosmological sequence of the matter dominance, deceleration-acceleration transition, and cosmic acceleration occur very naturally. Moreover, the accelerated universe can be asymptotically de Sitter or it may correspond to an exact Λ CDM cosmology. The study of perturbations around the cosmological solutions above has been performed too. There is no problem with including the usual matter with a specific equation of state into such consideration. In that case, one can construct versions of the Gauss-Bonnet gravities above where a matter dominance period and a deceleration-acceleration transition occur *only* in the presence of matter. It is also remarkable that, even in the case when such intermediate universe is not a solution of some modified Gauss-Bonnet gravity, it can actually be made so, at the price of introducing some compensating dark energy.

Since the Gauss-Bonnet term is usually induced in low-energy string theory, the functions $V(\phi)$ and $\xi_1(\phi)$ could contain the information about the string compactification and/or stringy nonperturbative effects. By the reconstruction program in this paper, we suggested the possible forms of $V(\phi)$ and $\xi_1(\phi)$, which are cosmologically viable.

Because of the fundamental role of the de Sitter space which appears in our scenario as the final state of the universe, special attention has been paid to such space. In particular, the one-loop effective action of $F(G)$ gravity has been found on the de Sitter background. This effective action was then used to derive stability criteria for the modified Gauss-Bonnet gravity theory. Some numerical examples showing explicit versions of modified Gauss-Bonnet gravity with a stable de Sitter vacuum have been presented.

The successful reconstruction of string-inspired, Gauss-Bonnet gravity from the universe expansion history, performed in the present work, shows that it actually represents a reasonable gravitational alternative for dark energy. Having in mind the promising results obtained in the comparison of such a theory with observational data (see, for example, Ref. [9]), it becomes clear that it deserves careful attention. Moreover, the theory successfully passes the check of the three-year WMAP observational data. Needless to say, as with any other alternative to general relativity at the current and/or future universe, additional accurate checks regarding the solar system tests should still be carried out. Nevertheless, even if some problems might be encountered, there will always remain a reasonable

chance that the situation improves by taking into account higher-order string corrections, as has been wisely indicated in Ref. [13]. The introduction of such higher-order string corrections in the above scenario will be discussed elsewhere.

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APPENDIX A: ONE-LOOP EFFECTIVE ACTION IN $F(G)$ GRAVITY ON DE SITTER SPACE

The important issue in any fundamental gravitational theory is stability issue, as it indicates if some of the highly symmetric spaces (flat, de Sitter or anti-de Sitter) could be the ground state. Having in mind that de Sitter space appears as classical solution of the above theory in the early as well as in the late universe we study stability of GB modified gravity in de Sitter space.

In order to study the stability of $F(G)$ gravity, one possibility is to calculate the one-loop effective action on the corresponding background. It will be here shown that, in the calculation of the one-loop effective action in higher-derivative gravity a multiplicative anomaly appears naturally. Thus, before starting the calculation we first review this concept in a general situation. The usual definition of the determinant of an operator, A , is done through its zeta function [29] $\zeta_A(s) = \sum_{i \in I} \lambda_i^{-s} = \text{tr} A^{-s}$, where analytic continuation in s to cover the domain of the complex plain to the left of the abscissa of convergence $s = s_0$ is naturally assumed. Recall that $s_0 = m/n$, namely, it is given by the dimension of the working space $m = \dim M$, divided by the order of the operator, $n = \text{ord} A$. For an elliptic operator of positive order on a compact manifold, satisfying the usual spectral condition (e.g. the presence of an Agmon-Nirenberg cut in the spectrum), the zeta function exists and has very nice general properties, as it is meromorphic with possible single poles on integer and specific fractional positions $s_k = (n - k)/m$, $k = 0, 1, 2, \dots, n - 1, n + 1, \dots$ of the negative real axis only [30]. The determinant of A is then given by [31]

$$\det {}_z A = \exp[-\zeta'_A(0)]. \quad (\text{A1})$$

This definition only depends on the homotopy class of the spectral cut.

Now, given the operators A , B , and AB , even if ζ_A , ζ_B , and ζ_{AB} exist, it turns out that, in general, $\det_{\zeta}(AB) \neq \det_{\zeta}A \det_{\zeta}B$. The multiplicative, or noncommutative, or determinant anomaly (also called defect of the determinant) is defined as

$$\delta(A, B) = \ln \left[\frac{\det_{\zeta}(AB)}{\det_{\zeta}A \det_{\zeta}B} \right] = -\zeta'_{AB}(0) + \zeta'_A(0) + \zeta'_B(0). \quad (\text{A2})$$

There is a useful formula due to Wodzicki for the multiplicative anomaly [32,33]. In terms of Wodzicki's residue: $\text{res}A = 2\text{Res}_{s=0} \text{tr}(A\Delta^{-s})$, Δ being the Laplacian operator, or equivalently, in local form using the symbol expansion, $\text{res}A = \int_{S^*M} \text{tra}_{-n}(x, \xi) d\xi$, with $S^*M \subset T^*M$ the co-sphere bundle on M , the multiplicative anomaly can be obtained as

$$\delta(A, B) = \frac{\text{res}\{\ln\sigma(A, B)\}^2}{2\text{ord}A\text{ord}B(\text{ord}A + \text{ord}B)}, \quad (\text{A3})$$

$$\sigma(A, B) := A^{\text{ord}B} B^{-\text{ord}A}.$$

Several implications of this multiplicative anomaly have been extensively discussed in the literature [34]. Here we consider a very different situation, never met before, namely, the case when one of the two operators slowly vanishes owing to the change of a certain parameter, a situation that may be not too unfrequent. The main issue here is to check if, in fact, the shrinking operator could leave some imprint on the surviving operator, through the multiplicative anomaly term [34]. To simplify notation we will drop the ζ from the determinants hereafter.

Such anomaly appears basically in higher-derivative gravities, like the R^2 theory, when one calculates the one-loop effective action in some background. This will precisely be case under consideration here. The anomaly may persist even after one of the operators shrinks, say as a consequence of time evolution or by the action of some other parameter, as we are now going to see. In fact, as an example, let us consider the uniparametric family of operators B_{ϵ} , where ϵ is a real parameter that will eventually go to zero, e.g., adiabatically. For further simplicity, let us just take $B_{\epsilon} = \epsilon B$, an overall constant factor. It is easy to see from its definition that the anomaly $\delta(A, B_{\epsilon})$ does not actually depend on ϵ , in fact

$$\delta(A, B_{\epsilon}) = \delta(A, B). \quad (\text{A4})$$

And, again from the definition of the anomaly (A2), it turns out that in the limit when the B_{ϵ} operator adiabatically

disappears, a contribution may remain, in the way that

$$\frac{\det(AB_{\epsilon})}{\det B_{\epsilon}} \Big|_{\epsilon \rightarrow 0} = \det A \cdot e^{\delta(A, B)}. \quad (\text{A5})$$

This is again an implication of the presence of the multiplicative anomaly and, on its turn, of the definition of the zeta determinant. It contributes, generically, an additional term to the determinant of the resulting operator after taking the limit.

Now to the physics. Corresponding to the scalar version of $F(G)$ gravity, we have the Euclidean action

$$S_E[\tilde{g}] = -\frac{1}{\kappa^2} \int d^4x \sqrt{\tilde{g}} \left[\tilde{R} - \frac{\epsilon \tilde{g}^{ij} \partial_i \tilde{\phi} \partial_j \tilde{\phi}}{2} + F(\tilde{\phi}) + F'(\tilde{\phi})(\tilde{G} - \tilde{\phi}) \right], \quad (\text{A6})$$

which is equivalent to the modified Gauss-Bonnet gravity as investigated in Ref. [14]. In Eq. (A6) $\tilde{\phi}$ is a scalar field which ‘‘on shell’’ becomes equal to the Gauss-Bonnet invariant \tilde{G} , \tilde{g}_{ij} is the metric, \tilde{R} the related scalar curvature, and finally $F(\tilde{\phi})$ is an arbitrary smooth potential. An interesting observation is that the multiplicative anomaly commutes with the limit of the small parameter $\epsilon \rightarrow 0$. We will confirm that this is the case below. It may persist after the limit is enforced.

As it has been shown in Ref. [14], to ensure the existence of de Sitter solutions the function F has to satisfy the condition

$$GF'(G) - F(G) = \frac{R}{2}, \quad (\text{A7})$$

where the quantities G and R are, respectively, the Gauss-Bonnet invariant and the scalar curvature related to the de Sitter constant solution.

Now we are going to consider small fluctuations around the constant curvature solution g_{ij} , associated with de Sitter solution, so we put

$$\tilde{g}_{ij} = g_{ij} + h_{ij}, \quad h = g^{ij} h_{ij}, \quad \tilde{\phi} = G + \phi, \quad (\text{A8})$$

and we develop the action (A6) around the constant curvature solution up to second order in h_{ij} and ϕ . As usual all tensor indices are lowered and risen by means of g_{ij} . After a straightforward calculation, taking into account that we are dealing with a maximally symmetric space, for the quadratic part and disregarding total derivatives we obtain

$$\begin{aligned}
 \mathcal{L}_2 = & \frac{-f_0 \hat{h}_{ij}^2}{4} - \frac{R \hat{h}_{ij}^2}{6} + \frac{f_1 R^2 \hat{h}_{ij}^2}{6} + \frac{\hat{h}_{ij} \Delta \hat{h}_{ij}}{4} + \frac{f_1 R \hat{h}_{ij} \Delta \hat{h}_{ij}}{6} + \frac{f_0 R \xi_i^2}{8} + \frac{R^2 \xi_i^2}{16} - \frac{31 f_1 R^3 \xi_i^2}{288} + \frac{f_0 \xi_i \Delta \xi_i}{2} + \frac{R \xi_i \Delta \xi_i}{4} \\
 & - \frac{17 f_1 R^2 \xi_i \Delta \xi_i}{36} - \frac{f_1 R \xi_i \Delta \Delta \xi_i}{6} + \frac{f_0 h^2}{16} - \frac{f_2 \phi^2}{2} + \frac{f_1 h^2 R^2}{48} - \frac{f_2 h \phi R^2}{6} - \frac{3 h \Delta h}{32} + \frac{5 f_1 h R \Delta h}{32} + \frac{\epsilon \phi \Delta \phi}{2} \\
 & - \frac{f_2 h R \Delta \phi}{4} + \frac{h R \Delta \sigma}{16} - \frac{f_1 h R^2 \Delta \sigma}{16} + \frac{f_2 \phi R^2 \Delta \sigma}{12} - \frac{f_0 R \sigma \Delta \sigma}{16} - \frac{R^2 \sigma \Delta \sigma}{32} + \frac{17 f_1 R^3 \sigma \Delta \sigma}{288} + \frac{3 h \Delta \Delta \sigma}{16} \\
 & - \frac{3 f_1 h R \Delta \Delta \sigma}{16} + \frac{f_2 \phi R \Delta \Delta \sigma}{4} - \frac{3 f_0 \sigma \Delta \Delta \sigma}{16} - \frac{R \sigma \Delta \Delta \sigma}{8} + \frac{3 f_1 R^2 \sigma \Delta \Delta \sigma}{16} - \frac{3 \sigma \Delta \Delta \Delta \sigma}{32} + \frac{f_1 R \sigma \Delta \Delta \Delta \sigma}{32}, \quad (\text{A9})
 \end{aligned}$$

where $f_k = F^{(k)}(G)$ ($k = 0, 1, 2$) is the derivative of the function $F(\tilde{\phi})$ evaluated at G , while \hat{h}_{ij} , ξ_k , σ , and h are the irreducible components of the symmetric tensor field h_{ij} . They are related by

$$h_{ij} = \hat{h}_{ij} + \nabla_i \xi_j + \nabla_j \xi_i + \nabla_i \nabla_j \sigma + \frac{1}{4} g_{ij} (h - \Delta \sigma), \quad (\text{A10})$$

and satisfy the conditions

$$\nabla_k \xi^k = 0, \quad \nabla_i \hat{h}^{ij} = 0, \quad g^{ij} \hat{h}_{ii} = 0. \quad (\text{A11})$$

Here ∇_k and $\Delta = g^{ij} \nabla_i \nabla_j$ are, respectively, the covariant derivative and the Laplace-Beltrami operators related to the de Sitter metric g_{ij} . It has to be remarked that Eq. (A9) coincide with the analog expression written in Ref. [14] when one puts $\phi = \tilde{G}$ and $\epsilon = 0$ (see Sec. IV in the cited paper).

As is well known, invariance under diffeomorphisms renders the operator in the (h, σ) sector uninvertible. One needs a gauge fixing term and a corresponding ghost compensating term. We consider the class of gauge conditions

$$\chi_k = \nabla_j h_{jk} - \frac{1 + \rho}{4} \nabla_k h,$$

parametrized by the real parameter ρ . As gauge fixing we choose the standard expression

$$\mathcal{L}_{gf} = \frac{1}{2} \chi^i G_{ij} \chi^j, \quad G_{ij} = \alpha g_{ij}, \quad (\text{A12})$$

and so the corresponding ghost Lagrangian reads [35]

$$\mathcal{L}_{gh} = G_{ij} B_i \frac{\delta \chi^j}{\delta \xi^k} C^k, \quad (\text{A13})$$

where C_k and B_k are the ghost and antighost vector fields, respectively, while $\delta \chi^k$ is the variation of the gauge condition due to an infinitesimal gauge transformation of the

field. It reads

$$\delta h_{ij} = \nabla_i \varepsilon_j + \nabla_j \varepsilon_i \Rightarrow \frac{\delta \chi^j}{\delta \varepsilon^j} = g_{ij} \Delta + R_{ij} + \frac{1 - \rho}{2} \nabla_i \nabla_j. \quad (\text{A14})$$

Neglecting total derivatives one has

$$\mathcal{L}_{gh} = \alpha B^k \left(\Delta + \frac{R}{4} \right) C_k, \quad (\text{A15})$$

and finally in irreducible components we obtain

$$\begin{aligned}
 \mathcal{L}_{gf} = & \frac{\alpha}{2} \left[\xi^k \left(\Delta + \frac{R}{4} \right)^2 \xi_k + \frac{3\rho}{8} h \left(\Delta + \frac{R}{3} \right) \Delta \sigma - \frac{\rho^2}{16} h \Delta h \right. \\
 & \left. - \frac{9}{16} \sigma \left(\Delta + \frac{R}{3} \right)^2 \Delta \sigma \right], \quad (\text{A16})
 \end{aligned}$$

$$\mathcal{L}_{gh} = \alpha \left[\hat{B}^i \left(\Delta + \frac{R}{4} \right) \hat{C}^j + \frac{\rho - 3}{2} b \left(\Delta - \frac{R}{\rho - 3} \right) \Delta c \right], \quad (\text{A17})$$

where ghost irreducible components are defined by

$$\begin{aligned}
 C_k &= \hat{C}_k + \nabla_k c, & \nabla_k \hat{C}^k &= 0, \\
 B_k &= \hat{B}_k + \nabla_k b, & \nabla_k \hat{B}^k &= 0.
 \end{aligned} \quad (\text{A18})$$

In order to compute the one-loop contributions to the effective action one has to consider the path integral for the bilinear part

$$\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_{gf} + \mathcal{L}_{gh} \quad (\text{A19})$$

of the total Lagrangian and take into account the Jacobian due to the change of variables with respect to the original ones. In this way, one gets [35,36]

$$\begin{aligned}
 Z^{(1)} &= (\det G_{ij})^{-1/2} \int D[h_{ij}] D[C_k] D[B^k] \exp \left(- \int d^4 x \sqrt{g} \mathcal{L} \right) \\
 &= (\det G_{ij})^{-1/2} \det J_1^{-1} \det J_2^{1/2} \int D[h] D[\hat{h}_{ij}] D[\xi^j] D[\sigma] D[\hat{C}_k] D[\hat{B}^k] D[c] D[b] \exp \left(- \int d^4 x \sqrt{g} \mathcal{L} \right), \quad (\text{A20})
 \end{aligned}$$

where the determinant of the operator G_{ij} for our choice is trivial, while J_1 and J_2 are the Jacobians due to the change of variables in the ghost and tensor sectors, respectively. They read [35]

$$J_1 = \Delta_0, \quad J_2 = \left(-\Delta_1 - \frac{R}{4}\right)\left(-\Delta_0 - \frac{R}{3}\right)\Delta_0, \quad (\text{A21})$$

Δ_0 and Δ_1 being the Laplacians acting on scalar and vector fields.

Now, a straightforward computation (disregarding the multiplicative anomaly [37]) leads to the following off-shell one-loop contribution to the ‘‘partition function’’

$$\begin{aligned} e^{-\Gamma^{(1)}} \equiv Z^{(1)} &= \det\left(-\Delta_1 - \frac{R}{4}\right)^{1/2} \det\left(-\Delta_0 - \frac{R}{2}\right) \det\left[-\Delta_2 - \frac{R(9f_0 + 4R - 12X)}{3(4f_0 + 3R - 4X)}\right]^{-1/2} \\ &\times \det\left[\left(-\frac{9\epsilon}{256} - \frac{21\epsilon f_0}{256R} + \frac{f_2^2 R^2}{512} + \frac{21\epsilon X}{256R}\right)\Delta_0^3 \right. \\ &+ \left(-\frac{17\epsilon f_0}{256} + \frac{9f_2}{256} + \frac{21f_0 f_2}{256R} - \frac{7\epsilon R}{256} + \frac{7f_2^2 R^3}{1536} + \frac{31\epsilon X}{512} - \frac{21f_2 X}{256R}\right)\Delta_0^2 \\ &+ \left(\frac{17f_0 f_2}{256} - \frac{9\epsilon f_0 R}{1024} + \frac{7f_2 R}{256} - \frac{3\epsilon R^2}{1024} + \frac{5f_2^2 R^4}{1536} - \frac{31f_2 X}{512} + \frac{3\epsilon R X}{512}\right)\Delta_0 \\ &\left. + \left(\frac{9f_0 f_2 R}{1024} + \frac{3f_2 R^2}{1024} + \frac{f_2^2 R^5}{1536} - \frac{3f_2 R X}{512}\right)\right]^{-1/2}. \end{aligned} \quad (\text{A22})$$

For convenience we have written this in the Landau gauge corresponding to $\rho = 1$ and $\alpha = \infty$ and moreover we have put $X = f_0 + R/2 - Gf_1$. Thus, $X = 0$ is the ‘‘on-shell’’ condition corresponding to the de Sitter solution.

As a check one can easily see that Eq. (A22) has the correct limit in the Einstein plus cosmological constant case, that is when $\epsilon = 0$, $f_2 = 0$, $f_0 = -1/2$. In fact in such a case the complicated expression in the scalar sector decouples and one obtains the well known result reported in Ref. [36]. Furthermore, when $\epsilon = 0$, it can be shown that also at the one-loop level such a model is equivalent to the $F(G)$ theory developed in Ref. [14], which confirms the commutativity with the multiplicative anomaly.

APPENDIX B: STABILITY CRITERIA FOR MODIFIED GAUSS-BONNET GRAVITY

In this Appendix, applying the results of the previous Appendix we study the stability issue for several specific models. In principle, one can study the one-loop effective action for the de Sitter space explicitly in terms of special functions [36]. This happens in some cases, where one is able to express the determinant, say $\det(-\Delta_0^3 + a_1\Delta_0^2 - a_2\Delta_0 + a_3)$, in terms of a product of more elementary determinants of lower-dimensional operators. It is there that the multiplicative anomaly explicitly appears. Fortunately, for the case under discussion we are able to carry out a detailed analysis of the stability conditions of the model without the need to go through all this cumbersome process. This is what we do here.

In Ref. [38], the one-loop effective action has been also used in deriving a stability criterion for the class of modified models described by the function $f(R)$. Such a crite-

ri-
on of stability has been obtained by imposing the absence of negative eigenvalues related to the Laplace like operators appearing in the regularized one-loop effective action and it has been independently confirmed in Ref. [39] within a pure classical approach, namely, involving the study of cosmological perturbations.

In the case we are discussing here the situation is more complicated from the technical view point, since the functional determinant one has to investigate involves an algebraic polynomial of third order in the Laplace operator. But, in principle, a stability criterion for such a class of modified Gauss-Bonnet models could be investigated along the same lines, namely, by requiring the vanishing of the imaginary part in the one-loop effective action or the absence of negative eigenvalues. Note that the equivalent analysis of cosmological perturbations in $F(G)$ gravity is extremely complicated [9]. However, such study may be very important cosmologically, since it may prove (or disprove) the usual belief that the current (almost) de Sitter dark energy era may be eternal.

In order to discuss this issue we take the limit $\epsilon \rightarrow 0$ and consider the on-shell condition $X = 0$. In principle the operator of the scalar sector in (A22) may be written in the factorized form

$$\begin{aligned} A(X) &= X^3 + a_1 X^2 + a_2 X + a_3 \\ &= (X - X_1)(X - X_2)(X - X_3), \end{aligned} \quad (\text{B1})$$

where now $X = -\Delta_0$ is a non-negative differential operator, X_i are the roots of the third order algebraic equation $A(X) = 0$ and the constant coefficients a_i are given by

$$\begin{aligned} a_1 &= -R\left(\frac{7}{3} + \frac{18}{f_2 R^3} + \frac{42f_0}{f_2 R^4}\right), \\ a_2 &= R^2\left(\frac{5}{3} + \frac{14}{f_2 R^3} + \frac{34f_0}{f_2 R^4}\right), \\ a_3 &= -R^3\left(\frac{1}{3} + \frac{3}{2f_2 R^3} + \frac{9f_0}{f_2 R^4}\right). \end{aligned} \tag{B2}$$

The nature of the roots depend on the discriminant, D , which reads

$$D = \left(\frac{3a_2 - a_1^2}{9}\right)^3 + \left(\frac{9a_1 a_2 - 27a_3 - 2a_1^3}{54}\right)^2. \tag{B3}$$

Depending on the sign of the discriminant one has the three cases:

$$6f_0 - R^2 f_1 - 3R = 0, \quad \frac{27f_0}{f_2} + \frac{9R}{f_2} + 2R^4 < 0,$$

$$\begin{aligned} &76\,204\,800f_0^4 + 136\,950\,912f_0^3R + 92\,161\,152f_0^2R^2 + 27\,527\,040f_0R^3 + 3079\,296R^4 + 6901\,200f_0^3f_2R^4 \\ &+ 9430\,128f_0^2f_2R^5 + 4292\,784f_0f_2R^6 + 651\,024f_2R^7 + 171\,975f_0^2f_2^2R^8 + 154\,794f_0f_2^2R^9 + 34815f_2^2R^{10} \\ &+ 1600f_0f_2^3R^{12} + 704f_2^3R^{13} \leq 0. \end{aligned} \tag{B4}$$

The first one is the on-shell condition, while the second and the third ones derive from $a_3 > 0$ and $D \geq 0$ respectively.

When the discriminant is negative all the roots are distinct and the condition $a_3 > 0$ does not ensure the operator $A(X)$ to be non-negative. Of course, a sufficient condition can be obtained by requiring all the roots to be negative, but in such a case, for technical reasons we did not find a reasonable stable model.

As an example let us consider the choice $F(\tilde{G}) = \alpha\sqrt{\tilde{G}} + \beta\tilde{G}^\gamma$, α and γ being arbitrary dimensionless parameters, while β has dimensions $[\text{mass}]^{1-2\gamma}$. This is the only dimensional parameter in the model and for this reason it remains free also after imposing the above restrictions. The on-shell condition gives

$$R^{2\gamma-1} = \frac{6\gamma^{-1}(6 + \sqrt{6}\alpha)}{2\beta}, \tag{B5}$$

where α, β, γ are assumed to have values such that $R > 0$, since we want a de Sitter solution. The other two conditions above fix the possible ranges of α and γ . In particular it can be seen that there are stable solutions for very small, positive values of both α and γ , only.

A careful analysis of the stability conditions (B4) leads to the following conclusions. First, the whole analysis can be done in terms of f_0 and f_2 , by the substitution of the first equation, namely

$$f_1 = \frac{3}{R}\left(\frac{2f_0}{R} - 1\right). \tag{B6}$$

- (a) $D > 0$, then one root is real while the other two form a couple of conjugated complex numbers, say $X_1 = \bar{X}_3$.
- (b) $D = 0$, then all the roots are real and at least two of them are equal, say $X_2 = X_3$.
- (c) $D < 0$, then the three roots are real and distinct.

As a consequence, in the first two cases the factorization reads $A(X) = (X - X_1)B(X)$, $B(X)$ being a non-negative differential operator and then one has stability of the Gauss-Bonnet modified models provided $X_1 < 0$ or, which is equivalent, $a_3 > 0$, since $X_1X_2X_3 = -a_3$. Then requiring $a_3 > 0$ and $D \geq 0$ we arrive at the following sufficient conditions for the stability of the model we are considering on the de Sitter background:

Then, an analysis of the roots of the last inequality, seen as a polynomial of f_0 at the bound of the second inequality, shows that there is a stability region for values of f_0 around $f_0 \approx -R/2$, that is $f_2 \approx -6/R$, and $f_2 \leq 9/4R^3$. This is depicted in the figures. All of them are plots of the last inequality in Eq. (B4), where the relevant domain for f_0 is identified (as we see there, an additional one exists but it requires very precise fine tuning). Figure 1(a) is obtained for f_2 on the first border of the first of the inequalities in Eq. (B4), which is two-sided, namely

$$f_2 < (-9/2R^3)(1 + 3f_0/R), \quad f_0 < -R/3, \tag{B7}$$

$$f_2 > (-9/2R^3)(1 + 3f_0/R), \quad f_0 > -R/3. \tag{B8}$$

Figure 1(b) shows the same plot but for f_2 well inside the region delimited by the first of the inequalities: $f_2 = 1/R^3$.

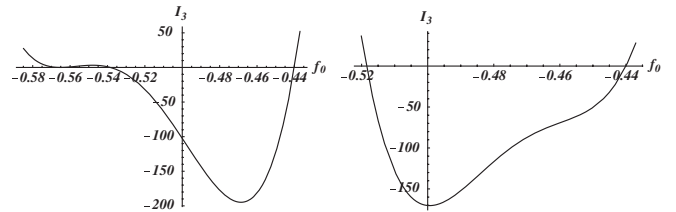


FIG. 1. Figure 1(a), left, plot of the left-hand side of the last inequality in Eq. (B4), which we call I_3 , where f_2 lies on the left border of the first stability domain (B7). Figure 1(b), right, same plot where $f_2 = 1/R^3$ lies inside the region delimited by the first of the inequalities in Eq. (B4), not far from the right end of the first stability domain.

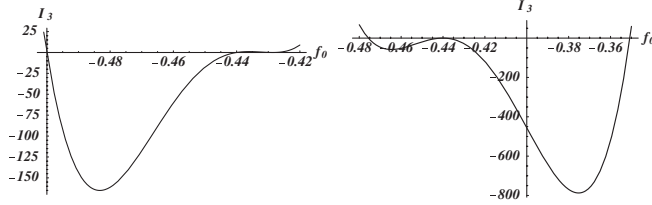


FIG. 2. Figure 2(a), left, plot of I_3 , where f_2 corresponds to the case $f_2 = 0$. Figure 2(b), right, same plot where $f_2 = -1/R^3$ lies close to the left end of the second stability domain (B8).

Figure 2(a) shows the stability region corresponding to the very particular case of a vanishing second derivative of $F(G)$, namely $f_2 = 0$, while Fig. 2(b) corresponds to the value $f_2 = -1/R^3$, which marks quite closely the end of the first stability domain (B7) for f_2 . We thus see that for

very reasonable values of the first derivatives of the function $F(G)$, we are inside the domain of convergence. To identify specific functions having these derivatives, however, is not an easy task, as the preceding example shows.

Similarly, one can study the stability conditions of the de Sitter universe which corresponds to the final era in the first of the reconstruction scenarios considered above, for other versions of string-inspired gravity. It is our impression, that this study, which recently became quite important due to the possible emergence of a (metastable) de Sitter vacuum in string theory, could prove to be less involved than the corresponding analysis of cosmological perturbations [9]. Indeed, even the classical stability study of higher-derivative gravity on the de Sitter background is already not that easy [40].

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