On Non-looping Term Rewriting

Yi Wang* and Masahiko Sakai

Graduate School of Information Science, Nagoya University, Furo-cho, Chikusa-ku, Nagoya, 4648603 Japan
{ywang80@trs.cm.,sakai@}is.nagoya-u.ac.jp

Abstract. Proving non-termination is important for instance if one wants to decide termination for given TRSs. Although the usual method is to find looping reduction sequences, there are non-looping infinite reduction sequences. We find some new interesting non-looping examples and propose new definitions of inner-looping sequence and normal sequence to classify them. We also show the undecidability of the existence of inner-looping sequence.

1 Introduction

Termination is one of the central properties of term rewriting systems (TRSs for short). We say a TRS terminates if it does not admit any infinite reduction sequences. Termination guarantees that any expression cannot be infinitely rewritten, and hence the existence of a normal form for it. Thus most researches on termination are for proving termination or for clarifying decidable classes. However, proving non-termination is also important for instance if one wants to decide termination for given TRSs.

An infinite reduction sequence often loops, that is, an instance of the starting term re-occurs as a subterm in the sequence. It is rather easy to detect loops and to give a proof of non-termination. However, some infinite reduction sequence may have no loop [5]. Its known that one-rule TRS that is non-terminating and admits no loop [8].

We give some new interesting examples and present new definitions of inner-looping sequence and normal sequence to classify them. We also show the undecidability of the existence of non-looping sequence.

2 Preliminaries

We assume the reader is familiar with the standard definitions of term rewriting systems [1]. A *signature* $\mathcal{F}$ is a set of function symbols, where every $f \in \mathcal{F}$ is associated with a non-negative integer by an arity function: $arity\colon \mathcal{F} \to \mathbb{N}(=\{0,1,2,\ldots\})$. The set of all *terms* built from a signature $\mathcal{F}$ and a countable infinite set $\mathcal{V}$ of *variables* such that $\mathcal{F} \cap \mathcal{V} = \emptyset$, is represented by $\mathcal{T}(\mathcal{F},\mathcal{V})$. We write $s = t$ when two terms s and t are identical.

The set of all *positions* in a term t is denoted by $\mathcal{P}os(t)$ and ε represents the root position. The *height* $|t|$ of a term t is 0 if t is a variable or a constant, and $1 + max(\{height(s_i) \mid i \in \{1,\ldots,m\}\})$ if $t = f(s_1,\ldots,s_m)$. Let C be a *context* with a hole $\Box$. We write $C[t]$ for the term obtained from C by replacing $\Box$ with a term t. A *substitution* θ is a mapping from $\mathcal{V}$ to $\mathcal{T}(\mathcal{F},\mathcal{V})$ such that the set $\mathrm{Dom}(\theta) = \{x \in \mathcal{V} \mid \theta(x) \neq x\}$ is finite. We usually identify a substitution θ with the set $\{x \mapsto \theta(x) \mid x \in \mathrm{Dom}(\theta)\}$ of variable bindings. We write $t\theta$ instead of $\theta(t)$.

A *rewrite rule* $l \to r$ is a directed equation which satisfies $l \notin \mathcal{V}$ and $\mathrm{Var}(r) \subseteq \mathrm{Var}(l)$. A *term rewriting system* TRS is a finite set of rewrite rules. The *reduction relation* $\to_R \subseteq \mathcal{T}(\mathcal{F},\mathcal{V}) \times \mathcal{T}(\mathcal{F},\mathcal{V})$ associated with a TRS R is defined as follows: $s \to_R t$ if there exist a

* Presently, with Financial Services Dept., Accenture Japan Ltd.

rewrite rule $l \to r \in R$, a substitution θ, and a context C such that $s = C[l\theta]$ and $t = C[r\theta]$. We say that s is reduced to t. The transitive closure of $\to_R$ is denoted by $\to_R^+$. The transitive and reflexive closure of $\to_R$ is denoted by $\to_R^*$. We also denote k-step reduction by $\to_R^k$.

For a TRS R, a term $t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ *terminates* if there is no infinite reduction sequence starting from t. We say that R terminates if every term terminates.

3 Loop and non-Loop

Infinite reductions are often composed of loops. A loop is a reduction where an instance of the starting term re-occurs as a subterm. It is obvious that a loop gives an infinite reduction. In fact, the usual way to deduce non-termination is to construct a loop.

Definition 1 (Loop). A *reduction sequence loops* if it contains $t \to_R^+ C[t\theta]$ for some context C, substitution θ and term t. A *TRS R admits a loop* if there is a looping reduction sequence of R.

Example 2. Let $R_1 = \{f(x) \to h(f(g(x)))\}$. We can construct the following reduction sequence: $t = f(x) \to h(f(g(x))) \to h(h(f(g(g(x))))) \to \cdots$ which loops with $C = h[\Box]$ and $\theta = \{x \mapsto g(x)\}$.

Definition 3 (Non-Looping TRS). A rewrite sequence is *non-looping* if it is infinite and does not contain any loop. A TRS is *non-looping* if it admits an non-looping sequence. A TRS is *properly non-looping* if it is non-looping and does not admit any looping sequence.

Example 4 ([4, 5]). The following TRS R_2 is non-looping.

$$R_2 = \begin{cases} b(c) \to d(c) \\ b(d(x)) \to d(b(x)) \\ a(d(x)) \to a(b(b(x))) \end{cases}$$

The TRS R_2 has an infinite rewrite sequence: $a(b(c)) \to^2 a(b(b(c))) \to^3 a(b(b(b(c)))) \to \cdots$.

4 Inner-Loop

We propose a new definition for a certain class of non-looping sequences which covers examples in the previous section. Moreover, we present some new interesting non-looping examples which also belongs to the class we proposed. Let $\Delta^i s\delta^i = \cdots \Delta[\Delta[\Delta[s\delta]\delta]\delta]\delta \cdots$, where context Δ and substitution δ repeat i times.

Definition 5 (Inner-Looping Sequence). Given a TRS R, let s be a term, an *inner-looping sequence* is of the form:

$$C[\Delta^{l_1} s\delta^{l_1}] \to_R^+ C[\Delta^{l_2} s\delta^{l_2}] \to_R^+ \cdots \qquad (*)$$

where C and Δ are contexts, δ is a substitution, $\{l_i\}$ is an infinite sequence of natural numbers.

Obviously, a looping sequence is an inner-looping sequence where $C = \Box$,

Definition 6 (Inner-Looping TRS). A TRS R is *inner-looping* if R admits an inner-looping sequence. A TRS R is *properly inner-looping* if R is inner-looping and does not admit any looping sequence.

A modified *Post's Correspondence Problem* (mPCP for short) is defined as follows:

Let $\{\langle u_i, v_i\rangle \in \Sigma^+ \times \Sigma^+ \mid 1 \le i \le n\}$ be a finite set of mPCP pairs. Does there exist an solution $u_1 u_{e_1} u_{e_2} \cdots u_{e_m} = v_1 v_{e_1} v_{e_2} \cdots v_{e_m}$ where $m \ge 0$ and $e_1, e_2, \ldots, e_m \in \{1, \ldots, n\}$?

Proposition 7. *The following TRSs are properly inner-looping.*

1. *R_2 in Example 4.*
2. $R_3 = \begin{cases} f(c, a(x), y) \to f(c, x, a(y)) \\ f(c, a(x), y) \to f(x, y, a^2(c)) \end{cases}$ *[8].*
3. *Let $\{\langle u_i, v_i\rangle \in \Sigma^+ \times \Sigma^+ \mid 1 \le i \le n\}$ be an instance of mPCP having a solution. Let $h_i \in \Sigma$. We write $h_1 h_2 h_3 \cdots h_n(c)$ for $h_1(h_2(h_3(\cdots h_n(c))))$. Let $u = h_1 h_2 h_3 \cdots h_n(c)$. We use notations $\overline{u} = h'_n h'_{n-1} h'_{n-2} \cdots h'_1(c)$ where h'_i is a fresh symbol that corresponds to h_i.*

$$\begin{aligned} R_4 = &\{f(\overline{u_1}(c), \overline{v_1}(c), z) \to f(u_1(z), u_1(z), u_1(z))\} \\ &\cup \{f(\overline{u_i}(x), \overline{v_i}(y), z) \to f(x, y, u_i(z)) \mid 2 \le i \le n\} \\ &\cup \{f(u_i(x), v_i(y), z) \to f(x, y, \overline{u_i}(z)) \mid 1 \le i \le n\} \end{aligned} \quad .$$

4.

$$R_5 = \begin{cases} f(a(x), y, w, z) \to f(x, y, w, a(z)) \\ f(x, a(y), w, z) \to f(x, y, w, a(z)) \\ f(a(c), a(c), w, z) \to f(w, z, z, a^2(c)) \end{cases} .$$

Proof. 1. We have an inner-looping sequence in the $(*)$ form: $C = a(\Box)$, $\Delta = b(\Box)$, $s = c$, $\delta = \emptyset$, $l_i = i$. The proof for non-existence of a looping sequence is found in [5].

2. We have an inner-looping sequence in the $(*)$ form: $C = f(c, a(c), \Box)$, $\Delta = a(\Box)$, $s = c$, $\delta = \emptyset$, $l_i = i$. The proof for non-existence of a looping sequence is found in [8].
3. Let $u_1 u_{e_1} u_{e_2} \cdots u_{e_m}(c)$ be a term corresponding to a solution, denote $t_p = u_{e_1} u_{e_2} \cdots u_{e_m} \overline{u_{e_m}} \cdots \overline{u_{e_2}}\, \overline{u_{e_1}}(\Box)$, then we have the following inner-looping sequence of R_5 in the $(*)$ form: $C = f(\overline{u_1}(c), \overline{v_1}(c), t_p[\overline{u_1}(\Box)])$, $\Delta = u_1(t_p[\overline{u_1}(\Box)])$, $s = c$, $\delta = \emptyset$, $l_i = 2^{i-1} - 1$.
   Consider an instance of mPCP $\{\langle a, aa\rangle, \langle ab, b\rangle\}$ having a solution aab, which leads to TRS R_6:

$$R_6 = \begin{cases} f(a'(c), a'a'(c), z) \to f(a(z), a(z), a(z)) & (1) \\ f(b'a'(x), b'(y), z) \to f(x, y, ab(z)) & (2) \\ f(a(x), aa(y), z) \to f(x, y, a'(z)) & (3) \\ f(ab(x), b(y), z) \to f(x, y, b'a'(z)) & (4) \end{cases} .$$

We have $t_p = abb'a'(\Box)$, $C = f(a'(c), a'a'(c), abb'a'a'(\Box))$, $\Delta = aabb'a'a'(\Box)$ and $s = c$. Indeed, it admits an inner-looping sequence:

$$\begin{aligned} &C[\Delta^0 s\delta] = f(a'(c), a'a'(c), abb'a'a'(c)) \\ &\to f(aabb'a'a'(c), aabb'a'a'(c), aabb'a'a'(c)) \\ &\to f(abb'a'a'(c), bb'a'a'(c), a'aabb'a'a'(c)) \\ &\to f(b'a'a'(c), b'a'a'(c), b'a'a'aabb'a'a'(c)) \\ &\to f(a'(c), a'a'(c), abb'a'a'aabb'a'a'(c)) = C[\Delta^1 s\delta] \\ &\to f(aabb'a'a'aabb'a'a'(c), aabb'a'a'aabb'a'a'(c), aabb'a'a'aabb'a'a'(c)) \\ &\to \cdots \ . \end{aligned}$$

The non-existence of a looping sequence of R_6 is shown as follows. Since non-innermost f's in a term do not contribute to infinite sequences, it is enough to consider terms in form of $t = f(t_1, t_2, t_3)$ with no f symbol inside. Rules except (1) decrease $|t_1|$ and increase $|t_3|$ but they do not change $|t_1|+|t_3|$. Hence, the rule (1) must be used infinitely many, which also requires the groundness of t. Since the rule (1) increases $|t_1| + |t_3|$ by $|t_3|$, we have no looping sequence.

4. An inner-looping sequence of R_5 is

$$C = f(a(c), a(c), \Box_p, \Box_q),\ \Delta = a[\Box],\ s = c,\ \delta = \emptyset,\ \{l_i\} = 1, 1, 2, 3, 5, 8, \ldots$$
$$C[\Delta^{l_i} s\delta^{l_i}]_p[\Delta^{l_{i+1}} s\delta^{l_{i+1}}]_q \to^{+}_{R_6} C[\Delta^{l_{i+1}} s\delta^{l_{i+1}}]_p[\Delta^{l_{i+2}} s\delta^{l_{i+2}}]_q\ .$$ [1]

□

It is worth pointing out that $\{l_i\}$ for R_5 is a Fibonacci Sequence.

Note that the TRS R_3 [8] is a little bit complex because the left-hand sides are the same for constructing an one-rule example $R_7 = \{f(c, a(x), y) \to g(f(c, x, a(y)), f(x, y, a^2(c)))\}$. The TRS $R_8 = \{f(a(x), y) \to f(x, a(y)),\ f(c, y) \to f(y, a(c))\}$ is a simpler example, which has a straightforward infinite inner-looping sequence $f(c, c) \to f(c, a(c)) \to^2 f(c, a^2(c)) \to^3 f(c, a^3(c)) \to^4 \cdots$. Here, we observe that the numbers of intermediate reduction steps increase in inner-looping sequences.

So far, we defined "inner-looping" property and showed the existence of properly inner-looping TRSs. It is easy to see that either looping or inner-looping property has some *special patterns* in its infinite rewrite sequence. So naturally we want to be able to answer the following question: is there some non-looping rewrite sequence *without any patterns at all*? We give the following definition inspired by normal numbers in mathematics [2]; real numbers whose digits show a random distribution with all digits appearing equally.

Definition 8 (Normal Sequence). Given TRS R, let $t_0 \to_R t_1 \to_R \cdots \to_R t_n \to_R \cdots$ be an infinite sequence starting from t_0, denoted by $\mathcal{S}$. Let $s \in B^*$ be a context of finite symbols in base $B \subseteq \mathcal{F}$. We say context s occurs in term t if $t = C[s[t']]$ for a context C and a term t'. Denote function $N(s, n)$ to be the number of times the context s occurs in t_n. We say the sequence $\mathcal{S}$ is *normal* in base B if $\lim_{n\to\infty} \frac{N(s,n)}{n} = \frac{1}{|B|^k}$ for every s with height k $(k = 1, 2, \ldots)$. A TRS R is *normal* if R admits a normal sequence.

Here "normal" says that when $n \to \infty$, in t_n every function symbol (context) shows a *random distribution* with all function symbols (contexts) appearing equally. Next proposition shows the existence of such a normal TRS.

Proposition 9. *TRS* $R_9 = R_{base} \cup R_{repeat} \cup R_{successor}$ *is normal.*

$$R_{base} = \begin{cases} f(a, x, 1(y)) \to 1(f(a, 1(x), y)) \\ f(a, x, 0(y)) \to 0(f(a, 0(x), y)) \\ f(a, x, \varepsilon) \to f(c, x, \varepsilon) \end{cases}$$
$$R_{repeat} = \begin{cases} f(b, 1(x), y) \to f(b, x, 1(y)) \\ f(b, 0(x), y) \to f(b, x, 0(y)) \\ f(b, \varepsilon, y) \to f(a, \varepsilon, y)) \end{cases}$$
$$R_{successor} = \begin{cases} f(c, 1(x), y) \to f(c, x, 0(y)) \\ f(c, 0(x), y) \to f(b, x, 1(y)) \\ f(c, \varepsilon, y) \to f(a, \varepsilon, 1(y))) \end{cases}$$

[1] This is a general case for inner-looping sequence by allowing context C to have "multi-holes".

Proof. Set base $B = \{0, 1\}$, $t_0 = f(a, \varepsilon, \varepsilon)$ starts a normal sequence of the form:

$$f(a,\varepsilon,\varepsilon) \to^* 1(f(a,1(\varepsilon),\varepsilon)) \to^* 110(f(a,01(\varepsilon),\varepsilon)) \to^* 11011(f(a,11(\varepsilon),\varepsilon)) \\ \to^* 11011100(f(a,001(\varepsilon),\varepsilon)) \to^* 11011100101(f(a,101(\varepsilon),\varepsilon)) \to^* \cdots .$$

□

It is well known that *Champernowne's Constant* [3]: $C_2 = 0.\underline{1}\,\underline{10}\,\underline{11}\,\underline{100}\,\underline{101}\cdots$ is a normal number. Notice that the sequence in TRS R_9 is imitating C_2.

At the end of this section, we state a negative result on the decidability of the existence of inner-looping sequences.

Theorem 10. *The inner-looping property and the properly inner-looping property for TRSs R are undecidable.*

Proof. It is known that the mPCP is undecidable. Considering strictly inner-looping TRS R_4 in Proposition 7, it can be proved that there is a non-looping sequence if and only if there exists a term $u_1 u_{e_2} u_{e_2} \cdots u_{e_m}(c)$ that is corresponding to a solution of mPCP. Consider the case that the given mPCP has no solution. As stated in the proof of Proposition 7, the first rule in R_4 must be used infinitely many for an inner-looping sequence. Thus it is easy to see its impossibility. Therefore, the theorem follows from the undecidability of mPCP. □

Note that the non-looping property is undecidable [7]. The existence of proper loops, $t \to^* C[t]$ with $C \neq \Box$, is shown to be undecidable by Otto [6].

Acknowledgement

We thank the anonymous referees for giving valuable comments. This work is partly supported by MEXT.KAKENHI #18500011 and #16300005.

References

1. F. Baader and T. Nipkow. Term rewriting and all that. *Cambridge University Press*, 1998.
2. D. H. Bailey and R. E. Crandall. Random Generators and Normal Numbers. *Experimental Mathematics*, 11:527–546, 2002.
3. D. G. Champernowne. The Construction of Decimals Normal in the Scale of Ten. *J. London Mathematical Society*, 8:254–260, 1933.
4. N. Dershowitz. Termination of rewriting. *J. Symb. Comput.*, 3:69–115, 1987.
5. A. Geser and H. Zantema. Non-looping String Rewriting. *Theoret. Informatics and Appl.*, 33:279–301, 1999.
6. F. Otto. The undecidability of self-embedding for finite semi-Thue and Thue systems. *Theoretical Computer Science*, 47:225–232, 1986.
7. D. Plaisted. The undecidability of self-embedding for term rewriting systems. *Information Processing Letters*, 20:61-64, 1985.
8. H. Zantema and A. Geser. Non-looping rewriting. *Technical Report Utrecht University*, UU-CS-1996-03, 1996.