Narrowing-based Simulation of Term Rewriting Systems with Extra Variables and its Termination Proof

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Abstract. Term rewriting systems (TRSs) are extended by allowing to contain extra variables in their rewrite rules. We call the extended systems EV-TRSs. They are ill-natured since every one-step reduction by their rules with extra variables is infinitely branching and they are not terminating. To solve these problems, this paper extends narrowing on TRSs into that on EV-TRSs and show that it simulates the reduction sequences of EV-TRSs as the narrowing sequences starting from ground terms. We prove the soundness of ground narrowing-sequences for the reduction sequences. We prove the completeness for the case of right-linear systems, and also for the case that no redex in terms substituted for extra variables is reduced in the reduction sequences. Moreover, we give a method to prove the termination of the simulation, extending the termination proof of TRSs using dependency pairs, to that of narrowing on EV-TRSs starting from ground terms.

1 Introduction

An extra variable is a variable appearing only in the right-hand side of a rewrite rule. Term rewriting systems (TRSs) are extended by allowing to contain extra variables in their rewrite rules. We call the extended systems *EV-TRSs*, especially *proper* EV-TRSs if they contain at least one extra variable. Proper EV-TRSs are ill-natured since every one-step reduction by their rules with extra variables is infinitely branching even up to renaming and none of them are terminating.

On the other hand, as a transformational approach to inverse computation of term rewriting, we have recently proposed an algorithm to generate a program computing the inverses of the functions defined by a given constructor TRS [10]. Unfortunately, the algorithm produces EV-TRSs in general. This fact gives rise to necessity of a simulation method of EV-TRSs.

This paper shows how to simulate the reduction sequences of EV-TRSs, and discusses the termination of the simulation. We first extend narrowing [6] on TRSs to that on EV-TRSs, restricting the substitutions for extra variables. In case of TRSs, the narrowing derivations starting from ground terms is just equivalent to the reduction sequences. This fact leads us to simulate the reduction sequences by the ground narrowing-sequences which are narrowing derivations starting from ground terms. Such a simulation solves the infinitely-branching problem, and terminates for some proper EV-TRSs. We prove the soundness of the ground narrowing-sequences for the reduction sequences of EV-TRSs. Then, we prove the completeness for the case of right-linear systems, and also for case that no redex in the terms substituted for extra variables is reduced in the reduction sequences. One of the typical instances of the latter case is a sequence constructed by substituting normal forms for extra variables. As a technique to prove termination of the proposed simulation, we extend the termination proof technique of TRSs using dependency pairs, proposed by T. Arts and J. Giesl [1], to that of narrowing on EV-TRSs (starting from ground terms).

This paper is organized as follows. In Section 3, we explain the idea for simulating EV-TRSs, define narrowing on EV-TRSs and prove the soundness and completeness. In Section 4, we discuss the termination of the simulations, i.e., narrowing starting from ground terms. Section 5 compares our results with the related works. We give the proofs of the theorems in the appendix.

2 Preparation

This paper follows the general notation of term rewriting [2, 7]. In this section, we briefly describe the notations used in this paper.

Let $\mathcal{F}$ be a signature and $\mathcal{X}$ be a countably infinite set of variables. The set of *terms* over $\mathcal{F}$ and $\mathcal{X}$ is denoted by $T(\mathcal{F},\mathcal{X})$. The set $T(\mathcal{F},\emptyset)$ of *ground terms* is simply written as $T(\mathcal{F})$. For a function symbol f, $\mathtt{arity}(f)$ denotes the number of arguments of f. The identity of terms s and t is denoted by $s \equiv t$. The set of variables in terms t_1,..., t_n is represented as $\mathcal{V}ar(t_1,...,t_n)$. The top symbol of a term t is denoted by $\mathtt{top}(t)$.

We use $\mathcal{O}(t)$ to denote the set of all positions of term t, and $\mathcal{O}_\mathcal{F}(t)$ and $\mathcal{O}_\mathcal{X}(t)$ to denote the set of function symbol positions and variable positions of t, respectively. For $p, q \in \mathcal{O}(t)$, we write $p \leq q$ if there exists p' satisfying $pp' = q$. The subterm at a position $p \in \mathcal{O}(t)$ is represented by $t|_p$. We use contexts with exactly one-hole $\Box$. When we explicitly write positions of $\Box$ in a context C, we write $C[\ ]_p$ where $p \in \mathcal{O}(C)$ and $C|_p \equiv \Box$. The notation $u \trianglelefteq t$ means that u is a subterm of t.

A *substitution* is a mapping σ from $\mathcal{X}$ to $T(\mathcal{F},\mathcal{X})$ such that $\sigma(x) \not\equiv x$ for finitely many $x \in \mathcal{X}$. We use σ, δ and θ to denote substitutions. Substitutions are naturally extended to mappings from $T(\mathcal{F},\mathcal{X})$ to $T(\mathcal{F},\mathcal{X})$ and $\sigma(t)$ is often written as $t\sigma$. We call $t\sigma$ an *instance* of t. The *composition* of σ and θ, denoted by $\sigma\theta$, is defined as $x\sigma\theta = \theta(\sigma(x))$. The *domain* and *range* of σ are defined as $\mathcal{D}om(\sigma) = \{\ x \in \mathcal{X} \mid x\sigma \not\equiv x\ \}$ and $\mathcal{R}an(\sigma) = \{\ x\sigma \mid x \in \mathcal{D}om(\sigma)\ \}$, respectively. The set of variables occurring in a term of $\mathcal{R}an(\sigma)$ is denoted by $\mathcal{VR}an(\sigma)$, i.e., $\mathcal{VR}an(\sigma) = \bigcup_{t\in\mathcal{R}an(\sigma)} \mathcal{V}ar(t)$. We write $\{x_1 \mapsto t_1,\ ...,\ x_n \mapsto t_n\}$

as σ if $\mathcal{D}om(\sigma) = \{x_1, ..., x_n\}$ and $x_i\sigma \equiv t_i$ for each i, and write $\emptyset$ instead of σ if $\mathcal{D}om(\sigma) = \emptyset$. We write $\sigma = \theta$ if $\mathcal{D}om(\sigma) = \mathcal{D}om(\theta)$ and $\sigma(x) \equiv \theta(x)$ for all $x \in \mathcal{D}om(\sigma)$. The *restriction* of σ to $X \subseteq \mathcal{X}$ is denoted by $\sigma|_X$, i.e., $\sigma|_X = \{x \mapsto t \mid x \in \mathcal{D}om(\sigma) \cap X, x\sigma \equiv t\}$. We write $\sigma \lesssim \sigma'$ if there exists θ satisfying $\sigma\theta = \sigma'$.

A *rewrite rule* is a pair (l, r), written as $l \to r$, where l ($\notin \mathcal{X}$) and r are terms. It may have a unique label ρ and be written as $\rho : l \to r$. Variables appearing only in the right-hand side of the rule ρ is called *extra variables* and the set of them are denoted by $\mathcal{EV}ar(\rho)$. An *EV-TRS* is a finite set of rewrite rules. Especially, it is called a *term rewriting system* if every rewrite rule $l \to r$ satisfies $\mathcal{V}ar(l) \supseteq \mathcal{V}ar(r)$. Let R be an EV-TRS. The *reduction relation* $\to_R$ is a binary relation on terms defined by $\to_R = \{(C[l\sigma], C[r\sigma]) \mid C \text{ is a context}, l \to r \in R\}$. When we explicitly specify the position p and the rule ρ in $s \to_R t$, we write $s \to_R^{[p,\rho]} t$ or $s \to_R^{p} t$. As usual $\xrightarrow{*}_R$ and $\xrightarrow{n}_R$ are a reflexive and transitive closure of $\to_R$, and the n-step reduction of $\to_R$, respectively. We call $s_0 \to_R s_1 \to_R \cdots$ a *reduction sequence* of R. R is said to be *right-linear* if the right-hand side of every rule is linear. A term t is *terminating* (*SN*, for short) with respect to R if there is no infinite reduction sequence of R starting from t. R is *terminating* (*SN*, for short) if every term is terminating with respect to R.

Terms s and t are *variants* if s and t are instances of each other. Letting $\to$ be a binary relation on terms, $\to$ is *finitely branching* if a set $\{ t \mid s \to t \}$ is finite up to renaming for any term s. Otherwise, it is *infinitely branching*. Let R be an EV-TRS. If R is a TRS then $\to_R$ is finitely branching. Otherwise, however, it is infinitely branching in general.

Example 1. The following R_1 is an EV-TRS;

$$R_1 = \{\ f(x, 0) \to s(x),\ \ g(x) \to h(x, y),\ \ h(0, x) \to f(x, x),\ \ a \to b\ \}.$$

A term $g(0)$ can be reduced by the second rule above to any of terms, $h(0, 0)$, $h(0, g(0))$, $h(0, f(0, 0))$ and so on. Thus, $\to_{R_1}$ is infinitely branching. □

3 Simulation by Ground Narrowing-sequences

In this section, we first explain the intuitive idea for simulating EV-TRSs. Then, we extend narrowing [6] on TRSs to that on EV-TRSs, and show how to simulate the reduction sequences of EV-TRSs using the ground narrowing-sequences. We prove the soundness of the ground narrowing-sequences for the reduction sequences of EV-TRSs, Then, we prove the completeness for the case of right-linear systems, and also for the EV-safe reduction sequences defined later.

Consider the sequences starting from $g(0)$ by R_1 in Example 1. Substituting a fresh variable for each extra variable makes it possible to focus on only a term such as $h(0, z)$ to which $g(0)$ is reduced by the second rule. In addition, narrowing enables z to act as an arbitrary term. Then, narrowing with such a restriction constructs the sequence in Fig.1 (a) which represents the infinitely many possibly infinite sequences in Fig.1 (b). That's why we extend narrowing

$$
\begin{array}{lllllll}
\text{(a)} & g(0) & \rightsquigarrow_{R_1} & h(0,z) & \rightsquigarrow_{R_1} & f(z,z) \rightsquigarrow_{\{z\mapsto 0\}} {}_{R_1} & s(0) \\
\text{(b)} & & & h(0,a) & \to_{R_1} & \cdots & \\
& & \nearrow_{R_1} & & & & \\
& g(0) & \to_{R_1} & h(0,0) & \to_{R_1} & f(0,0) \quad \to_{R_1} & s(0) \\
& & \searrow_{R_1} & h(0,g(a)) & \to_{R_1} & \cdots & \\
& & \vdots & \vdots & \ddots & &
\end{array}
$$

Fig. 1. (a) Simulation we propose, and (b) the sequences we simulate.

to simulate EV-TRSs.

On the other hand, since variables in the reduction sequences (not in narrowing sequences) act as constants, it is enough to treat terms in the sequences as ground terms by introducing new constants. For instance, the ground term $f(c_x, 0)$ represents the term $f(x, 0)$ with variable x. Thus, considering the reduction sequences starting from only ground terms covers those starting from any terms.

Next, we define narrowing on EV-TRSs, following the idea above. A *unifier* of terms s and t is a pair (σ, σ') of substitutions such that $s\sigma \equiv t\sigma'$[3]. The *most general unifier* of s and t, denoted by $\mathtt{mgu}(s,t)$, is a unifier (σ, σ') of s and t such that $\sigma \lesssim \theta$ and $\sigma' \lesssim \theta'$ for all unifier (θ, θ') of s and t.

Definition 1. *Let R be an EV-TRS. A term s is said to be* narrowable *into a term t with a substitution δ, a position $p \in \mathcal{O}(s)$ and a rewrite rule $\rho \in R$, written as $s \rightsquigarrow_{\delta}{}^{[p,\rho]}_{R}\, t$, if there exist a context C, a term u and a substitution σ such that*

(a) $p \in \mathcal{O}_{\mathcal{F}}(s)$, $s \equiv C[u]_p$, $t \equiv C\delta[r\sigma]_p$, $(\delta, \sigma) = \mathtt{mgu}(u, l)$ and $\mathcal{VR}an(\delta) \cap (\mathcal{V}ar(s) \setminus \mathcal{D}om(\delta)) = \emptyset$,
(b) $x\sigma \in (\mathcal{X} \setminus \mathcal{V}ar(s, u\delta))$ for all $x \in \mathcal{EV}ar(\rho)$, and
(c) $x \not\equiv y$ implies $x\sigma \not\equiv y\sigma$ for any $x, y \in \mathcal{EV}ar(\rho)$,

where $\rho : l \to r$. We assume $\mathcal{D}om(\delta) \subseteq \mathcal{V}ar(u)$. We call $\rightsquigarrow_R$ narrowing *by R. Note that p and ρ may be omitted like as $s \rightsquigarrow_{\delta}{}_R\, t$ or $s \rightsquigarrow_{\delta}{}^{p}_{R}\, t$.*

The above extension is done by adding the conditions (b) and (c) on extra variables to the ordinary definition of narrowing. We write $s \overset{n}{\rightsquigarrow}_{\delta}{}_R\, t$ or $s \overset{*}{\rightsquigarrow}_{\delta}{}_R\, t$ if there exists a narrowing derivation $s \equiv t_0 \rightsquigarrow_{\delta_0}{}_R\, t_1 \rightsquigarrow_{\delta_1}{}_R \cdots \rightsquigarrow_{\delta_{n-1}}{}_R\, t_n \equiv t$, called *narrowing sequence*, such that $\delta = \delta_0\delta_1 \cdots \delta_{n-1}$. If $n = 0$ then $\delta = \emptyset$. Note that δ may be omitted like as $s \overset{n}{\rightsquigarrow}_R\, t$ or $s \overset{*}{\rightsquigarrow}_R\, t$ if $\delta = \emptyset$. Especially, narrowing sequences starting from ground terms are said to be *ground*. From the definition

[3] Usually, unifiers are defined as single substitutions under the condition with no common variable in both terms. But we define unifiers as pairs of substitutions to eliminate renaming variables of rewrite rules in narrowing and to simplify treatments of variables in the proofs of theorems.

$$
\begin{array}{lll}
R_2 = \{ & add^{\#}(y) \to tp_2(0,y), & add^{\#}(s(z)) \to u_1(add^{\#}(z)), \\
& u_1(tp_2(x,y)) \to tp_2(s(x),y), & add^{\#}(add(x,y)) \to tp_2(x,y), \\
& mul^{\#}(0) \to tp_2(0,y), & mul^{\#}(0) \to tp_2(x,0), \\
& mul^{\#}(s(z)) \to u_2(add^{\#}(z)), & u_2(tp_2(w,y)) \to u_3(mul^{\#}(w),y), \\
& u_3(tp_2(x,s(y)),y) \to tp_2(s(x),s(y)), & mul^{\#}(mul(x,y)) \to tp_2(x,y) \quad \}.
\end{array}
$$

Fig. 2. The EV-TRS computing the inverses of addition and multiplication.

of narrowing, it is clear that $\underset{\delta}{\rightsquigarrow}_R$ are finitely branching for any EV-TRS R. It is also clear that $\to_R \; = \; \rightsquigarrow_R$ on ground terms for any TRS R.

Here, we show an example of the simulation by narrowing on EV-TRSs.

Example 2. The system computing inverse images $add^{\#}$ and $mul^{\#}$ of addition and multiplication, respectively, of two natural numbers is resulted in the EV-TRS seen in Fig.2 [10]. Considering the narrowing sequences starting from $mul^{\#}(s^4(0))$, there exist only 16 finite-paths up to renaming. This means that all solutions of $mul^{\#}(s^4(0))$ are found in finite time and space. One of such paths is as follows;

$$
\begin{array}{l}
mul^{\#}(s^4(0)) \overset{*}{\rightsquigarrow}_{R_2} u_3(u_3(mul^{\#}(0), s(0)), s(0)) \rightsquigarrow_{R_2} u_3(u_3(tp_2(0,y), s(0)), s(0)) \\
\qquad \underset{\{y \mapsto s^2(0)\}}{\rightsquigarrow}{}_{R_2} u_3(tp_2(s(0), s^2(0)), s(0)) \rightsquigarrow_{R_2} tp_2(s^2(0), s^2(0)). \qquad \square
\end{array}
$$

The following theorem shows the soundness whose proof is similar to that on TRSs [6].

Theorem 1. *Let R be an EV-TRS. For all $s \in T(\mathcal{F})$, $t \in T(\mathcal{F},\mathcal{X})$ and a substitution δ, $s \overset{*}{\underset{\delta}{\rightsquigarrow}}_R t$ implies $s\delta \overset{*}{\to}_R t$.*

Here, we introduce the notion of EV-safety of the reduction sequences. We say that a reduction sequence is *EV-safe* if no redex in the terms substituted for extra variables is reduced in that sequence. In the beginning of the appendix, a precise definition of this notion is found. Followings are results on the completeness.

Theorem 2. *Let R be an EV-TRS. For all $s, t \in T(\mathcal{F})$, the EV-safe sequence $s \overset{*}{\to}_R t$ implies a term t' and a substitution θ such that $s \overset{*}{\underset{\delta}{\rightsquigarrow}}_R t'$ and $t \equiv t'\theta$.*

Theorem 3. *Let R be a right-linear EV-TRS. For all $s, t \in T(\mathcal{F})$, $s \overset{*}{\to}_R t$ implies a linear term t' and a substitution θ such that $s \overset{*}{\underset{\delta}{\rightsquigarrow}}_R t'$ and $t \equiv t'\theta$.*

Since variables in target terms of reduction can be considered as constants and the simulation of EV-TRSs is done by ground narrowing-sequences, we focused on the ground reduction sequences in the above theorems.

The following example shows that the completeness does not hold in general, and also shows that the conditions in Theorem 2 and 3 are essential and necessary.

Example 3. Consider the sequence starting from $g(0)$ by R_1 in Example 1 again. We have the non-EV-safe reduction-sequence $g(0) \to_{R_3} h(0,a) \to_{R_3} f(a,a) \to_{R_3} f(a,b)$. However, this sequence cannot be simulated by narrowing. In fact, $g(0)$ is not narrowable to any term matchable with $f(a,b)$. $\square$

4 Termination of EV-TRSs' Simulation

In order to give the termination of the simulation by ground narrowing-sequences, we extend the termination proof technique of TRSs using dependency pairs, proposed by T. Arts and J. Giesl [1], to that of narrowing starting from ground terms.

Let R be an EV-TRS and t be a term. We say that t is *N-SN* with respect to R if there exists no infinite narrowing sequence starting from it. R is said to be *N-SN* if all terms are N-SN with respect to R. R is *N-GSN* if all ground terms are N-SN with respect to R. Since the proposed simulation is done by narrowing starting from ground terms, it is enough to consider N-GSN. Conversely, since most of EV-TRSs (even TRSs) are not N-SN, results on N-SN have very restrictive power. The following proposition holds obviously because the ground narrowing and reduction sequences on TRSs are equivalent.

Proposition 1. *A TRS is terminating if and only if it is N-GSN.*

The following proposition associated with N-SN and N-GSN also holds obviously.

Proposition 2. *If an EV-TRS is N-SN then it is also N-GSN.*

The converse of the above does not hold. For example, considering a TRS $R_4 = \{\ d(0) \to 0,\ \ d(s(x)) \to s(s(d(x)))\ \}$, this is N-GSN but not N-SN.

Let R be an EV-TRSs over a signature $\mathcal{F}$. The set of *defined symbols* of R is defined as $\mathcal{D}_R = \{\ \mathtt{top}(l) \mid l \to r \in R\ \}$, and the set of *constructors* of R as $\mathcal{C}_R = \mathcal{F} \setminus \mathcal{D}_R$. To define the dependency pairs, we prepare a fresh function symbol F not in a signature $\mathcal{F}$ for each defined symbol f. We call F the *capital symbol* of f. This paper uses small letters for function symbols in $\mathcal{F}$ and uses the string obtained by replacing the first letter of a defined symbol with the corresponding capital letter. For example, we use *Abc* as the capital symbol of a defined symbol *abc*. The set $\overline{\mathcal{D}}_R$ of capital symbols determined by the set $\mathcal{D}_R$ is defined as $\overline{\mathcal{D}}_R = \{\, F \mid f \in \mathcal{D}_R \,\}$. Moreover, we define $\overline{\mathcal{F}} = \mathcal{F} \cup \overline{\mathcal{D}}_R$. The definition of dependency pairs of EV-TRSs is the same as that of TRSs.

Definition 2. *Let R be an EV-TRS. The pair $\langle F(s_1,...,s_n), G(t_1,...,t_m)\rangle$ is called a* dependency pair *of R if there are a rewrite rule $f(s_1,...,s_n) \to r \in R$ and a subterm $g(t_1,...,t_m) \trianglelefteq r$ with $g \in \mathcal{D}_R$. The set of all dependency pairs of R is denoted by $\mathcal{DP}_R$.*

Let $\langle s,t\rangle \in \mathcal{DP}_R$. We also call variables only in t (i.e.,in $\mathcal{V}ar(t) \setminus \mathcal{V}ar(s)$) extra variables, and write $\mathcal{EV}ar(\langle s,t\rangle)$ as the set of all extra variables of $\langle s,t\rangle$.

4.1 Termination Proof of Reduction of TRSs

We first explain briefly the main part of the termination proof technique of TRSs' reduction using dependency pairs [1]. The notion of chain is as follows.

Definition 3 ([1]). *Let R be a TRS and $\langle s_1,t_1\rangle, \langle s_2,t_2\rangle, ... \in \mathcal{DP}_R$. The sequence $\langle s_1,t_1\rangle\langle s_2,t_2\rangle\cdots$ of dependency pairs is called an $R\&\mathcal{DP}_R$-*reduction-chain *(R-*chain*, for short) if there exist a term $s'_i \in T(\overline{\mathcal{F}},\mathcal{X})$ and a substitution σ_i such that $t_i\sigma_i \xrightarrow{*}_R s'_{i+1}$ and $s_i\sigma_i \equiv s'_i$ for each $i > 0$.*

Then, there is the following theorem that relates between non-existence of infinite chains and termination of TRSs.

Theorem 4 ([1]). *A TRS R is SN if and only if there is no infinite R-chain.*

To guarantee the non-existence of infinite chains, the following theorem is useful.

Theorem 5 ([1]). *Let R be a TRS over a signature $\mathcal{F}$. There is no infinite R-chain if and only if there is a quasi-reduction order*[4] $\succsim$ *on $T(\overline{\mathcal{F}}, \mathcal{X})$ such that*

- $l \succsim r$ *for every rule* $l \to r \in R$, *and*
- $s \succ t$ *for every dependency pair* $\langle s, t \rangle \in \mathcal{DP}_R$.

To find such an order, argument filtering functions are used.

Definition 4 ([8]). *An* argument filtering (AF) *is a function π such that for any $f \in \mathcal{F}$, $\pi(f)$ is either an integer i or a list $[i_1, ..., i_m]$ of integers where $0 \le m, i \le \mathtt{arity}(f)$ and $1 \le i_1 < \cdots < i_m \le \mathtt{arity}(f)$. Note that we assume $\pi(f) = [1, ..., n]$, where $n = \mathtt{arity}(f)$, if $\pi(f)$ is not defined. and also for each capital symbol. We can naturally extend π over terms as follows;*

- $\pi(x) = x$ *where* $x \in \mathcal{X}$,
- $\pi(f(t_1, ..., t_n)) = \pi(t_i)$ *where* $\pi(f) = i$,
- $\pi(f(t_1, ..., t_n)) = f(\pi(t_{i_1}), ..., \pi(t_{i_m}))$ *where* $\pi(f) = [i_1, ..., i_m]$.

Moreover, π is extended over an EV-TRS R and the set of its dependency pairs as $\pi(R) = \{\pi(l) \to \pi(r) \mid l \to r \in R\}$ and $\pi(\mathcal{DP}_R) = \{\langle \pi(s), \pi(t) \rangle \mid \langle s, t \rangle \in \mathcal{DP}_R\}$.

This paper assumes that $\pi(f)$ is not an integer but in form $[i_1, ..., i_m]$ for any defined symbol f, and also for any capital symbol. We say that such an AF function is *simple*. Order using AF function π is defined as follows;

- $s \succsim_\pi t$ if and only if $\pi(s) \succ \pi(t)$ or $\pi(s) \equiv \pi(t)$, and
- $s \succ_\pi t$ if and only if there are a contest C such that $\pi(s) \succ C[\pi(t)]$, or $C \not\equiv \Box$ and $\pi(s) \equiv C[\pi(t)]$.

Dependency graphs [1] combined with the above techniques are more powerful.

Example 4. Consider the following TRS;

$$R_5 = \{\, minus(x, 0) \to x, \;\; minus(s(x), s(y)) \to minus(x, y), \\ quot(0, s(y)) \to 0, \;\; quot(s(x), s(y)) \to s(quot(minus(x, y), s(y))) \,\}.$$

The dependency pairs of R_5 are as follows;

$$\mathcal{DP}_{R_5} = \{\, \langle M(s(x), s(y)), M(x, y) \rangle, \;\; \langle Q(s(x), s(y)), M(x, y) \rangle, \\ \langle Q(s(x), s(y)), Q(minus(x, y), s(y)) \rangle \qquad \},$$

[4] A *quasi-order* $\succsim$ is a reflexive and transitive relation, and $\succsim$ is called *well-founded* if its strict part $\succ$ is well-founded. A *quasi-reduction order* $\succsim$ is a well-founded quasi-order such that $\succsim$ is compatible with contexts and both $\succsim$ and $\succ$ are closed under substitutions.

$$
\begin{array}{ccccccc}
 & \langle s_1,\ t_1 \rangle & & \langle s_2,\ t_2 \rangle & & \langle s_3,\ t_3 \rangle & \cdots \\
 & \substack{(\delta_1,\sigma_1)=\\ \mathtt{mgu}(s'_1,s_1)} \vdots \quad \vdots & & \substack{(\delta_2,\sigma_2)=\\ \mathtt{mgu}(s'_2,s_2)} \vdots \quad \vdots & & \substack{(\delta_3,\sigma_3)=\\ \mathtt{mgu}(s'_3,s_3)} \vdots \quad \vdots & \\
\left(T(\overline{\mathcal{F}}) \ni \exists s_0 \overset{*}{\underset{\delta'_0}{\leadsto}}_R \right) s'_1 & t_1\sigma_1 \overset{*}{\underset{\delta'_1}{\leadsto}}_R & s'_2 & t_2\sigma_2 \overset{*}{\underset{\delta'_2}{\leadsto}}_R & s'_3 & t_3\sigma_3 \overset{*}{\underset{\delta'_3}{\leadsto}}_R & \cdots
\end{array}
$$

Fig. 3. A (ground) R-narrowing-chain.

where M and Q are abbreviations of $Minus$ and $Quot$, respectively. Let π_5 be an AF function with $\pi_5(minus) = [1]$. Then, the following inequalities are satisfied by the recursive path order (rpo) with a precedence $quot > s > m$ and $Q > M$;

$$
\begin{array}{rlrl}
minus(x) & \succsim_{\pi_5} x, & minus(x(x)) & \succsim_{\pi_5} minus(x), \\
quot(0, s(y)) & \succsim_{\pi_5} 0, & quot(s(x), s(y)) & \succsim_{\pi_5} s(quot(minus(x), s(y))), \\
M(s(x), s(y)) & \succsim_{\pi_5} M(x, y), & Q(s(x), s(y)) & \succsim_{\pi_5} Q(minus(x), s(y)), \\
Q(s(x), s(y)) & \succsim_{\pi_5} M(x, y). & &
\end{array}
$$

Therefore, R_5 is terminating by Theorem 4, 5. □

4.2 Termination Proof of Narrowing on EV-TRSs

Here, we extend the termination proof in Subsection 4.1 to that of narrowing.

We first extend the chain constructed by reduction to that by narrowing.

Definition 5. *Let R be an EV-TRS and $\langle s_1, t_1\rangle, \langle s_2, t_2\rangle, \ldots \in \mathcal{DP}_R$. The sequence $\langle s_1, t_1\rangle\langle s_2, t_2\rangle \cdots$ of dependency pairs is called an $R\&\mathcal{DP}_R$-*narrowing-chain *(R-*narrowing-chain*, for short) if there exist a term $s'_i \in T(\overline{\mathcal{F}}, \mathcal{X})$ and the most general unifier $(\delta_i, \sigma_i) = \mathtt{mgu}(s'_i, s_i)$ such that $t_i\sigma_i \overset{*}{\underset{\delta'_i}{\leadsto}}_R s'_{i+1}$ for each $i > 0$ where $x\sigma_i$ is a fresh variable for all $x \in \mathcal{EV}ar(\langle s_i, t_i\rangle)$ (see Fig.3). Especially, it is said to be* ground*, written as $s_0\langle s_1, t_1\rangle\langle s_2, t_2\rangle \cdots$, if there exists some ground term $s_0 \in T(\overline{\mathcal{F}})$ such that $s_0 \overset{*}{\underset{\delta'_0}{\leadsto}}_R s'_1$.*

Then, we obtain the following theorem that corresponds to Theorem 4.

Theorem 6. *Let R be an EV-TRS.*

(a) R is N-SN if and only if there is no infinite R-narrowing-chain.
(b) R is N-GSN if and only if there is no infinite ground R-narrowing-chain.

In the case of the termination proof of TRSs, we can do it by finding a reduction order to ensure no infinite chain. However, it is difficult to find such a order on proper EV-TRSs. Hence, we use AF functions again to eliminate extra variables.

Let R be an EV-TRS and π be an AF function. We say that π *eliminates all extra variables* of R and $\mathcal{DP}_R$ if $\mathcal{V}ar(\pi(l)) \supseteq \mathcal{V}ar(\pi(r))$ for all rules $l \to r \in R$ and $\mathcal{V}ar(\pi(s)) \supseteq \mathcal{V}ar(\pi(t))$ for all dependency pairs $\langle s, t\rangle \in \mathcal{DP}_R$. We extend

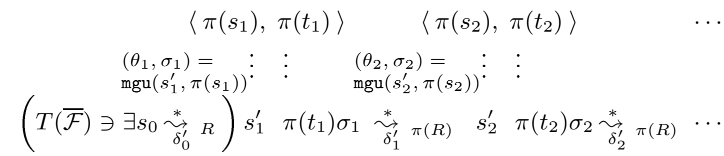

Fig. 4. A (ground) $\pi(R\&\mathcal{DP}_R)$-narrowing-chain.

the notion of (ground) $R\&\mathcal{DP}_R$-narrowing-chains into (ground) $\pi(R\&\mathcal{DP}_R)$-*narrowing-chains*, which is obtained by replacing R and $\mathcal{DP}_R$ with $\pi(R)$ and $\pi(\mathcal{DP}_R)$, respectively, in Definition 5 (see Fig.4).

No infinite $\pi(R\&\mathcal{DP}_R)$-narrowing-chain gives us the following theorem.

Theorem 7. *Let R be an EV-TRS and π be a simple AF function that eliminates all extra variables of R and $\mathcal{DP}_R$. If there exists no infinite $\pi(R\&\ \mathcal{DP}_R)$-narrowing-chain then R is N-GSN. Moreover, if $\pi(t)$ is a ground term for all $\langle s,t\rangle \in \mathcal{DP}_R$ then R is N-SN.*

If there is no extra variable in $\pi(R)$ and $\pi(\mathcal{DP}_R)$, we can check whether an infinite $\pi(R\&\mathcal{DP}_R)$-narrowing-chain exists, by using the termination proof techniques in Subsection 4.1 similarly to the case of TRSs. Note that this theorem is also usable to check whether ordinary narrowing of TRSs is terminating, although the result is very restrictive and narrowing sequences seldom terminate. For example, even a simple TRS $\{\ f(s(x)) \to f(x)\ \}$ which terminates is not N-SN, since term $f(y)$ with variable y leads to an infinite narrowing sequence.

Example 5. Consider R_1 in Example 1 again. The set of its dependency pairs is $\mathcal{DP}_{R_1} = \{\ \langle G(x), H(x,y)\rangle,\ \ \langle H(0,x), F(x,x)\rangle\ \}$.

The sequence $\langle G(x), H(x,y\rangle\langle H(0,x), F(x,x)\rangle$ is an R_1-narrowing-chain and $G(0)\langle G(x), H(x,y\rangle\langle H(0,x), F(x,x)\rangle$ is the ground one.

Let π_1 be a simple AF function with $\pi_1(h) = \pi_1(H) = \pi_1(s) = \pi_1(g) = \pi_1(G) = \pi_1(f) = \pi_1(F) = [\ ]$. Then, we have $\pi_1(R_1) = \{\ f \to s,\ \ g \to h,\ \ h \to f,\ \ a \to b\ \}$ and $\pi_1(\mathcal{DP}_{R_1}) = \{\ \langle G, H\rangle,\ \ \langle H, F\rangle\ \}$. It is clear that no infinite $\pi_1(R_1\&\mathcal{DP}_{R_1})$-narrowing-chain exists. Moreover, all right-hand sides of dependency pairs are ground. Therefore, R_1 is N-SN. □

Example 6. Consider the EV-TRS $R_6 = \{\ a \to d(c(y))\ \} \cup R_4$. Let π_6 be an AF function with $\pi_6(c) = [\,]$. Here, we have $\pi_6(R_6) = \{a \to d(c), d(0) \to 0, d(s(x)) \to s^2(d(x))\}$ and $\pi_6(\mathcal{DP}_{R_6}) = \{\ \langle A, D(c)\rangle, \langle D(s(x)), D(x)\rangle\}$. The inequalities $a \succsim_{\pi_6} d(c)$, $d(0) \succsim_{\pi_6} 0$, $d(s(x)) \succsim_{\pi_6} d(x)$, $A \succ_{\pi_6} D(c)$, and $D(s(x)) \succ_{\pi_6} D(x)$ are satisfied by the rpo with $a > d$ and $A > D$. Therefore, R_6 is N-GSN. It is clear that R_6 is not N-SN. □

The following corollary is a little weaker but easier to use than Theorem 7.

Corollary 1. *Let R be an EV-TRS and π be a simple AF function that eliminates all extra variables of R and $\mathcal{DP}_R$ and satisfying $\pi(\mathcal{DP}_R) = \mathcal{DP}_{\pi(R)}$. If*

$\pi(R)$ is terminating then R is N-GSN. Moreover, if every subterm u of $\pi(r)$ with $\mathtt{top}(u) \in \mathcal{D}_R$ is ground for all $l \to r \in R$ then R is N-SN.

In Example 5, we have $\pi_1(\mathcal{DP}_{R_1}) = \mathcal{DP}_{\pi_1(R_1)}$, and the right-hand side of all rules in $\pi_1(R_1)$ is ground. Hence, Corollary 1 is usable to prove that R_1 is N-GSN and N-SN.

5 Related Works

In studies on normalizing reduction strategies [3, 4, 12, 11], several kinds of EV-TRSs as approximations of TRSs are used. Arbitral reduction systems [3] are formalized as EV-TRSs whose right-hand sides are extra variables. They introduced an Ω-reduction system to simulate the reduction sequence, which is a special case of narrowing extended in this paper. Although they have termination, the theorems in Section 4 does not work to show their termination. The reason is that argument filtering method in this paper cannot eliminate all extra variables of collapsing rules. To overcome this problem is one of future works.

There are some studies on narrowing of conditional TRSs (CTRSs) with extra variables [5, 9]. The targets of their results are 3-CTRSs, in which every extra variable must appear in condition parts. On the other hand, EV-TRSs are not 3-CTRSs but 4-CTRSs, CTRSs with no restrictions. In addition, the CTRS, from which our motivating EV-TRS R_2 in Fig.2 is obtained by transformation, is not 3-CTRS but 4-CTRS.

Acknowledgments

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A EV-safe Reduction Sequences on EV-TRSs

Here, we give a precise definition of the EV-safe reduction sequences on EV-TRSs. Let t be a term and x be a variable. The set of positions that x occurs in t is denoted by $\mathcal{O}_x(t)$; $\mathcal{O}_x(t) = \{\, p \mid p \in \mathcal{O}_{\mathcal{X}}(t), t|_p \equiv x \,\}$. Let $P, Q \subseteq \mathcal{O}(t)$. We write $P \leq Q$ if for all $q \in Q$ there exists some $p \in P$ such that $p \leq q$. The set $P\backslash p$ is defined as $P\backslash p = \{\, q \mid pq \in P \,\}$. The minimum set of P is defined as $min(P) = \{\, p \mid p \in P, \neg(\exists q \in P, q < p) \,\}$. For example, $min(\{11, 1, 2\}) = \{1, 2\}$. We define the minimum set of union of P and Q as $P \sqcup Q = min(P \cup Q)$, and the minimum set of intersection of P and Q as $P \sqcap Q = \{\, p \mid p \in min(P), (\, \exists q \in min(Q), q \leq p \,) \,\} \cup \{\, q \mid q \in min(Q), (\, \exists p \in min(P), p \leq q \,) \,\}$. For example, $\{11, 22\} \sqcup \{112, 2, 31\} = \{11, 2, 31\}$ and $\{11, 22\} \sqcap \{112, 2, 31\} = \{112, 22\}$.

We give the notion of the transition of positions at one-step reduction, adding the positions of extra variables.

Definition 6. *Let a rewrite rule $\rho : l \to r$, let P be a set of positions and p be a position. We write $P \Rightarrow^{[p,\rho]} Q$ if there is no position q in P such that $q \leq p$, and Q satisfies the following;*

$$Q = \{\, q \mid q \in P, p \not\leq q \,\} \sqcup (\textstyle\bigsqcup_{x \in \mathcal{EV}ar(\rho)} \{\, pq \mid r|_q \equiv x \,\}) \\ \sqcup (\textstyle\bigsqcup_{x \in \mathcal{V}ar(l)} \{\, pqw \mid r|_q \equiv x, w \in (\textstyle\bigsqcap_{q' \in \mathcal{O}_x(l)} P\backslash pq') \,\}).$$

This notion of transition is similar to that of descendants that follows redex positions [4]. We use a set of positions, such as P and Q, to represent positions, under which reductions are prohibited. The notation of $P \Rightarrow^{[p,\rho]} Q$ shows the transition in the one-step reduction at the position p by the rule ρ. Now, we define EV-safety as follows.

Definition 7. *Let R be an EV-TRS, $\rho_i : l_i \to r_i \in R$. We say that the reduction sequence $s_0 \to_R^{[p_0,\rho_0]} s_1 \to_R^{[p_1,\rho_1]} \cdots$ is* EV-safe, *written as $P_0 : s_0 \to_R^{[p_0,\rho_0]} P_1 : s_1 \to_R^{[p_1,\rho_1]} \cdots$, if there are set $P_0, P_1, ...$ of positions such that (a) $P_0 = \emptyset$, and (b) for each $i \geq 0$, $P_{i+1} \subseteq \mathcal{O}(s_{i+1})$ and there exists $Q_{i+1} \subseteq \mathcal{O}(s_{i+1})$ with $P_i \Rightarrow^{[p_i,\rho_i]} Q_{i+1}$ and $P_{i+1} \leq Q_{i+1}$.*

Example 7. Consider R_1 in Example 1. The sequence $g(0) \to_{R_1} h(0,0) \to_{R_1} f(0,0) \to_{R_1} s(0)$ is EV-safe because of $\emptyset : g(0) \to_{R_1} \{2\} : h(0,0) \to_{R_1} \emptyset : f(0,0) \to_{R_1} \emptyset : s(0)$. On the other hand, the sequence $g(0) \to_{R_1} h(0,a) \xrightarrow{*}_{R_1} h(0,b)$ does not since the subterm a of $h(0,a)$ is reduced. □

B Proofs

Proof of Theorem 1 Let s and t be terms and δ be a substitution. We prove by induction on n that $s \overset{n}{\underset{\delta}{\rightsquigarrow}}_R t$ implies $s\delta \xrightarrow{*}_R t$.

Since the case of $n = 0$ is trivial, we assume that $s \overset{n-1}{\underset{\delta}{\rightsquigarrow}}_R u \underset{\delta'}{\rightsquigarrow}_R^{[p,\rho]} t$. Letting $\rho : l \to r \in R$, there exist a context C, a term u' and a substitution σ such that $u \equiv C[u']_p \underset{\delta'}{\rightsquigarrow}_R C\delta'[r\sigma]_p \equiv t$ and $(\delta', \sigma) = \mathtt{mgu}(u', l)$. From the definition of most general unifiers, we have $u'\delta' \equiv l\sigma$. We have $s\delta \xrightarrow{*}_R u$ by induction hypothesis. Then, it follows from the stability of reduction that $s\delta\delta' \xrightarrow{*}_R u\delta'$. Therefore, we have $s\delta\delta' \xrightarrow{*}_R u\delta' \equiv C[u']_p\delta' \equiv C\delta'[u'\delta']_p \equiv C\delta'[l\sigma]_p \to_R C\delta'[r\sigma]_p \equiv t$. □

We prepare the following lemmas. The following can be easily proved.

Lemma 1. *Let (σ, σ') be the most general unifier of terms s and t. For any unifier (θ, θ') of s and t, there exists a substitution δ such that $s\sigma\delta \equiv s\theta$ and $t\sigma'\delta \equiv t\theta'$.*

Lemma 2. *Let R be an EV-TRS. Let $P : s\theta \to_R P' : t$ and $P \leq \mathcal{O}_{\mathcal{X}}(s)$. Then, there is t' and θ' such that $s \underset{\delta}{\rightsquigarrow}_R t'$, $t \equiv t'\theta'$ and $P' \leq \mathcal{O}_{\mathcal{X}}(t')$.*

Proof. We assume that $s\theta \to_R^{[p,\rho]} t$ where $\rho : l \to r \in R$. Then, $s\theta \equiv C[l\sigma]_p$ and $t \equiv C[r\sigma]_p$, where we assume $\mathcal{D}om(\theta) \cap \mathcal{D}om(\sigma) = \emptyset$ and $\mathcal{D}om(\theta) = \mathcal{V}ar(s)$ without loss of generality.

From EV-safety, we have $p \in \mathcal{O}(s) \setminus \mathcal{O}_{\mathcal{X}}(s)$. Since there exist a context $C'[\;]_p$ and a term v such that $s \equiv C'[v]_p$, we have $s\theta \equiv C'\theta[v\theta]_p \equiv C[l\sigma]_p$. It follows from $v\theta \equiv l\sigma$ that (θ, σ) is a unifier of v and l. Let (δ, σ') be the most general unifier such that $x\sigma' \in \mathcal{X} \setminus (\mathcal{V}ar(s) \cup \mathcal{VR}an(\delta))$ for every $x \notin \mathcal{V}ar(l)$. Then, we have $s \equiv C'[v]_p \underset{\delta}{\rightsquigarrow}_R C'\delta[r\sigma']_p$. On the other hand, it follows from $\mathcal{V}ar(C'[\;]) \subseteq \mathcal{V}ar(s) = \mathcal{D}om(\theta)$ and Lemma 1 that $r\sigma'\theta' \equiv r\sigma$ and $C'\delta\theta'[\;]_p \equiv C'\theta[\;]_p \equiv C[\;]_p$. Hence, $t \equiv C[r\sigma]_p \equiv C'\delta\theta'[r\sigma'\theta']_p \equiv (C'\delta[r\sigma']_p)\theta'$, which conclude the first part of the proof by taking $t' \equiv C'\delta[r\sigma]_p$.

Now, we show that $P' \leq \mathcal{O}_{\mathcal{X}}(t')$. Let $q \in \mathcal{O}_{\mathcal{X}}(t')$. Consider the case that $p \not\leq q$. Since $q \not\leq p$ from $t' \equiv C'\delta[r\sigma']_p$, we have $q \in \mathcal{O}_{\mathcal{X}}(C'\delta[\;]_p)$. There exists $q' \leq q$ such that $q' \in \mathcal{O}_{\mathcal{X}}(C'[\;]_p) \subseteq \mathcal{O}_{\mathcal{X}}(s)$. It follows from $p \in \mathcal{O}_{\mathcal{X}}(s)$ that

$p' \le q'$ for some $p' \in P$. Thus, $p' \in P'$ follows from $P \Rightarrow^{[p,\rho]} P'$. We have shown $p' \le q$ and $p' \in P'$. Consider the case that $p \le q$. If q was introduced by an extra variable, that is $q = pq'$ and $r|_{q'} \equiv x \in \mathcal{EV}ar(\rho)$ for some q', we have $pq' \in P'$. Otherwise, q was moved via the reduction, that is $q = pq'w$ and $r|_{q'} \equiv y$ for some $y \in \mathcal{V}ar(l)$ and $w \in \mathcal{O}_{\mathcal{X}}(y\sigma')$. Then, we can show $p' \le pq'w$ for some $p' \in P'$, from the fact that there exists $p'' \in P$ satisfying $p'' \le pq'w$ for all q' such that $l|_{q'} \equiv y$. □

Proof of Theorem 2 From Lemma 2, we can easily prove the following claim by induction on n; if $P : s'\theta \xrightarrow{n}_R P' : t$ and $P \le \mathcal{O}_{\mathcal{X}}(s)$, then $s' \overset{*}{\leadsto}_R t'$ and $t'\theta' \equiv t$ for some t' and θ'. □

Proof of Theorem 3 We prove by induction on n that $s \xrightarrow{n}_R t$ implies a linear term t' and a substitution θ such that $s \overset{*}{\underset{\delta}{\leadsto}}_R t'$ and $t \equiv t'\theta$. The case of $n = 0$ is trivial. Suppose $s \xrightarrow{n-1}_R u \to_R t$. By induction hypothesis, there exist a linear term u' and a substitution θ' such that $s \overset{*}{\leadsto}_R u'$ and $u'\theta' \equiv u$. Suppose $u \equiv C[l\sigma]_p \to_R^{[p,\rho]} C[r\sigma] \equiv t$ where $\rho : l \to r \in R$. Consider the case $p \in \mathcal{O}(u') \setminus \mathcal{O}_{\mathcal{X}}(u')$. Then, we have $u' \equiv C'[v]_p$ for some $C'[\,]$ and v, and also have $u'\theta' \equiv C'\theta'[v\theta']_p \equiv C[l\sigma]_p$. In similar to the proof of Lemma 2, we can show that $u' \equiv C'[v]_p \underset{\delta}{\leadsto}_R C'\delta[r\sigma']_p$ and $t \equiv (C'\delta[r\sigma']_p)\theta'$, where $(\delta, \sigma') = \mathtt{mgu}(v, l)$. Here, we can show the linearity of $t' \equiv C'\delta[r\sigma']_p$ from the linearity of u' and r. Consider the case $p \notin \mathcal{O}(u') \setminus \mathcal{O}_{\mathcal{X}}(u')$. Then, we have $u'|_q \equiv y$ and $p = qq'$ for some $y \in \mathcal{V}ar(u')$, q and q'. From the linearity of u', we have $u' \equiv C'[y]_q$ and $y\theta' \equiv C''[l\sigma]_{q'}$ for some $C'[\,]_q$ and $C''[\,]_{q'}$. Let $\theta = \theta'|_{\mathcal{D}om(\theta')\setminus\{y\}} \cup \{y \mapsto C''[r\sigma]_{q'}\}$. Then, θ is a substitution, and we have $s \overset{*}{\leadsto}_R u'$ and $u'\theta \equiv C'\theta[y\theta]_q \equiv C[r\sigma]_p \equiv t$. □

Let R be an EV-TRS and t be a term. We say that t is *almost terminating* with respect to $\leadsto_R$ if there exists an infinite narrowing sequence starting from t and every proper subterm of t is N-SN with respect to R. It is clear that t has an almost-terminating subterm with respect to $\leadsto_R$ if there is an infinite narrowing sequence starting from t. Let $s \leadsto_R^q t$. Then, we write $s \leadsto_R^{p<} t$ if $p < q$, and write $s \leadsto_R^{p\le} t$ if $p \le q$. We abbreviate the sequence $a_{i,1}, ..., a_{i,n_i}$ as $\boldsymbol{a}_i$.

Proof of Theorem 6 We prove here only the claim (b) since the proof of (a) is similar to (b).

We first show the *only if*-part by constructing an infinite ground R-narrowing-chain from an infinite ground sequence. We assume that R is not N-GSN. Then, there exists an infinite ground narrowing-sequence. Let s_0 be an almost-terminating ground-term with respect to $\leadsto_R$, and $s_0 \equiv f_1(\boldsymbol{u}_0)$. Then, we have $f_1(\boldsymbol{u}_0) \overset{*}{\leadsto}{}_R^{\varepsilon<} f_1(\boldsymbol{v}_1) \equiv s'_1 \underset{\delta_1}{\leadsto}{}_R^{[\varepsilon,\rho_1]} r_1\sigma_1 \leadsto_R \cdots$ where $\rho_1 : f_1(\boldsymbol{w}_1)(\equiv l_1) \to r_1 \in R$ and $(\delta_1, \sigma_1) = \mathtt{mgu}(s'_1, l_1)$. Since $\boldsymbol{v}_1$ are N-SN, $x\sigma_1$ is N-SN for any $x \in \mathcal{D}om(\sigma_1)$. Hence, there is a subterm $t_1 \equiv f_2(\boldsymbol{u}_1)$ of r_1 such that $t_1\sigma_1$ is almost terminating with respect to $\leadsto_R$. Since $t_1\sigma_1$ is almost terminating with respect to $\leadsto_R$, as similar as the case of s_0, we have $t_1\sigma_1 \equiv f_2(\boldsymbol{u}_1\sigma_1) \overset{*}{\leadsto}{}_R^{\varepsilon<} f_2(\boldsymbol{v}_2) \equiv s'_2 \underset{\delta_2}{\leadsto}{}_R^{[\varepsilon,\rho_2]} r_2\sigma_2 \leadsto_R \cdots$ where $\rho_2 : f_2(\boldsymbol{w}_2)(\equiv l_2) \to r_2 \in R$ and $(\delta_2, \sigma_2) = \mathtt{mgu}(s'_2, l_2)$. Since $\boldsymbol{v}_2$ are N-SN, $x\sigma_2$ is also N-SN for any $x \in \mathcal{D}om(\sigma_2)$. Hence, there is a subterm $t_2 \equiv f_3(\boldsymbol{u}_2)$ of r_2 such that $t_2\sigma_2$ is almost terminating with respect to $\rightsquigarrow_R$. Here, $\langle F_1(\boldsymbol{w}_1), F_2(\boldsymbol{u}_2)\rangle, \langle F_2(\boldsymbol{w}_2), F_3(\boldsymbol{u}_3)\rangle \in \mathcal{DP}_R$ follow from ρ_1 and ρ_2. Since $u_{0,i} \overset{*}{\rightsquigarrow}_R v_{1,i}$, $(\delta_1, \sigma_1) = \mathtt{mgu}(s'_1, l_1)$, $u_{1,i} \overset{*}{\rightsquigarrow}_R v_{2,i}$ and $(\delta_2, \sigma_2) = \mathtt{mgu}(s'_2, l_2)$, we have a ground chain $F_1(\boldsymbol{u}_0)\langle F_1(\boldsymbol{w}_1), F_2(\boldsymbol{u}_2)\rangle\langle F_2(\boldsymbol{w}_2), F_3(\boldsymbol{u}_3)\rangle$.

By repeating similarly to the above, we obtain an infinite ground chain $F_1(\boldsymbol{u}_0)\langle F_1(\boldsymbol{w}_1), F_2(\boldsymbol{u}_2)\rangle\langle F_2(\boldsymbol{w}_2), F_3(\boldsymbol{u}_3)\rangle \cdots$.

We prove *if*-part by constructing an infinite ground narrowing-sequence from an infinite ground R-narrowing-chain $F_1(\boldsymbol{u}_0)\langle F_1(\boldsymbol{w}_1), F_2(\boldsymbol{u}_2)\rangle\langle F_2(\boldsymbol{w}_2), F_3(\boldsymbol{u}_3)\rangle \cdots$. From the definition of R-narrowing-chain, there are a term $F_i(\boldsymbol{v}_i)$ and the most general unifier $(\delta_i, \sigma_i) = \mathtt{mgu}(F_i(\boldsymbol{v}_i), F_i(\boldsymbol{w}_i))$ such that $F_i(\boldsymbol{u}_i)\sigma_{i-1} \overset{*}{\rightsquigarrow}_R F_i(\boldsymbol{v}_i)$, where $F_1(\boldsymbol{u}_0)\sigma_0 \equiv F_1(\boldsymbol{u}_0)$. From the construction of dependency pairs, we have $\rho_i : f_i(\boldsymbol{w}_i) \to C_i[f_{i+1}(\boldsymbol{u}_{i+1})] \in R$. Hence, we can easily construct an infinite ground narrowing-sequence $f_1(\boldsymbol{u}_0) \overset{*}{\rightsquigarrow}_R f_1(\boldsymbol{v}_1) \underset{\delta_1}{\rightsquigarrow}{}^{[\varepsilon,\rho_1]}_R C_1\delta_1[f_2(\boldsymbol{u}_1)]_{p_1}$ $\overset{*}{\rightsquigarrow}{}^{p_1<}_R C_1\delta_1[f_2(\boldsymbol{v}_2)] \underset{\delta_2}{\rightsquigarrow}{}^{[p_1,\rho_2]}_R C_1\delta_1[C_2\delta_2[f_3(\boldsymbol{u}_2)\sigma_2]_{p_2}]_{p_1} \rightsquigarrow_R \cdots$. □

Let π be a simple AF function and θ be a substitution. We define the substitution θ_π as $\theta_\pi = \{ x \mapsto \pi(t) \mid x \in \mathcal{D}om(\theta), x\theta \equiv t \}$. Let t be a term. It is clear that $\pi(t\theta) \equiv \pi(t)\theta_\pi$ holds.

Lemma 3. *Let R be an EV-TRS, π be a simple AF function that eliminates all extra variables of R and $\mathcal{DP}_R$. For any $s, t \in T(\mathcal{F}, \mathcal{X})$, if $s \underset{\delta}{\overset{*}{\rightsquigarrow}}_R t$ and $\pi(s)$ is ground then $\pi(s) \overset{*}{\rightsquigarrow}_{\pi(R)} \pi(t)$ and $\pi(s) \overset{*}{\to}_{\pi(R)} \pi(t)$.*

Proof. We first prove by induction on n that $s \overset{n}{\rightsquigarrow}_R t$ and $\pi(s) \in T(\mathcal{F})$ imply $\pi(s) \overset{*}{\rightsquigarrow}_{\pi(R)} \pi(t)$. In case of $n = 0$, it is trivial. We assume that $s \underset{\delta}{\rightsquigarrow}{}^{[p,\rho]}_R u \underset{\delta'}{\overset{n-1}{\rightsquigarrow}}_R t$ and $\pi(s)$ is ground, where $\rho : l \to r \in R$. Then, there are $C[\ ]_p$ and s' such that $s \equiv C[s']_p$, $(\delta, \sigma) = \mathtt{mgu}(s', l)$ and $u \equiv C\delta[r\sigma]_p$. Consider the case that □ in $C[\ ]$ is eliminated by π. Let $\pi(C[\ ]_p) \equiv u'$. Then, we have $\pi(s) \equiv \pi(C[s']_p) \equiv u'$ and $\pi(u) \equiv \pi(C\delta[r\sigma]_p) \equiv \pi(C\delta[\ ]_p) \equiv \pi(C[\ ]_p)\delta_\pi \equiv u'\delta_\pi$. Since $\pi(s)$ is ground, u' is also ground. Then, we have $\pi(u) \equiv u'\delta_\pi \equiv u'$. By induction hypothesis, we have $\pi(u) \overset{*}{\rightsquigarrow}_{\pi(R)} \pi(t)$. Therefore, $\pi(s) \equiv \pi(u) \overset{*}{\rightsquigarrow}_{\pi(R)} \pi(t)$. Consider the otherwise. Let $\pi(C[\ ]_p) \equiv C'[\ ]_q$. Then, we have $\pi(s) \equiv \pi(C[s']_p) \equiv (\pi(C[\ ]_p)[\pi(s')]_q \equiv C'[\pi(s')]_q$ and $\pi(u) \equiv \pi(C\delta[r\sigma]_p) \equiv (\pi(C\delta[\ ]_p))[\pi(r\sigma)]_q \equiv C'\delta_\pi[\pi(r\sigma)]_q$. Since $\pi(s)$ is ground, $C'[\ ]_q$ is also ground. Then, it follows from $C'\delta_\pi \equiv C'$ that $\pi(s) \equiv C'[\pi(r\sigma)]_q$. We also have $\pi(l) \to \pi(r) \in \pi(R)$. It follows from the assumption that $\mathcal{V}ar(\pi(l)) \supseteq \mathcal{V}ar(\pi(r))$. Then, $\pi(r\sigma)$ is ground. Since $\pi(u) \equiv C'[\pi(r\sigma)]$ is also ground, we have $\pi(u) \overset{*}{\rightsquigarrow}_{\pi(R)} \pi(t)$ by induction hypothesis. On the other hand, $\pi(s')\delta_\pi \equiv \pi(l)\sigma_\pi$ follows from $s'\delta \equiv l\sigma$, $\pi(s'\delta) \equiv \pi(s')\delta_\pi$ and $\pi(l\sigma) \equiv \pi(l)\sigma_\pi$. Since $\pi(s')$ is ground, we have $\pi(s')\delta_\pi \equiv \pi(s') \equiv \pi(l)\sigma_\pi$. Therefore, we have the sequence $\pi(s) \equiv C'[\pi(s')]_q \equiv C'[\pi(l)\sigma_\pi]_q \rightsquigarrow_{\pi(R)} C'[\pi(r)\sigma_\pi]_q \equiv C'[\pi(r\sigma)]_q \equiv \pi(u) \overset{*}{\rightsquigarrow}_{\pi(R)} \pi(t)$. Since $\pi(R)$ is a TRS and $\pi(s)$ is ground, it is clear that $\pi(s) \overset{*}{\rightsquigarrow}_{\pi(R)} \pi(t)$ if and only if $\pi(s) \overset{*}{\to}_{\pi(R)} \pi(t)$. □

Proof of Theorem 7 By Theorem 6, there exists an infinite R-narrowing-chain when R is not N-GSN. The first part can be easily proved by constructing an infinite ground $\pi(R\&\mathcal{DP}_R)$-narrowing-chain from an infinite ground R-narrowing-chain, using Lemma 3. The second part can be proved similarly. □