ON THE MEAN SQUARE OF STANDARD L-FUNCTIONS ATTACHED TO IKEDA LIFTS

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ABSTRACT. By using estimates on the frequency of large values of the Riemann zeta-function and modular L-functions attached to the full modular group $SL(2,\mathbb{Z})$, we prove sharp upper and lower estimates of the mean square of standard L-functions attached to Siegel cusp forms which are Ikeda lifts, on boundaries and the central line of the critical strip. The mean square of spinor L-functions attached to Saito-Kurokawa lifts is also studied.

1. Introduction and statement of main results

Let f be a holomorphic normalized Hecke-eigen cusp form of weight 2κ with respect to the full modular group $SL(2,\mathbb{Z})$, and ν be a positive integer with $\nu \equiv \kappa (\mathrm{mod}\, 2)$. Ikeda [5] proved the existence of a Hecke-eigen Siegel cusp form F_0 of weight $\kappa+\nu$ with respect to the Siegel modular group $Sp(2\nu,\mathbb{Z})$. This Siegel cusp form F_0 is called the Ikeda lift of f, and the associated standard L-function $L(s,F_0,\mathrm{st})$ can be decomposed as

$$L(s,F_0,\mathrm{st}) = \zeta(s)\prod_{j=1}^{2\nu} L(s+\kappa+\nu-j,f), \tag{1.1}$$

where $\zeta(s)$ is the Riemann zeta-function and

$$L(s,f)=\sum_{n=1}^{\infty} a(n)n^{-s}$$

is the Hecke L-function associated with f.

The first author [12] proved rather sharp estimates of the mean square of $L(s,F_0,\mathrm{st})$. Let

$$I(\sigma;T)=\int_1^T |L(\sigma+it,F_0,\mathrm{st})|^2 dt$$

for $T \geq 2$. The "critical strip" for $L(s,F_0,\mathrm{st})$ is $-\nu+1/2 \leq \sigma \leq \nu+1/2$. Moreover, the study of $I(\sigma;T)$ for $-\nu+1/2 \leq \sigma < 1/2$ can be reduced to the study of the case $1/2 < \sigma \leq \nu+1/2$ by the functional equation. Hence we can restrict our consideration to the case $1/2 \leq \sigma \leq \nu+1/2$. Then Theorem 5 of [12] reads as follows. The symbol $A \ll B$ means $A=O(B)$, and $A \asymp B$ means $A \ll B \ll A$.

Theorem A. *Let ℓ denote a positive integer. If $\nu+1 \le \ell \le 2\nu$, we have*

$$I(\sigma;T) \asymp T^{2(2\nu-\ell)(\ell+1-2\sigma)+1} \quad (-\nu+\ell<\sigma<-\nu+\ell+1/2), \tag{1.2}$$

$$T^{2(2\nu-\ell)^2+1}(\log T)^{-4} \ll I(-\nu+\ell+1/2;T) \ll T^{2(2\nu-\ell)^2+1}(\log T)^4. \tag{1.3}$$

If $\nu+2 \le \ell \le 2\nu$, we have

$$I(\sigma;T) \asymp T^{2(2\nu-\ell)(\ell+1-2\sigma)+4(\ell-\nu-\sigma)+1} \quad (-\nu+\ell-1/2<\sigma<-\nu+\ell), \tag{1.4}$$

$$I(-\nu+\ell;T) \asymp T^{2(2\nu-\ell)(2\nu+1-\ell)+1}\log T. \tag{1.5}$$

Furthermore, we have

$$T^{2\nu(\nu-1)+1}(\log T)^{-1/3}(\log\log T)^{-2/3} \ll I(1;T) \ll T^{2\nu(\nu-1)+1}(\log T)^{7/3} \tag{1.6}$$

and

$$T^{2\nu^2+1}(\log T)^{-7} \ll I(1/2;T) \ll T^{2\nu^2+1}(\log T)^9. \tag{1.7}$$

For $1/2<\sigma<1$, we only have an upper bound of $I(\sigma;T)$; see (3.32) of [12].

It is difficult to obtain such sharp estimates for standard L-functions of general Siegel cusp forms. For example, by using the result of Duke, Howe and Li [1], it can be shown that

$$\int_1^T |L(1/2+it,F,\mathrm{st})|^2 dt \ll T^{(16/3)\nu^2+(10/3)\nu+1/2+\varepsilon} \tag{1.8}$$

for any Hecke-eigen Siegel cusp form F with respect to $Sp(2\nu,\mathbb{Z})$, where (and in what follows) ε is an arbitrarily small positive number. Moreover, if ν is a power of 2, then the exponent on the right-hand side of (1.8) can be replaced by $4\nu^2+3\nu+1/2+\varepsilon$ ((5.16) of Kohnen, Sengupta and the second author [10]). But these estimates are still much worse than (1.7). The reason why we can obtain such sharp estimates as above in the Ikeda lift case is surely the existence of decomposition (1.1). The usefulness of such kind of decompositions in mean value problems was first noticed by the second author [16] for Rankin-Selberg L-functions, and then developed in a more general setting by the first author [12].

Among (1.2) — (1.7) proved in [12], the results (1.2), (1.4) and (1.5) determine completely the real order of $I(\sigma;T)$ up to constant factors. On the other hand, the results (1.3), (1.6) and (1.7) still includes ambiguity of log-factors. It is the purpose of the present paper to remove this ambiguity to determine the order of $I(-\nu+\ell+1/2;T)$ $(\nu+1\le\ell\le 2\nu)$, $I(1;T)$ and $I(1/2;T)$ almost completely, that is, up to loglog-factors.

Theorem 1. *For $\nu+1\le\ell\le 2\nu$, we have*

$$I(-\nu+\ell+1/2;T)\asymp T^{2(2\nu-\ell)^2+1}. \tag{1.9}$$

Theorem 2. *We have*

$$T^{2\nu(\nu-1)+1}\log T(\log\log T)^{-2}\ll I(1;T)\ll T^{2\nu(\nu-1)+1}\log T(\log\log T)^2. \tag{1.10}$$

Theorem 3. *We have*

$$T^{2\nu^2+1}\log T(\log\log T)^{-8}\ll I(1/2;T)\ll T^{2\nu^2+1}\log T(\log\log T)^8. \tag{1.11}$$

Combining these results with (1.2), (1.4) and (1.5), it is now natural to propose the following conjecture.

Conjecture. The asymptotic relations

$$I(\sigma;T)\sim CT^{2(2\nu-\ell)(\ell+1-2\sigma)+1}\quad(-\nu+\ell<\sigma\le-\nu+\ell+1/2) \tag{1.12}$$

for $\nu+1\le\ell\le 2\nu$,

$$I(\sigma;T)\sim CT^{2(2\nu-\ell)(\ell+1-2\sigma)+4(\ell-\nu-\sigma)+1}\quad(-\nu+\ell-1/2<\sigma<-\nu+\ell) \tag{1.13}$$

for $\nu+2\le\ell\le 2\nu$,

$$I(-\nu+\ell;T)\sim CT^{2(2\nu-\ell)(2\nu+1-\ell)+1}\log T \tag{1.14}$$

for $\nu+1\le\ell\le 2\nu$, and

$$I(1/2;T)\sim CT^{2\nu^2+1}\log T, \tag{1.15}$$

would hold, where $A\sim B$ means $A/B\to 1$ as $T\to\infty$ and C is a certain constant depending on σ.

In Section 2 we will show Theorem 1, which is actually easy. This part is just a simple supplement of [12]. The main body of the present paper is the proof of Theorems 2 and 3, for which we apply the method of a paper of Ramachandra and the second author [15]. In Section 3 we will give a lemma, which is a generalization of Lemma 3.2 of [15]. Theorems 2 and 3 will be proved in Section 4. In the final section we will discuss further examples (Theorems 4, 5 and 6), including the case of spinor L-functions attached to Saito-Kurokawa lifts.

2. Proof of Theorem 1

To prove Theorems 1, 2 and 3, it is sufficient to prove the same estimates for

$$I(\sigma;T,2T)=\int_T^{2T}|L(\sigma+it,F_0,\mathrm{st})|^2dt$$
$$=\int_T^{2T}\left|\zeta(\sigma+it)\prod_{j=1}^{2\nu}L(\sigma+\kappa+\nu-j+it,f)\right|^2dt. \tag{2.1}$$

In this section we consider (2.1) for $\sigma=-\nu+\ell+1/2$ ($\nu+1\le\ell\le 2\nu$), that is

$$I(-\nu+\ell+1/2;T,2T)$$
$$=\int_T^{2T}\left|\zeta(-\nu+\ell+1/2+it)\prod_{j=1}^{2\nu}L(\kappa+\ell-j+1/2+it,f)\right|^2dt. \tag{2.2}$$

Then by (3.35) and (3.36) of [12] we have

$$|\zeta(-\nu+\ell+1/2+it)|\asymp 1$$

and

$$|L(\kappa+\ell-j+1/2+it,f)|\asymp 1 \qquad (j\le\ell-1).$$

For $j\ge\ell+2$, by the functional equation we have

$$|L(\kappa+\ell-j+1/2+it,f)|$$
$$\asymp(|t|+2)^{2(j-\ell-1/2)}|L(\kappa-\ell+j-1/2-it,f)|\asymp(|t|+2)^{2(j-\ell-1/2)}.$$

Therefore, when $\ell=2\nu$, all factors but $|L(\kappa+1/2+it,f)|$ in the integrand on the right-hand side of (2.2) are $\asymp 1$. Hence

$$I(\nu+1/2;T,2T)\asymp\int_T^{2T}|L(\kappa+1/2+it,f)|^2dt, \tag{2.3}$$

which is $\asymp T$ by (3.41) of [12]. This is Theorem 1 for $\ell=2\nu$.

When $\nu+1\le\ell\le 2\nu-1$ (which occurs only in case $\nu\ge 2$), the right-hand side of (2.2) is

$$\asymp\prod_{j=\ell+2}^{2\nu}T^{4(j-\ell-1/2)}\int_T^{2T}|L(\kappa-1/2+it,f)L(\kappa+1/2+it,f)|^2dt, \tag{2.4}$$

where the product is 1 when $\ell=2\nu-1$. Using the functional equation we have

$$|L(\kappa-1/2+it,f)|\asymp(|t|+2)|L(\kappa+1/2-it,f)|. \tag{2.5}$$

But, since f is normalized Hecke-eigen, all Fourier coefficients of f are real, so the right-hand side of (2.5) is equal to $(|t|+2)|L(\kappa+1/2+it,f)|$. Also we see that $\sum_{j=\ell+2}^{2\nu}4(j-\ell-1/2)=2(2\nu-\ell)^2-2$. Hence from (2.4) we obtain

$$I(-\nu+\ell+1/2;T,2T)\asymp T^{2(2\nu-\ell)^2}\int_T^{2T}|L(\kappa+1/2+it,f)|^4dt. \tag{2.6}$$

Let

$$\tilde{L}(s,f) = L(s+\kappa-1/2,f) = \sum_{n=1}^{\infty} \tilde{a}(n)n^{-s} \qquad \left(\tilde{a}(n) = a(n)n^{1/2-\kappa}\right).$$

Ivić [8] proved that

$$\int_1^T |\tilde{L}(\sigma+it,f)|^4 dt = A(\tilde{L}^2)T + O(T^{16(1-\sigma)/5+\varepsilon}) \tag{2.7}$$

for $3/4 \le \sigma \le 1$, where

$$A(\tilde{L}^2) = \sum_{n=1}^{\infty} \left| \sum_{d|n} \tilde{a}(d)\tilde{a}(n/d) \right|^2 n^{-2\sigma}.$$

In particular, the case $\sigma = 1$ of (2.7) implies that the integral on the right-hand side of (2.6) is $\asymp T$. This completes the proof of Theorem 1.

Remark Ivić [8] deduced the above (2.7) from the result

$$\inf \left\{ b \ge 0 \;\middle|\; \int_1^X \left(\sum_{n \le X} \sum_{d|n} \tilde{a}(d)\tilde{a}(n/d) \right)^2 dx \ll X^{1+2b+\varepsilon} \right\} = \frac{3}{8}, \tag{2.8}$$

which Ivić himself proved in [7]. By using (2.8), we can obtain (2.7) immediately from Theorem 2 of [12].

3. A lemma on large values of L-functions

Let

$$L(s) = \prod_p (1-a_1(p)p^{-s})^{-1} \cdots (1-a_k(p)p^{-s})^{-1}, \tag{3.1}$$

where the product runs over all prime numbers and $a_j(p)$ $(1 \le j \le k)$ are complex numbers. We assume that $|a_j(p)| \le A$ $(A > 0)$ for any j and p. Then $L(s)$ is convergent absolutely for $\sigma > 1$ and define a non-zero holomorphic function there. We assume that

(I) There exists an $\ell_1 > 0$ such that $L(s)$ can be continued meromorphically to $\sigma > 1 - \ell_1$, holomorphic except for a possible pole at $s = 1$, and satisfies

$$L(s) = O((|t|+2)^{C_1}) \qquad (C_1 > 0) \tag{3.2}$$

in this region.

Let $1 - \ell_1 < \sigma < 1$, and denote by $N_L(\sigma, T)$ the number of zeros of $L(s)$ (counted with multiplicity) in the region $\Re s \ge \sigma$, $|\Im s| \le T$. We further assume that

(II) For any $\eta > 0$, the estimate

$$N_L(\sigma,T) = O(T^{C_2(1-\sigma)+\eta}) \qquad (C_2 > 0) \tag{3.3}$$

holds.

In this section, as a preparation for the proof of Theorems 2 and 3, we prove the following

Lemma. *We assume* (I) *and* (II). *Let* $T \geq 100$, $\varepsilon > 0$, *and let* $t_1, \ldots, t_R$ *be real numbers satisfying* $T/2 < t_1 < \cdots < t_R \leq T$, $|t_{r+1} - t_r| \geq 1$ $(1 \leq r \leq R-1)$, *and*

$$|\log L(1 + it_r)| > kA \log\log\log T - \log \varepsilon \qquad (1 \leq r \leq R). \tag{3.4}$$

Then, there exists a $\delta = \delta(\varepsilon)$, *which tends to* 0 *when* ε *tends to* 0, *for which*

$$R \ll T^{(C_2+1)\delta}(\log T)^3 \tag{3.5}$$

holds where the implied constant in (3.5) is independent of δ.

In the case $L(s) = \zeta(s)$, this lemma was given as Lemma 3.2 of Ramachandra and the second author [15]. The aim of [15] is to apply this lemma to the problem of evaluating the error term in the asymptotic formula for $\sum_{n \leq x} d(n)^2$, where $d(n)$ is the divisor function. Kühleitner and Nowak [11] obtained the assertion of the lemma for Dedekind zeta-functions of arbitrary number fields. Their method is a generalization of the argument in [15]. Here, also as a generalization of the argument in [15], we present a proof of the above general form of the lemma.

Let δ $(< \ell)$ be a small positive number, and take $\sigma = 1 - \delta$ and $\eta = \delta$ in (3.3). We obtain

$$N_L(1 - \delta, T) = O(T^{(C_2+1)\delta}). \tag{3.6}$$

For any zero $\rho = \beta + i\gamma$ of $L(s)$ in the region

$$B_1 = \{\sigma + it \mid \sigma \geq 1 - \delta, T - 2(\log T)^3 \leq t \leq 2T + 2(\log T)^3\},$$

define

$$B_\rho = \{\sigma + it \mid \sigma \geq 1 - \delta, \gamma - (\log T)^3 \leq t \leq \gamma + (\log T)^3\},$$
$$B_\rho^* = \{\sigma + it \mid \sigma \geq 1 - \delta, \gamma - 2(\log T)^3 \leq t \leq \gamma + 2(\log T)^3\},$$
$$B = \{\sigma + it \mid \sigma \geq 1 - \delta, T \leq t \leq 2T\},$$

and

$$\tilde{B} = B \setminus \bigcup_{\rho \in B_1} B_\rho, \qquad \tilde{B}^* = B \setminus \bigcup_{\rho \in B_1} B_\rho^*.$$

Then $B_\rho \subset B_\rho^*$ and $B_1 \supset B \supset \tilde{B} \supset \tilde{B}^*$. For any $\tilde{s} \in \tilde{B}$, $L(s)$ has no zero in the region

$$U(\tilde{s}) = \{\sigma + it \mid \sigma \geq 1 - \delta, |t - \Im\tilde{s}| \leq (\log T)^3/2\},$$

hence $\log L(s)$ is well-defined there. Denote by $D(r)$ the disc of radius r whose center is $2 + i\Im\tilde{s}$. Then $D(1+\delta) \subset U(\tilde{s})$, and for any $s \in D(1+\delta)$, we have

$$\Re \log L(s) = \log |L(s)| \leq C \log T \qquad (C > 0)$$

by (3.2). Hence, applying Borel-Carathéodory's theorem (see Section 5.5 of Titchmarsh [18]) we obtain

$$|\log L(s)| \ll \delta^{-1} \log T \tag{3.7}$$

for any $s \in D(1+\delta/2)$. Especially (3.7) holds for any s with $\Im s = \Im \tilde{s}$, $\Re s \geq 1 - \delta/2$. But $\tilde{s}$ is an arbitrary point of $\tilde{B}$, so (3.7) holds for any $s \in \tilde{B} \cap \{\sigma \geq 1-\delta/2\}$. Therefore, if $s \in \tilde{B}^* \cap \{\sigma \geq 1 - \delta/2\}$, then (3.7) holds for any point in the region

$$V(s) = \{\sigma + it \mid \sigma \geq 1 - \delta/2, |t - \Im s| \leq (\log T)^3\}.$$

Now, we fix $s \in \tilde{B}^* \cap \{\sigma \geq 1 - \delta/8\}$, and let $w = u + iv$ be complex. At first we assume $u = \delta/4$. Then $\Re(s+w) > 1$, so we can use (3.1) to obtain

$$\log L(s+w) = -\sum_{j=1}^{k} \sum_{p} \log(1 - a_j(p)p^{-s-w}) = \sum_{j=1}^{k} \sum_{p} \sum_{m=1}^{\infty} \frac{a_j(p)^m}{mp^{m(s+w)}}.$$

Hence

$$\begin{aligned} &\frac{1}{2\pi i} \int_{\delta/4-i\infty}^{\delta/4+i\infty} \log L(s+w)\Gamma(w)X^w dw \\ &= \sum_{j=1}^{k} \sum_{p} \sum_{m=1}^{\infty} \frac{a_j(p)^m}{mp^{ms}} \cdot \frac{1}{2\pi i} \int_{\delta/4-i\infty}^{\delta/4+i\infty} \Gamma(w) \left(\frac{X}{p^m}\right)^w dw \\ &= \sum_{j=1}^{k} \sum_{p} \sum_{m=1}^{\infty} \frac{a_j(p)^m}{mp^{ms}} \cdot \exp(-p^m/X), \end{aligned} \tag{3.8}$$

where $X = (\log T)^{8/\delta}$. Since $a_j(p) = O(1)$, the contribution of the part corresponding to $m \geq 2$ on the right-hand side is $O(1)$ (whose implied constant depends on k), hence

$$\frac{1}{2\pi i} \int_{\delta/4-i\infty}^{\delta/4+i\infty} \log L(s+w)\Gamma(w)X^w dw = \sum_{p} \frac{c(p)e^{-p/X}}{p^s} + O(1), \tag{3.9}$$

where $c(p) = a_1(p) + \cdots + a_k(p)$.

When $|v| \geq (\log T)^3$, we have

$$\begin{aligned} \log L(s+\delta/4+iv) &= \sum_{j=1}^{k} \sum_{p} \sum_{m=1}^{\infty} \frac{a_j(p)^m}{mp^{m(s+\delta/4+iv)}} \\ \ll \sum_{p} \frac{c(p)}{p^{\sigma+\delta/4}} + O(1) &\ll \sum_{p} \frac{1}{p^{\sigma+\delta/4}} \ll \log\zeta(\sigma+\delta/4) \ll \log(\delta^{-1}). \end{aligned}$$

Using this estimate and Stirling's formula, we see that the contribution of the part $|v| \geq (\log T)^3$ to the left-hand side of (3.9) is

$$\ll \log(\delta^{-1}) X^{\delta/4} \int_{(\log T)^3}^{\infty} e^{-\pi|v|/2} |v|^{\delta/4-1/2} dv \ll \log(\delta^{-1}) \exp(-c(\log T)^3) \tag{3.10}$$

with a certain $c > 0$.

We deform the path of the remaining integral to the oriented polygonal path joining $\delta/4 - i(\log T)^3$, $-\delta/8 - i(\log T)^3$, $-\delta/8 + i(\log T)^3$, and $\delta/4 + i(\log T)^3$ in that order. When w is on this polygonal path, the point $s+w$ is in $V(s)$, so we can use (3.7) for $|\log L(s+w)|$. Hence the contribution of horizontal segments is

$$\ll \delta^{-1}(\log T)e^{-\pi(\log T)^3/2}\int_{-\delta/8}^{\delta/4}(\log T)^{3(u-1/2)}X^u du \ll \delta^{-1}\exp(-c(\log T)^3),$$

and the contribution of the vertical segment at $u = -\delta/8$ is

$$\ll \delta^{-1}(\log T)X^{-\delta/8}\int_{-(\log T)^3}^{(\log T)^3} e^{-\pi|v|/2}(|v|+1)^{-\delta/8-1/2}dv \ll \delta^{-1}.$$

Since the residue at $w = 0$ is $\log L(s)$, from the above estimates and (3.9), (3.10) we find that

$$\log L(s) = \sum_p \frac{c(p)e^{-p/X}}{p^s} + O(\delta^{-1}). \tag{3.11}$$

Now let $\Re s = 1$. Then from (3.11) we have

$$\log L(1+it) = \sum_{p\le X/2}\frac{c(p)e^{-p/X}}{p^{1+it}} + O\left(\sum_{p>X/2}\frac{e^{-p/X}}{p}\right) + O(\delta^{-1}). \tag{3.12}$$

It is easy to see that the second term on the right-hand side is $O(1)$. The first term on the right-hand side is

$$\sum_{p\le X/2}\frac{c(p)}{p^{1+it}}(1+O(p/X)) = \sum_{p\le X/2}\frac{c(p)}{p^{1+it}} + O(1).$$

Therefore we have

$$\log L(1+it) = \sum_{p\le X/2}\frac{c(p)}{p^{1+it}} + O(\delta^{-1}) \tag{3.13}$$

for any $s = 1+it \in \tilde{B}^*$. Since $|c(p)| \le kA$, we have

$$\begin{aligned}|\log L(1+it)| &\le kA\sum_{p\le X/2}\frac{1}{p} + \frac{\beta_1}{\delta}\\ &= kA(\log\log(X/2)+O(1)) + \frac{\beta_1}{\delta}\\ &\le kA\log\log\log T + \frac{\beta_2}{\delta}\end{aligned} \tag{3.14}$$

for $s = 1+it \in \tilde{B}^*$, where β_1, β_2 are absolute positive constants. The inequalities (3.4) and (3.14) contradict with each other if we choose

$$\delta = \delta(\varepsilon) = \frac{2\beta_2}{\log(\varepsilon^{-1})}. \tag{3.15}$$

Therefore, under this choice of δ, we find that the points $1+it_r$ satisfying (3.4) do not belong to $\tilde{B}^*$, hence

$$1+it_r \in \bigcup_{\rho\in B_1} B_\rho^* \qquad (1\le r\le R).$$

But the width of each B_ρ^* is $4(\log T)^3$, and the number of zeros of $L(s)$ in B_1 is $O(T^{(C_2+1)\delta})$ by (3.6). Hence the total number R of the points t_r should satisfy

$$R \ll T^{(C_2+1)\delta}(\log T)^3,$$

which is the assertion of the lemma.

4. Proof of Theorems 2 and 3

We begin with the proof of Theorem 2. Since

$$|L(\kappa+\nu-j+1+it,f)| \asymp \begin{cases} 1 & (j\le\nu), \\ (|t|+2)^{2(j-\nu-1)} & (j\ge\nu+2), \end{cases}$$

we have

$$\begin{aligned} I(1;T,2T) &\asymp \prod_{j=\nu+2}^{2\nu} T^{4(j-\nu-1)} \int_T^{2T} |\zeta(1+it)L(\kappa+it,f)|^2 dt \\ &= T^{2\nu(\nu-1)} \int_T^{2T} |\zeta(1+it)L(\kappa+it,f)|^2 dt. \end{aligned} \tag{4.1}$$

Denote the integral on the right-hand side by $S(1)$. Let j be an integer, $[T]\le j\le[2T]$, and t_j the point at which $|\zeta(1+it)|$ $(j\le t\le j+1)$ attains its maximal value. Let J_1 be the set of all j for which

$$|\log\zeta(1+it_j)| > \log\log\log T - \log\varepsilon \tag{4.2}$$

holds, and J_2 the set of all other j. Then

$$S(1) \le \sum_{j=[T]}^{[2T]} |\zeta(1+it_j)|^2 \int_j^{j+1} |L(\kappa+it,f)|^2 dt = \sum_{j\in J_1} + \sum_{j\in J_2}. \tag{4.3}$$

This kind of division of the sum has been used in the proof of Lemma 3.3 of [15]. We apply the lemma in Section 3 to the Riemann zeta-function (hence $k=A=1$) to see that the number of elements of J_1 is $O(T^{4\delta})$, because we can choose $C_2=2$ for $11/14\le\sigma\le1$ by Jutila [9]. We recall that f is a holomorphic Hecke-eigen cusp form of weight 2κ. Hence, using the estimate

$$\zeta(1+it) \ll (\log t)^{2/3} \tag{4.4}$$

of Vinogradov-Korobov (see (6.7) of [6]) and

$$L(\kappa+it,f) \ll (|t|+2)^{1/3+\varepsilon} \tag{4.5}$$

of Good [4], we obtain

$$\sum_{j\in J_1} \ll T^{4\delta+2/3+2\varepsilon}(\log T)^{4/3} \ll T^{2/3+\varepsilon'}. \tag{4.6}$$

Here, and in what follows, ε' denotes a positive quantity which tends to 0 when ε tends to 0, and is not necessarily the same at each occurrence.

If $j \in J_2$, then

$$\Big|\log|\zeta(1+it_j)|\,\Big| = \Big|\Re\log\zeta(1+it_j)\Big| \le \Big|\log\zeta(1+it_j)\Big| \le \log\log\log T - \log\varepsilon,$$

so

$$\begin{aligned}|\zeta(1+it_j)| &= \exp(\log|\zeta(1+it_j)|)\\ &\le \exp(\log\log\log T - \log\varepsilon) \ll \log\log T.\end{aligned} \tag{4.7}$$

Hence

$$\sum_{j\in J_2} \ll (\log\log T)^2 \int_{[T]}^{[2T]+1} |L(\kappa+it,f)|^2 dt, \tag{4.8}$$

which is $O(T\log T(\log\log T)^2)$, because

$$\int_T^{2T} |L(\kappa+it,f)|^2 dt \asymp T\log T \tag{4.9}$$

(Good [3]). Substituting this estimate and (4.6) into (4.3), we obtain

$$S(1) \ll T\log T(\log\log T)^2. \tag{4.10}$$

Next, let $[T]+1 \le j \le [2T]-1$, and t_j' the point at which $|\zeta(1+it)|$ $(j \le t \le j+1)$ attains its minimal value. Let J_1' be the set of all j for which

$$|\log\zeta(1+it_j')| > \log\log\log T - \log\varepsilon \tag{4.11}$$

holds, and J_2' the set of all other j. Then

$$S(1) \ge \sum_{j=[T]+1}^{[2T]-1} |\zeta(1+it_j')|^2 \int_j^{j+1} |L(\kappa+it,f)|^2 dt = \sum_{j\in J_1'} + \sum_{j\in J_2'}. \tag{4.12}$$

Again by the lemma we see that the number of elements of J_1' is $O(T^{4\delta})$. Hence, similarly to (4.6), we have

$$\sum_{j\in J_1'} \ll T^{2/3+\varepsilon'}. \tag{4.13}$$

On the other hand, if $j \in J_2'$, then

$$\Big|\log|\zeta(1+it_j')|^{-1}\,\Big| = \Big|-\Re\log\zeta(1+it_j')\Big| \le \Big|\log\zeta(1+it_j')\Big| \le \log\log\log T - \log\varepsilon,$$

so

$$|\zeta(1+it_j')|^{-1} \ll \log\log T. \tag{4.14}$$

Therefore

$$\sum_{j\in J_2'} \gg (\log\log T)^{-2} \sum_{j\in J_2'} \int_j^{j+1} |L(\kappa+it,f)|^2 dt$$
$$= (\log\log T)^{-2} \left\{ \int_{[T]+1}^{[2T]} |L(\kappa+it,f)|^2 dt - \sum_{j\in J_1'} \int_j^{j+1} |L(\kappa+it,f)|^2 dt \right\}. \tag{4.15}$$

The first term in the curly parenthesis is $\asymp T\log T$ by (4.9), while the second term in the curly parenthesis is (again by using (4.5))

$$\ll \sum_{j\in J_1'} T^{2/3+2\varepsilon} \ll T^{2/3+\varepsilon'}.$$

Hence the right-hand side of (4.15) is $\gg T\log T(\log\log T)^{-2}$. Combining this with (4.12) and (4.13), we obtain

$$S(1) \gg T\log T(\log\log T)^{-2}. \tag{4.16}$$

This and (4.10) imply the assertion of Theorem 2.

The basic structure of the proof of Theorem 3 is similar. Since f is Hecke-eigen, $\tilde{L}(s,f)$ has the Euler product expansion of the form (3.1) with $k=2$, $A=1$. It is well known that $\tilde{L}(s,f)$ can be continued to an entire function, and of polynomial order with respect to $|t|$. Moreover, $\tilde{L}(s,f)$ is an example of "general L-function" in the sense of Perelli [14], so (3.3) is valid with $C_2=2$ for $3/4\le\sigma\le 1$, by Theorem 5 of [14]. Therefore we can apply the lemma in Section 3 to obtain

$$R \ll T^{4\delta}. \tag{4.17}$$

Now consider $I(1/2;T,2T)$. In this case

$$|L(\kappa+\nu-j+1/2+it,f)| \asymp \begin{cases} 1 & (j\le \nu-1), \\ (|t|+2)^{2(j-\nu-1/2)} & (j\ge\nu+2), \end{cases}$$

hence we have

$$I(1/2;T,2T) \asymp \prod_{j=\nu+2}^{2\nu} T^{4(j-\nu-1/2)}$$
$$\times \int_T^{2T} |\zeta(1/2+it)L(\kappa+1/2+it,f)L(\kappa-1/2+it,f)|^2 dt.$$

Since $\sum_{j=\nu+2}^{2\nu} 4(j-\nu-1/2) = 2\nu^2-2$, by using (2.5) we have

$$I(1/2;T,2T) \asymp T^{2\nu^2} \int_T^{2T} |\zeta(1/2+it)|^2 |L(\kappa+1/2+it,f)|^4 dt. \tag{4.18}$$

We denote the integral on the right-hand side by $S(1/2)$. Let $t_j(f)$ be the point at which $|L(\kappa+1/2+it,f)|$ $(j\leq t\leq j+1)$ attains its maximal value. Let $J_1(f)$ be the set of all j, $[T]\leq j\leq [2T]$, for which

$$|\log L(\kappa+1/2+it_j(f),f)|>2\log\log\log T-\log\varepsilon \tag{4.19}$$

holds, and $J_2(f)$ the set of all other j. Then, similarly to (4.3), we have

$$\begin{aligned} S(1/2)&\leq \sum_{j=[T]}^{[2T]}|L(\kappa+1/2+it_j(f),f)|^4\int_j^{j+1}|\zeta(1/2+it)|^2dt \\ &=\sum_{j\in J_1(f)}+\sum_{j\in J_2(f)}. \end{aligned} \tag{4.20}$$

Estimate (4.17) implies that the number of elements of $J_1(f)$ is $O(T^{4\delta})$. Hence, using

$$L(\kappa+1/2+it,f)\ll(\log(|t|+2))^2 \tag{4.21}$$

((3.42) of [12]) and the classical estimate

$$\zeta(1/2+it)\ll(|t|+2)^{1/6},$$

we see that the first sum on the right-hand side of (4.20) is $O(T^{1/3+\varepsilon'})$. If $j\in J_2(f)$, then, similarly to (4.7), we have

$$|L(\kappa+1/2+it_j(f),f)|\ll(\log\log T)^2. \tag{4.22}$$

Hence, noting the well known formula

$$\int_T^{2T}|\zeta(1/2+it)|^2dt\asymp T\log T,$$

we find that the second term on the right-hand side of (4.20) is estimated as $O(T\log T(\log\log T)^8)$, so the same upper bound holds for $S(1/2)$.

As for the lower bound, we again proceed similarly to the proof of Theorem 2. Define $t_j'(f)$ analogously to t_j'. Corresponding to (4.14), we can show

$$|L(\kappa+1/2+it_j'(f),f)|^{-1}\ll(\log\log T)^2,$$

and using this, we obtain $S(1/2)\gg T\log T(\log\log T)^{-8}$. This completes the proof of Theorem 3.

5. Further examples

The method developed in the present paper can be applied to various other situations. In this section we describe briefly some typical examples.

First, we consider the mean square of the function $(\zeta(s))^h L(s,F_0,st)$, where h is any real number. Let

$$I(\sigma,h;T)=\int_1^T\left|\zeta(\sigma+it)^hL(\sigma+it,F_0,st)\right|^2dt. \tag{5.1}$$

Theorem 4. *For any real h, we have*

$$T^{2\nu(\nu-1)+1}\log T(\log\log T)^{-2|h+1|} \ll I(1,h;T) \\ \ll T^{2\nu(\nu-1)+1}\log T(\log\log T)^{2|h+1|}. \tag{5.2}$$

We can prove this theorem quite similarly to Theorem 2, so we omit the details of the proof. It is to be noted that, in the proof of the case $h+1<0$, J_1', J_2' are used for the upper bound, while J_1, J_2 are used for the lower bound. In this case, instead of (4.4), we use the inequality

$$|\zeta(1+it)| \gg (\log t)^{-2/3}(\log\log t)^{-1/3}$$

(Section 6.19 of Titchmarsh [19]) to estimate the sums on J_1, J_1'.

Now we recall the well known conjecture

$$\int_1^T |\zeta(1/2+it)|^{2k}dt \asymp T(\log T)^{k^2}, \tag{C_k}$$

where k is any positive integer. This conjecture has been hitherto established only for $k=1,\ 2$.

Theorem 5. *Let h be a positive integer. Then we have*

$$T^{2\nu^2+1}(\log T)^{(h+1)^2}(\log\log T)^{-8} \ll I(1/2,h;T) \\ \ll T^{2\nu^2+1}(\log T)^{(h+1)^2}(\log\log T)^{8} \tag{5.3}$$

unconditionally for $h=1$, under the assumption (C_3) for $h=2$, and under the assumptions (C_{h+1}) and (C_{h+2}) for $h\geq 3$.

This theorem can be proved similarly to Theorem 3. Define $I(1/2,h;T,2T)=I(1/2,h;2T)-I(1/2,h;T)$. Analogously to (4.18), we obtain

$$I(1/2,h;T,2T) \asymp T^{2\nu^2}\int_T^{2T} |\zeta(1/2+it)|^{2(h+1)}|L(\kappa+1/2+it,f)|^4 dt. \tag{5.4}$$

Consider the upper bound. The integral on the right-hand side can be divided into two sums, on $J_1(f)$ and on $J_2(f)$. The latter can be estimated, by using (4.22) and (C_{h+1}) (hence unconditionally if $h=1$), as

$$O\left(T(\log T)^{(h+1)^2}(\log\log T)^8\right).$$

The sum on $J_1(f)$ is estimated as $O(T^{2(h+1)\theta_1+\varepsilon})$, where θ_1 is any number for which $\zeta(1/2+it) \ll (|t|+2)^{\theta_1+\varepsilon}$ holds for any real t. To prove the second inequality of (5.3), it is sufficient to show that

$$2(h+1)\theta_1 < 1. \tag{5.5}$$

Since the best known value of θ_1 is smaller than 1/6, (5.5) follows unconditionally for $h = 1,\ 2$. For $h \geq 3$, we assume (C_{h+2}). From this assumption it easily follows that $\theta_1 \leq 1/2(h+2)$, which clearly implies (5.5). The first inequality of (5.3) can be deduced similarly.

Next we discuss the mean square of spinor L-functions. The explicit form of spinor L-functions attached to Ikeda lifts has been shown by Murakawa [13] and Schmidt [17], but the general form of attached spinor L-functions is rather complicated, involving powers of symmetric power L-functions, and it seems to be not easy to obtain sharp mean square estimates of them by the present method. However, in the simplest case $\nu = 1$, our method can be applied successfully. In this case the lift is actually the classical Saito-Kurokawa lift, and the attached spinor L-function is

$$L(s, F_0, \mathrm{spin}) = L(s, f)\zeta(s-\kappa)\zeta(s-\kappa+1) \tag{5.6}$$

(see Section 6 of Eichler-Zagier [2]). The critical strip of this function is $\kappa - 1 \leq \sigma \leq \kappa + 1$, so it is enough to consider the mean square for $\kappa \leq \sigma \leq \kappa + 1$. Define

$$I^*(\sigma; T) = \int_1^T |L(\sigma + it, F_0, \mathrm{spin})|^2 dt.$$

Theorem 6. *We have*

$$I^*(\sigma; T) = AT + O\left(T^{2(\kappa+1-\sigma)+\varepsilon}\right) \tag{5.7}$$

with a certain positive constant A for $\kappa + 1/2 < \sigma \leq \kappa + 1$,

$$T \log T (\log\log T)^{-4} \ll I^*(\kappa + 1/2; T) \ll T \log T (\log\log T)^4, \tag{5.8}$$

$$I^*(\sigma; T) \ll T^{2(1-\sigma+\kappa)+\min\{4\theta_1(\sigma-\kappa), 2\theta_2(2\kappa+1-2\sigma)\}+\varepsilon} \tag{5.9}$$

for $\kappa < \sigma < \kappa + 1/2$, where θ_2 is any number for which $L(\kappa + it, f) \ll (|t| + 2)^{\theta_2+\varepsilon}$ holds for any real t, and

$$T^2 \log T (\log\log T)^{-4} \ll I^*(\kappa; T) \ll T^2 \log T (\log\log T)^4. \tag{5.10}$$

The proof of (5.9) (which is actually valid for $\kappa < \sigma \leq \kappa + 1/2$) is similar to that of (3.32) of [12]. Also, similarly to the case $\ell = 2\nu$ of (3.27) of [12], we find that $I^*(\sigma; T) \asymp T$ for $\kappa + 1/2 < \sigma \leq \kappa + 1$. These results imply

$$\int_{-\infty}^{\infty} |\tilde{L}(\sigma + it, F_0, \mathrm{spin})|^2 \cdot \frac{dt}{|\sigma + it|^2} < \infty$$

for any $\sigma > 0$, where $\tilde{L}(s, F_0, \mathrm{spin}) = L(s + \kappa, F_0, \mathrm{spin})$. Therefore by Theorem 2 of [12] we obtain (5.7). Finally, we can show (5.8) and (5.10) analogously to Theorem 3 and Theorem 2, respectively.

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