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An analytical approximation method for the stabilizing solution of the Hamilton-Jacobi equation based on stable manifold theory

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Abstract— In this paper, an analytical approximation approach for the stabilizing solution of the Hamilton-Jacobi equation using stable manifold theory is proposed. The proposed method gives approximated flows on the stable manifold of the associated Hamiltonian system and provides approximations of the stable Lagrangian submanifold. With this method, the closed loop stability is guaranteed and can be enhanced by taking higher order approximations. A numerical example shows the effectiveness of the method.

I. INTRODUCTION

When analyzing a control system or designing a feedback control, one often encounters certain types of equations that dominate fundamental properties of the control problem at hand. It is the Riccati equation for linear systems and the Hamilton-Jacobi equation plays the same role in nonlinear systems. For example, an optimal feedback control can be derived from a solution of a Hamilton-Jacobi equation [18] and H^∞ feedback controls are obtained by solving one or two Hamilton-Jacobi equations [2], [15], [28], [29]. Closely related to optimal control and H^∞ control is the notion of dissipativity, which is also characterized by a Hamilton-Jacobi equation (see, e.g., [13], [32]). Some active areas of research in recent years are the factorization problem [3], [4] and the balanced realization problem [10] and the solutions of these problems are again represented by Hamilton-Jacobi equations (or, inequalities). Contrary to the well-developed theory and computational tools for the Riccati equation, which are widely applied, the Hamilton-Jacobi equation is still an impediment to practical applications of nonlinear control theory.

In [19], [11], [22], [12] various series expansion techniques are proposed to obtain approximate solutions of the Hamilton-Jacobi equation. With these methods, one can calculate sub-optimal solutions using a few terms for simple nonlinearities. Although higher order approximations are possible to obtain for more complicated nonlinearities, their computations are often time-consuming and there is no guarantee that resulting controllers show better performance. Another approach is through successive approximation, where the Hamilton-Jacobi equation is reduced to a sequence of first order linear partial differential equations. The convergence of the algorithm is proven in [17]. In [6] an explicit technique to find approximate solutions to the

sequence of partial differential equation is proposed using the Galerkin spectral method and in [31] the authors propose a modification of the successive approximation method and apply the convex optimization technique. The advantage of the Galerkin method is that it is applicable to a larger class of systems, while the disadvantages are that it requires the initial iterate to satisfy certain conditions which are difficult to confirm and the multi-dimensional integration which can be significantly time-intensive. The state-dependent Riccati equation approach is proposed in [14], [21] where a nonlinear function is rewritten in a linear-like representation. In this method, feedback control is given in a power series form and has a similar disadvantage to the series expansion technique in that it is useful only for simple nonlinearities. A technique that employs open-loop controls and their interpolation is used in [20]. The drawback is that the interpolation of open-loop controls for each point in discretized state space is time-consuming and the computational cost grows exponentially with the state space dimension. A partially related research field to approximate solutions of the Hamilton-Jacobi equation is the theory of viscosity solutions. It deals with general Hamilton-Jacobi equations for which classical (differentiable) solutions do not exist. For introductions to viscosity solutions see, for instance, [5], [8], [9] and for an application to an H^∞ control problem, see [27]. The finite-element and finite-difference methods are studied for obtaining viscosity solutions. They, however, require discretization of state space, which can be a significant disadvantage.

Another direction in the research for the Hamilton-Jacobi equation is to study the geometric structure and the properties of the equation itself and its exact solutions. The papers [28] and [29] give a sufficient condition for the existence of the stabilizing solution using symplectic geometry. In [25], the geometric structure of the Hamilton-Jacobi equation is studied showing the similarity and difference with the Riccati equation. See also [30] for the treatment of the Hamilton-Jacobi equation as well as recently developed techniques in nonlinear control theory such as the theory of port-Hamiltonian systems. Recently, the authors proposed a Hamiltonian perturbation approach to obtain an approximation of the stabilizing solution when the uncontrolled part of the system is integrable[26].

In this paper, we attempt to develop a method to approximate the stabilizing solution of the Hamilton-Jacobi equation based on the geometric research in [28], [29] and [25]. The main object of the geometric research on the Hamilton-Jacobi equation is the associated Hamiltonian system. However, most approximation research papers mentioned above do

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not explicitly consider Hamiltonian systems, although it is well-known that the Hamiltonian matrix plays a crucial role in the calculation of the stabilizing solution for the Riccati equation. One of our purposes in this paper is to fill in this gap. The approach taken in this paper is based on stable manifold theory (see, e.g., [7], [24]). Using the fact that the stable manifold of the associated Hamiltonian system is a Lagrangian submanifold and its generating function corresponds to the stabilizing solution, which is shown in [28], and modifying stable manifold theory, we analytically give the solution sequence that converges to the solution of the Hamiltonian systems on the stable manifold. Thus, each element of the sequence approximates the Hamiltonian flow on the stable manifold and the feedback control constructed from the each element may serve as an approximation of the desired feedback. It should be mentioned that computation methods of stable manifolds in dynamical systems are being developed and a comprehensive survey of the recent results in this area can be found in [16]. The proposed method in this paper, however, is different from the above numerical methods in that it gives analytical expressions of the approximated flows on stable manifolds, which may have considerable potential for control system design that often leads to high dimensional Hamiltonian systems.

The organization of this paper is as follows. In §II, the theory of 1st-order partial differential equations is reviewed in the framework of symplectic geometry, stressing the one-to-one correspondence between solution and Lagrangian submanifold. In §III, a special type of solution, called the stabilizing solution, is introduced and the geometric theory for the Riccati equation is also reviewed. In §IV, an analytical approximation algorithm for the stable Lagrangian submanifold is proposed, using a modification of stable manifold theory. In §V, we illustrate a numerical example showing the effectiveness of the proposed method and discuss some computational issues.

II. Review of the theory of 1st-order partial differential equations

In this section we outline, by using the symplectic geometric machinery, the essential parts of the theory of partial differential equations of first order.

Let us consider a partial differential equation of the form

$$(\text{PD}) \qquad F(x_1, \cdots, x_n, p_1, \cdots, p_n) = 0,$$

where F is a C^∞ function of $2n$ variables, $x_1, \cdots, x_n$ are independent variables, z is an unknown function and $p_1 = \partial z / \partial x_1, \cdots, p_n = \partial z / \partial x_n$. Let M be an n dimensional space for $(x_1, \cdots, x_n)$. We regard the $2n$ dimensional space for $(x, p) = (x_1, \cdots, x_n, p_1, \cdots, p_n)$ as the cotangent bundle T^*M of M. T^*M is a symplectic manifold with symplectic form $\theta = \sum_{i=1}^n dx_i \wedge dp_i$.

Let $\pi : T^*M \to M$ be the natural projection and $V \subset T^*M$ be a hypersurface defined by $F = 0$. Define a submanifold

$$\Lambda_Z = \{(x, p) \in T^*M \mid p_i = \partial z / \partial x_i(x), i = 1, \cdots, n\}$$

for a smooth function $z(x)$. Then, $z(x)$ is a solution of (PD) if and only if $\Lambda_Z \subset V$. Furthermore, $\pi|_{\Lambda_Z} : \Lambda_Z \to M$ is a diffeomorphism and Λ_Z is a Lagrangian submanifold because $\dim \Lambda_Z = n$ and

$$\theta|_{\Lambda_Z} = 0.$$

Conversely, it is well-known (see, e.g. [1], [23]) that for a Lagrangian submanifold Λ passing through $q \in T^*M$ on which $\pi|_\Lambda : \Lambda \to M$ is a diffeomorphism, there exists a neighborhood U of q and a function $z(x)$ defined on $\pi(U)$ such that

$$\Lambda \cap U = \{(x, p) \in U \mid p_i = \partial z / \partial x_i(x), i = 1, \cdots, n\}.$$

Therefore, finding a solution of (PD) is equivalent to finding a Lagrangian submanifold $\Lambda \subset V$ on which $\pi|_\Lambda : \Lambda \to M$ is a diffeomorphism.

Let $f_1 = F$. To construct such a Lagrangian submanifold passing through $q \in T^*M$, and hence to obtain a solution defined on a neighborhood of $\pi(q)$, it suffices to find functions $f_2, \cdots, f_n \in \mathscr{F}(T^*M)$ with $df_1(q) \wedge \cdots \wedge df_n(q) \neq 0$ such that $\{f_i, f_j\} = 0$ $(i, j = 1, \cdots, n)$, where $\{\cdot, \cdot\}$ is the Poisson bracket, and

$$\left| \frac{\partial(f_1, \cdots, f_n)}{\partial(p_1, \cdots, p_n)} \right| (q) \neq 0. \qquad (1)$$

Using these functions, equations $f_1 = 0$, $f_j = $ constant, $j = 2, \ldots, n$ define a Lagrangian submanifold $\Lambda \subset V$. Note that the condition (1) implies, by the implicit function theorem, that $\pi|_\Lambda$ is a diffeomorphism on some neighborhood of q.

Since $\{F, \cdot\}$ is the Hamiltonian vector field X_F with Hamiltonian F, the functions $f_2, \cdots, f_n$ above are first integrals of X_F. The ordinary differential equation that gives the integral curve of X_F is the Hamilton's canonical equations

$$\begin{cases} \dfrac{dx_i}{dt} = \dfrac{\partial F}{\partial p_i} \\[2mm] \dfrac{dp_i}{dt} = -\dfrac{\partial F}{\partial x_i} \end{cases} \qquad (i = 1, \cdots, n), \qquad (2)$$

and therefore, we seek $n-1$ commuting first integrals of (2) satisfying (1).

III. The stabilizing solution of the Hamilton-Jacobi equation

Let us consider the Hamilton-Jacobi equation often encountered in nonlinear control theory

$$(\text{HJ}) \qquad H(x, p) = p^T f(x) - \frac{1}{2} p^T R(x) p + q(x) = 0,$$

where $f : M \to \mathbb{R}^n$, $R : M \to \mathbb{R}^{n \times n}$, $q : M \to \mathbb{R}$ are all C^∞, and $R(x)$ is a symmetric matrix for all $x \in M$. We also assume that f and q satisfy $f(0) = 0$, $q(0) = 0$ and $\frac{\partial q}{\partial x}(0) = 0$. In what follows, we write $f(x)$, $q(x)$ as $f(x) = Ax + O(|x|^2)$, $q(x) = \frac{1}{2} x^T Q x + O(|x|^2)$ where A is an $n \times n$ real matrix and $Q \in \mathbb{R}^{n \times n}$ is a symmetric matrix. The stabilizing solution of (HJ) is defined as follows.

Definition 1: A solution $z(x)$ of (HJ) is said to be the stabilizing solution if $p(0) = 0$ and 0 is an asymptotically stable equilibrium of the vector field $f(x) - R(x)p(x)$, where $p(x) = (\partial z/\partial x)^T(x)$.

It will be important to understand the notion of the stabilizing solution in the framework of symplectic geometry described in the previous section. Suppose that we have the stabilizing solution $z(x)$ around the origin. Then, the Lagrangian submanifold corresponding to $z(x)$ is

$$\Lambda_Z = \{(x, p) \,|\, p = \partial z/\partial x(x)\} \subset T^* M.$$

Λ_Z is invariant under the Hamiltonian flow of

$$\begin{cases} \dot{x} = f(x) - R(x)p \\ \dot{p} = -\dfrac{\partial f}{\partial x}(x)^T p + \dfrac{\partial (p^T R(x)p)}{\partial x}^T - \dfrac{\partial q}{\partial x}^T. \end{cases} \quad (3)$$

To see this invariance, one needs to show that the second equation identically holds on Λ_Z, which can be done by taking the derivative of (HJ) after replacing p with $p(x)$. Note that the left-hand side in the second equation of (3) restricted to Λ_Z is $(\partial p/\partial x)(f(x) - R(x)p(x))$. The first equation is exactly the vector field in Definition 1. Therefore, the stabilizing solution is the Lagrangian submanifold on which π is a diffeomorphism and the Hamiltonian flow associated with $H(x, p)$ is asymptotically stable.

We assume the following throughout this paper.

Assumption 1: The Riccati equation obtained by linearizing (HJ)

$$PA + A^T P - PR(0)P + Q = 0$$

has a solution $P = \Gamma$ such that $A - R(0)\Gamma$ is stable.

IV. ANALYTICAL APPROXIMATION OF THE STABLE LAGRANGIAN SUBMANIFOLD

A. Diagonalization of linear Hamiltonian systems

Extracting the linear part in (HJ), (3) can be written as

$$\begin{pmatrix} \dot{x} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} A & -R(0) \\ -Q & -A^T \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix} + \text{higher order terms.} \quad (4)$$

Using a suitable linear coordinate transformation

$$\begin{pmatrix} x' \\ p' \end{pmatrix} = T \begin{pmatrix} x \\ p \end{pmatrix} \quad (5)$$

(4) can be written as

$$\begin{pmatrix} \dot{x}' \\ \dot{p}' \end{pmatrix} = \begin{pmatrix} A - R(0)\Gamma & 0 \\ 0 & -(A - R(0)\Gamma)^T \end{pmatrix} \begin{pmatrix} x' \\ p' \end{pmatrix} + \text{higher order terms.} \quad (6)$$

B. Approximation of stable manifolds

We consider the following system (A in this subsection corresponds to $A - R(0)\Gamma$ in the previous subsection).

$$\begin{cases} \dot{x} = Ax + f(t, x, y) \\ \dot{y} = -A^T y + g(t, x, y) \end{cases} \quad (7)$$

Assumption 2: A is a stable $n \times n$ real matrix and it holds that $\|e^{At}\| \leqslant ae^{-bt}$, $t \geqslant 0$ for some constants $a > 0$ and $b > 0$.

Assumption 3: $f, g : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous and satisfy the following.

i) For all $t \in \mathbb{R}$, $|x| + |y| < l$ and $|x'| + |y'| < l$,

$$|f(t, x, y) - f(t, x', y')| \leqslant \delta_1(l)(|x - x'| + |y - y'|).$$

ii) For all $t \in \mathbb{R}$, $|x| + |y| < l$ and $|x'| + |y'| < l$,

$$|g(t, x, y) - g(t, x', y')| \leqslant \delta_2(l)(|x - x'| + |y - y'|),$$

where $\delta_j : [0, \infty) \to [0, \infty)$, $j = 1, 2$ are continuous and monotonically increasing on $[0, L_j]$ for some constants $L_1, L_2 > 0$.

Furthermore, there exist constants $M_1, M_2 > 0$ such that $\delta_j(l) \leqslant M_j l$ holds on $[0, L_j]$ for $j = 1, 2$.

Let us define the sequences $\{x_n(t, \xi)\}$ and $\{y_n(t, \xi)\}$ by

$$\begin{cases} x_{n+1} = e^{At}\xi + \displaystyle\int_0^t e^{A(t-s)} f(s, x_n(s), y_n(s))\, ds \\ y_{n+1} = -\displaystyle\int_t^\infty e^{-A^T(t-s)} g(s, x_n(s), y_n(s))\, ds \end{cases} \quad (8)$$

for $n = 0, 1, 2, \ldots$, and

$$\begin{cases} x_0 = e^{At}\xi \\ y_0 = 0 \end{cases} \quad (9)$$

with arbitrary $\xi \in \mathbb{R}^n$.

The following theorem states that the sequences $\{x_n(t, \xi)\}$, $\{y_n(t, \xi)\}$ are the approximating solutions to the exact solution of (7) on the stable manifold with the property that each element of the sequences is convergent to the origin. Due to space limitation, we did not include the proof.

Theorem 2: Under Assumptions 2 and 3, $x_n(t, \xi)$ and $y_n(t, \xi)$ are convergent to zero for sufficiently small $|\xi|$, that is, $x_n(t, \xi)$, $y_n(t, \xi) \to 0$ as $t \to \infty$ for all $n = 0, 1, 2, \ldots$. Furthermore, $x_n(t, \xi)$ and $y_n(t, \xi)$ are uniformly convergent to a solution of (7) on $[0, \infty)$ as $n \to \infty$. Let $x(t, \xi)$ and $y(t, \xi)$ be the solution obtained as the limits of $x_n(t, \xi)$ and $y_n(t, \xi)$, respectively. Then, it holds that $x(t, \xi)$, $y(t, \xi) \to 0$ as $t \to \infty$.

C. The approximation algorithm

For (6), Assumption 1 implies Assumption 2 and Assumption 3 is satisfied if f, R and q in (HJ) are sufficiently smooth. Thus, we propose the following procedure for approximation of $\partial V/\partial x$.

Algorithm:

(i) Construct the sequences (8) for (6) and obtain the sequences $\{x_n(t, \xi)\}$, $\{p_n(t, \xi)\}$ in the original coordinates using (5).

(ii) Form an $n - 1$ dimensional manifold in ξ-space, say, $(\xi_1(\eta_1, \ldots, \eta_{n-1}), \ldots, \xi_n(\eta_1, \ldots, \eta_{n-1}))$. Eliminate t and $n - 1$ variables $\eta_1, \ldots, \eta_{n-1}$ from $x = x_n(t, \xi(\eta_1, \ldots, \eta_{n-1}))$, $p = p_n(t, \xi(\eta_1, \ldots, \eta_{n-1}))$ to get $p = \pi_n(x)$.

(iii) The function π_n serves as an approximation of $\partial V/\partial x$.

Remark 4.1: (i) Typically, one can choose an $n-1$-sphere as the $n-1$ dimensional initial manifold.

(ii) Elimination of variables is an algebraic operation but not necessarily easy to carry out in practice. An effective use of software is required for this purpose. In §V, we interpolate the values of p_n for sample points of x_n to get the function $\pi_n(x)$. Furthermore, $\pi_n(x)$ depends on the initial manifold in ξ-space. The dependence, however, can be smaller by taking larger n.

(iii) What one obtains from the algorithm is equivalent to approximations of the stable Lagrangian submanifold. They, in general, do not satisfy the integrability condition for finite n. Therefore, it is difficult to get an approximation of the generating function for the Lagrangian submanifold in a geometric manner. However, since we have the analytical expression of the approximations, we can write down approximations of generating function as described below. Let us consider the following optimal control problem.

$$\dot{x} = f(x) + g(x)u$$
$$J = \int_0^\infty L(x(t), u(t))\, dt,$$

where L takes the form of, for example, $L = (h(x)^T h(x) + u^T u)/2$. The optimal feedback control is given by

$$u = -\frac{1}{2}g(x)^T \frac{\partial V}{\partial x}(x)^T,$$

where $V(x)$ is the stabilizing solution of the corresponding Hamilton-Jacobi equation. By the algorithm, the n-th approximation of the Lagrangian submanifold is obtained from

$$\begin{cases} x = x_n(t, \xi) \\ p = p_n(t, \xi) \end{cases}$$

and the n-th approximation of the optimal feedback can be described with t and ξ as

$$u_n(t, \xi) = -\frac{1}{2}g(x_n(t, \xi))^T p_n(t, \xi).$$

Since the generating function is the minimum value of J for each ξ, its approximation can be written as

$$V_n(\xi) = \int_0^\infty L(x_n(t, \xi), u_n(t, \xi))\, dt.$$

Similar expressions of the generating function for the H^∞ problem may be possible using dissipative system theory[32].

V. NUMERICAL EXAMPLE

Let us consider the 1-dimensional nonlinear optimal control problem;

$$\dot{x} = x - x^3 + u$$
$$J = \int_0^\infty \frac{q}{2}x^2 + \frac{r}{2}u^2\, dt.$$

The Hamilton-Jacobi equation for this problem is

$$H = p(x - x^3) - \frac{1}{2r}p^2 + \frac{q}{2}x^2 = 0$$

and the Hamilton's canonical equations are

$$\begin{cases} \dot{x} = x - x^3 - \dfrac{1}{r}p \\ \dot{p} = -(1 - 3x^2)p - qx. \end{cases} \tag{10}$$

A. The stable manifold approximation method

The coordinate transformation that diagonalizes the linear part of (10) is

$$\begin{pmatrix} x \\ p \end{pmatrix} = T \begin{pmatrix} x' \\ p' \end{pmatrix},$$
$$T = \begin{pmatrix} 1 & -(1 + \sqrt{1 + q/r}) \\ r + \sqrt{r^2 + qr} & q \end{pmatrix}.$$

The equations in the new coordinates are

$$\begin{pmatrix} \dot{x}' \\ \dot{p}' \end{pmatrix} = \begin{pmatrix} -\sqrt{1 + q/r}\, x' \\ \sqrt{1 + q/r}\, p' \end{pmatrix} + \begin{pmatrix} -x(x', p')^3 \\ 3x(x', p')^2 p(x', p') \end{pmatrix},$$

where

$$x(x', p') = x' - (1 + \sqrt{1 + q/r})p',$$
$$p(x', p') = (r + \sqrt{r^2 + qr})x' + qp'.$$

We construct the sequences (8) with

$$f(x', p') = -x(x', p')^3,$$
$$g(x', p') = 3x(x', p')^2 p(x', p'),$$

and $q = 1$, $r = 1$. From $x_n(t, \xi)$ and $p_n(t, \xi)$, the relation of x_n and p_n is obtained by eliminating t, which will be denoted as $p = \pi_n(x)$. We note that $\pi_n(x)$ depends on ξ. The approximated feedback functions are $u = -(1/r)\pi_n(x) = -\pi_n(x)$.

Figures 1-3 show the results of calculation for $\pi_n(x)$. To guarantee the convergence of the solution sequence (8), $|\xi|$ has to be small enough. If $|\xi|$ is too large, the sequence is not convergent (see, Fig. 1 and Fig. 3). If $|\xi|$ is small and only positive t is used in $x_n(t, \xi)$ and $p_n(t, \xi)$, then the resulting trace in the $x-p$ plane is short, hence, the function $\pi_n(x)$ is defined in a small set around the origin. Therefore, we substitute negative values in t to extend the trace toward the opposite direction. This, however, creates a divergent effect on the sequence (see, Fig. 1 and Fig. 2). And this effect becomes smaller as n increases.

In Fig. 4, the calculation result by the Hamiltonian perturbation approach in [26] is shown. Since the integrable nonlinearity is fully taken into account in this approach, the feedback function is better approximated in the region further from the origin. Also, we showed the result by the Taylor series expansion of order $n = 6$ in Fig. 4.
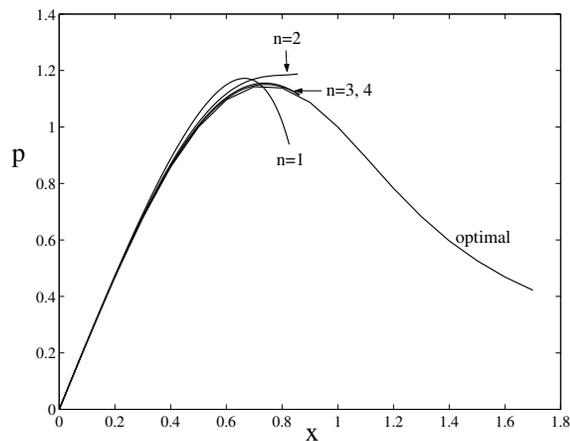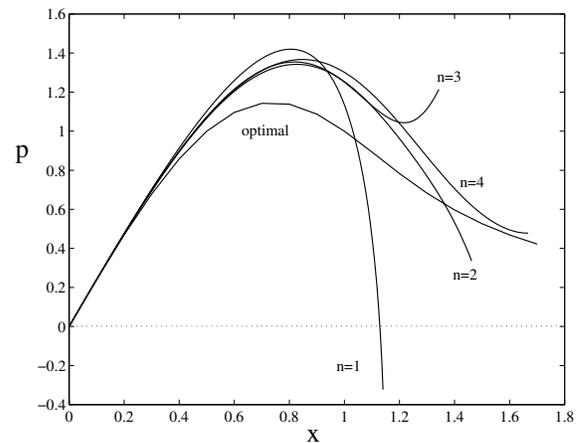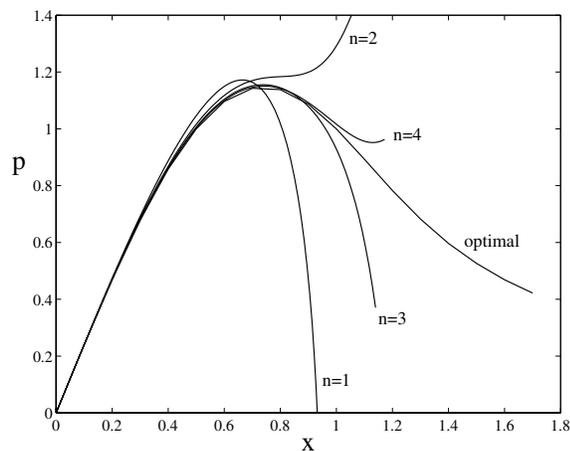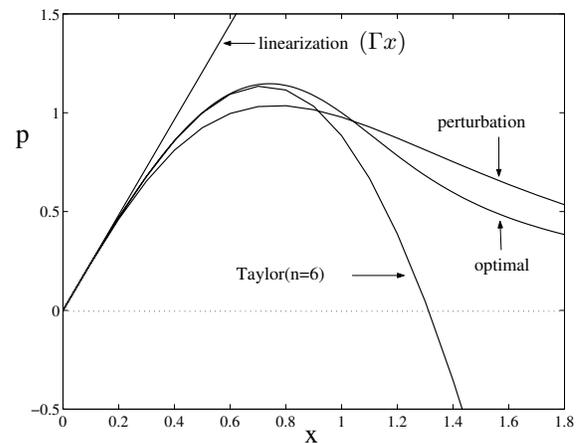
Fig. 1. $\xi = 0.42$ and extended to the negative time -0.5

Fig. 2. $\xi = 0.42$ and extended to the negative time -0.8

Fig. 3. $\xi = 0.6$ and extended to the negative time -0.5

Fig. 4. Perturbation and Taylor expansion solutions

VI. CONCLUDING REMARKS

In this paper, we proposed an analytical approximation approach for the stabilizing solution of the Hamilton-Jacobi equation using stable manifold theory. The proposed method gives approximated flows on the stable manifold of the associated Hamiltonian system as functions of time and initial states. Feedback controls are calculated by elimination of variables. Since this method focuses on the stable manifold of the Hamiltonian system, the closed loop system stability is guaranteed and can be enhanced by taking higher order approximation.

The elimination of variables is not necessarily easy in practice, although it is an algebraic operation. Effective software use should be discussed in the future research.

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APPENDIX

A. Outline of the proof of Theorem 2

From Assumptions 2 and 3, the following inequalities are derived. (In this section, we leave out the dependence of x_n and y_n on ξ for the sake of simplicity.)

- If $|x| + |y| \leqslant L_1$, then

$$|f(t, x, y)| \leqslant \delta_1(|x| + |y|)(|x| + |y|)$$
$$\leqslant M_1(|x| + |y|)^2.$$

If $|x| + |y| \leqslant L_2$, then

$$|g(t, x, y)| \leqslant \delta_2(|x| + |y|)(|x| + |y|)$$
$$\leqslant M_2(|x| + |y|)^2.$$

- If $|x|, |x'| \leqslant \bar{x}$ and $|y|, |y'| \leqslant \bar{y}$ for some positive constants $\bar{x}, \bar{y}$ satisfying $\bar{x} + \bar{y} \leqslant L_1$, then

$$|f(t, x, y) - f(t, x', y')|$$
$$\leqslant \delta_1(\bar{x} + \bar{y})(|x - x'| + |y - y'|)$$
$$\leqslant M_1(\bar{x} + \bar{y})(|x - x'| + |y - y'|).$$

If $|x|, |x'| \leqslant \bar{x}$ and $|y|, |y'| \leqslant \bar{y}$ for some positive constants $\bar{x}, \bar{y}$ satisfying $\bar{x} + \bar{y} \leqslant L_2$, then

$$|g(t, x, y) - g(t, x', y')|$$
$$\leqslant \delta_1(\bar{x} + \bar{y})(|x - x'| + |y - y'|)$$
$$\leqslant M_2(\bar{x} + \bar{y})(|x - x'| + |y - y'|).$$

(i) By taking limit in (8),

$$x(t) = e^{At}\xi + \int_0^t e^{A(t-s)} f(s, x(s), y(s))\, ds$$
$$y(t) = -\int_t^\infty e^{-A(t-s)} g(s, x(s), y(s))\, ds,$$

from which one can see that $x(t)$ and $y(t)$ satisfy (7).

(ii) For each $n = 0, 1, 2, \ldots$, $x_n(t)$ and $y_n(t)$ have the following estimates;

$$|x_n(t)| \leqslant \alpha_n e^{-bt}, \quad |y_n(t)| \leqslant \beta_n e^{-2bt},$$

where α_n and β_n are the constants defined by

$$\begin{cases} \alpha_{n+1} = \dfrac{2aM_1}{b}(\alpha_n{}^2 + \beta_n{}^2) + a|\xi| \\ \beta_{n+1} = \dfrac{2aM_2}{3b}(\alpha_n{}^2 + \beta_n{}^2) \\ \alpha_0 = a|\xi|, \quad \beta_0 = 0. \end{cases}$$

(iii) For sufficiently small $|\xi|$, $\{\alpha_n\}$ and $\{\beta_n\}$ are bounded and monotonically increasing sequences and therefore, $\lim_{n\to\infty} \alpha_n =: \underline{\alpha}$, $\lim_{n\to\infty} \beta_n =: \underline{\beta}$ exists. Furthermore, it holds that $\underline{\alpha}, \underline{\beta} \to 0$ when $|\xi| \to 0$.

(iv) The following inequalities hold

$$|x_n(t) - x_{n+1}(t)| \leqslant \gamma_n e^{-bt}$$
$$|y_n(t) - y_{n+1}(t)| \leqslant \varepsilon_n e^{-2bt},$$

where $\{\gamma_n\}$, $\{\varepsilon\}$ are the positive sequences defined by

$$\begin{cases} \gamma_{n+1} = \dfrac{a(\underline{\alpha} + \underline{\beta})M_1}{b}(\gamma_n + \varepsilon_n) \\ \varepsilon_{n+1} = \dfrac{a(\underline{\alpha} + \underline{\beta})M_2}{3b}(\gamma_n + \varepsilon_n) \\ \gamma_1 = \dfrac{a^3 M_1 |\xi|^2}{b}, \quad \varepsilon_1 = \dfrac{a^3 M_2 |\xi|^2}{3b}. \end{cases}$$

(v) For sufficiently small $|\xi|$, $\{\gamma_n\}$ and $\{\varepsilon_n\}$ are monotonically decreasing sequences and $\lim_{t\to\infty} \gamma_n = \lim_{t\to\infty} \varepsilon_n = 0$.