On computation of nonlinear balanced realization and model reduction

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***Abstract*— In this paper a computational algorithm for nonlinear balanced realization and model reduction based on Taylor series expansion is proposed. This algorithm requires recursive computations with respect to the order of the Taylor series in which we need to solve linear equations with unknown parameters in each step. Furthermore, the proposed method is applied to a double pendulum system. Some numerical simulations demonstrate the effectiveness of the proposed algorithm.**

I. INTRODUCTION

The nonlinear extension of the state-space concept of balanced realizations has been introduced in [1] by extending the controllability and observability Gramians to nonlinear functions called controllability and observability functions. After this paper, many results on nonlinear balancing and model reduction have been reported, e.g. [2], [3], [4], [5], [6]. In particular, *singular value functions* which are nonlinear state-space extension of the Hankel singular values in the linear case play an important role in the nonlinear Hankel theory. However, the original characterization in [1] was incomplete in the sense that they are not unique and the resulting model reduction procedure gives different reduced models according to the choice of different set of singular value functions.

One of the authors proposed a new characterization of Hankel singular value functions which have closer relationship to the gain structure of the Hankel operator in [7]. Further a new balanced realization and model reduction method has been proposed in [8], [7], [9], [10] in which almost unique reduced order models are obtained. However, in this method, we need to solve several partial differential equations to obtain the balanced realization. Moreover, only the existence of C^∞ solutions are guaranteed and how to compute them was not clear. Generally, to obtain a closed-form solution of these equations is quite difficult. Therefore, we should resolve computational difficulties arising from computing balanced realization in practical viewpoint. Concerning this point, Newman et al. [3], [4] proposed an effective computation algorithm based on the theory of stochastically excited dynamical systems and also found a closed form solution of a partial differential equation to be solved in balanced realization for a special class of systems. However, their method relies on the old (less precise) version of balanced realization in [1] and we need more precise investigation to obtain the newly developed balanced realization [10]. In this paper, we propose a computation algorithm for the new balanced realization and truncation based on Taylor series expansion. Our method requires analytic state space realization of input-affine continuous-time nonlinear systems.

There are two sets of partial differential equations to be solved in balanced truncation procedure. One is a set of a Hamilton-Jacobi-Bellman equation and a Lyapunov equation to obtain the controllability and observability functions, and the other is a first order partial differential equation with respect to the controllability and observability functions to obtain a coordinate transformation converting the system into the balanced realization. How to solve Hamilton-Jacobi-Bellman equations have been investigated by many researchers and here we adopt a classical result [11] based on Taylor series expansion. With respect to the latter partial differential equation, we also apply a Taylor series expansion to both the new coordinates and the Hankel singular value functions. This yields a procedure to determine the unknown coefficients of the Taylor series recursively. The proposed algorithm can always be executed in a recursive manner and we only need to solve linear equations in each step. Furthermore, the proposed algorithm is applied to a double pendulum example and a reduced order model is calculated using 4th oder Taylor series approximation. Some numerical simulations demonstrate the effectiveness of the proposed method. These results also reveals the effectiveness of the nonlinear balanced truncation method proposed in [8], [7], [9], [10].

II. NONLINEAR BALANCED REALIZATION AND MODEL REDUCTION

Here we briefly refer to the preliminary results which are the basis of the results reported in this paper. Let us consider an input-affine continuous-time nonlinear system

$$\Sigma : \begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases} \qquad (1)$$

with $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^r$. Here we need the following assumptions for the plant.

Assumption A1 Jacobian linearization of Σ is controllable, observable and asymptotically stable. Furthermore the Hankel singular values of the linearized system are distinct.

For this system, controllability function $L_c(x^0)$ and ob-

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servability function $L_o(x^0)$ are defined as follows

$$L_c(x^0) \;:=\; \inf_{\substack{u \in L_2(-\infty,0] \\ x(-\infty)=0,\, x(0)=x^0}} \frac{1}{2}\|u\|_{L_2}^2 \tag{2}$$

$$L_o(x^0) \;:=\; \frac{1}{2}\|y\|_{L_2}^2, \quad x(0)=x^0,\ u(t)\equiv 0. \tag{3}$$

These functions can be obtained by solving a pair of Hamilton-Jacobi equations.

Theorem 1: [1] Consider the system Σ in (1). Suppose that Assumption A1 holds and that there exists a neighborhood of W and a smooth observability function $L_o(x)$ on W. Then $L_o(x)$ is the unique smooth solution of the Hamilton-Jacobi equation

$$\frac{\partial L_o(x)}{\partial x} f(x) + \frac{1}{2}\ h(x)^{\mathrm{T}} h(x) = 0. \tag{4}$$

Furthermore, suppose that there exists a smooth controllability function $L_c(x)$ on W. Then $L_c(x)$ is the unique solution of the Hamilton-Jacobi equation

$$\frac{\partial L_c(x)}{\partial x} f(x) + \frac{1}{2}\ \frac{\partial L_c(x)}{\partial x} g(x) g(x)^{\mathrm{T}} \frac{\partial L_c(x)}{\partial x}^{\mathrm{T}} = 0 \tag{5}$$

such that $\dot{x} = -(f + gg^{\mathrm{T}}(\partial L_c/\partial x)^{\mathrm{T}})$ is asymptotically stable about 0 on W.

After calculating the controllability and observability functions, we need to find a coordinate transformation converting a system into the following *input-normal/output-diagonal* realization which is a more precise version compared with the original result in [1].

Theorem 2: [10] Consider the operator Σ with the state-space realization (1). Suppose that Assumption A1 holds. Then there exist a neighborhood W of the origin and a coordinate transformation $x = \Phi(z)$ on W converting the system into the following form

$$L_c(\Phi(z)) \;=\; \frac{1}{2} z^{\mathrm{T}} z \tag{6}$$

$$L_o(\Phi(z)) \;=\; \frac{1}{2}\sum_{i=1}^{n} z_i^2 \sigma_i(z_i)^2. \tag{7}$$

Based on this input-normal (output-diagonal) realization, we can easily obtain a *real* balanced realization by re-scaling the coordinate axes.

Corollary 1: [10] Consider the operator Σ with the state-space realization (1). Suppose that Assumption A1 holds. Then there exist a neighborhood W of the origin and a coordinate transformation $x = \Phi(z)$ on W converting the system into the following form

$$L_c(\Phi(z)) \;=\; \frac{1}{2}\sum_{i=1}^{n} \frac{z_i^2}{\bar{\sigma}_i(z_i)} \tag{8}$$

$$L_o(\Phi(z)) \;=\; \frac{1}{2}\sum_{i=1}^{n} z_i^2 \bar{\sigma}_i(z_i). \tag{9}$$

Proof: Suppose that the system is already balanced in the sense of Theorem 2 without loss of generality. Corollary can be proved by just defining the new singular value functions

$$\bar{\sigma}_i(z_i) := \sigma_i(\phi_i(z_i)) \tag{10}$$

with a coordinate transformation $x = \Phi(z) = (\phi_1(z_1), \phi_2(z_1), \ldots, \phi_n(z_n))$ where $z_i = \phi_i^{-1}(x_i) := x_i\sqrt{\sigma_i(x_i)}$. ∎

Here the functions $\sigma_i(\cdot)$'s and $\bar{\sigma}_i(\cdot)$'s are called *Hankel singular value functions*. Once we obtain a coordinate transformation $x = \Phi(z)$ yielding a balanced realization (or input-normal realization), we can obtain a reduced order model by truncating the important states. This method is known as *balanced truncation*.

Let us describe the system Σ on the new coordinates z

$$\Sigma : \begin{cases} \dot{z} = \bar{f}(z) + \bar{g}(z)u \\ y = \bar{h}(z) \end{cases}.$$

Here we suppose that Hankel singular value functions are in order, that is,

$$\min_{s=\pm c} \sigma_i(s) > \max_{s=\pm c} \sigma_{i+1}(s).$$

Divide the state-space into two subspaces $z = (z^a, z^b)$ with $z^a = (z_1, \ldots, z_k)$ and $z^b = (z_{k+1}, \ldots, z_n)$ and suppose that singular value function σ_k is much greater than σ_{k+1}. That is, the state variables in the z^a subspace have a bigger contribution to the input-output behavior than those of z^b. Then we can get a k-th order reduced order model by truncating the dynamics on the z^a subspace as follows

$$\Sigma^a : \begin{cases} \dot{z}^a = \bar{f}(z^a, 0) + \bar{g}(z^a, 0)u \\ y = \bar{h}(z^a, 0) \end{cases}. \tag{11}$$

The reduced order model Σ^a thus obtained preserves several important properties of the original system such as controllability, observability and stability.

Theorem 3: [12] Consider the system Σ in (1) and the divided system (11). Suppose that the system is already balanced in the sense of Theorem 2 (or Corollary 1). Then the reduced system Σ^a is balanced in the sense of Theorem 2 (Corollary 1 respectively), and

$$\sigma_i^a(z_i^a) \;=\; \sigma_i(z_i^a) \quad i \in \{1, 2, \ldots, k\} \tag{12}$$

hold with σ_i^a's the singular value functions of the system Σ^a. In particular, if $W = \mathbb{R}^n$, then

$$\|\Sigma^a\|_H = \|\Sigma\|_H. \tag{13}$$

Here $\|\Sigma\|_H$ denotes the Hankel norm of Σ, that is, the L_2-gain of the Hankel operator of Σ. In the rest of this paper, we will derive a method how to compute this model reduction procedure based on Taylor series approximation.

III. Computation Method for Balanced Truncation

A. Hamilton-Jacobi equations

First of all, we need to obtain Hamilton-Jacobi equations (4) and (5) in order to compute the reduced order model for a given plant. Since it is difficult to obtain closed solutions for those partial differential equations in general, we need to employ a method proposed in [11] to obtain approximate solutions based on Taylor series expansions. To this end, we adopt the following assumption.

Assumption A2 The functions $f(x)$, $g(x)$ and $h(x)$ are analytic.

Assumption A2 implies that the functions $f(x)$, $g(x)$ and $h(x)$ have a Taylor series expansion

$$\begin{aligned} f(x) &= Ax + f^h(x)\ , \quad g(x) = B + g^h(x) \\ h(x) &= Cx + h^h(x) \end{aligned}$$

where $f^h(x)$, $g^h(x)$ and $h^h(x)$ describe the higher order terms. The controllability and observability functions can be described in a similar way

$$L_c(x) = \frac{1}{2} x^{\mathrm{T}} P^{-1} x + L_c^h(x) \qquad (14)$$

$$L_o(x) = \frac{1}{2} x^{\mathrm{T}} Q x + L_o^h(x). \qquad (15)$$

Here P and Q are the controllability and observability Gramians for the Jacobian linearization (A, B, C) of Σ and those matrices can be obtained by Lyapunov equations

$$QA + A^{\mathrm{T}} Q + C^{\mathrm{T}} C = 0 \qquad (16)$$

$$PA^{\mathrm{T}} + AP + B\ B^{\mathrm{T}} = 0 \qquad (17)$$

(or Riccati equations) related to the Hamilton-Jacobi equations (4) and (5). $L_c^h(x)$ and $L_o^h(x)$ describe the higher order terms. The coefficients of the higher terms L_c^h and L_o^h can be determined recursively by substituting (14) and (15) for the Hamilton-Jacobi equations (4) and (5). Please see [11] for details.

B. Balancing coordinate transformation

Here a procedure to compute the coordinate transformation $x = \Phi(z)$ yielding the input-normal realization in Theorem 2 is proposed. Once it is obtained the balanced realization in Corollary 1 can be obtained straightforwardly by (10) as in its proof.

$$L_c(\Phi(z)) = \frac{1}{2} z^{\mathrm{T}} z\ , \quad L_o(\Phi(z)) = \frac{1}{2} \sum_{i=1}^{n} z_i^2 \sigma(z_i)^2 \qquad (18)$$

It is noticed that although there exists a C^∞ coordinate transformation $x = \Phi(z)$ in Theorem 2, the existence of an analytic one is not known. Further, its uniqueness is not examined either. Here we assume that there exists an analytic coordinate transformation and describe it as follows

$$x = \Phi(z) = Tz + \Phi^h(z) \qquad (19)$$

where T is a nonsingular matrix and $\Phi^h(x)$ denotes the higher order terms. Substitute this equation for (18) we obtain

$$\begin{aligned} L_c(\Phi(z)) &= \frac{1}{2} z^{\mathrm{T}} T^{\mathrm{T}} P^{-1} T z + \frac{1}{2} z^{\mathrm{T}} T^{\mathrm{T}} P^{-1} \Phi^h(z) \\ &\quad + \frac{1}{2} \Phi(z)^{h\mathrm{T}} P^{-1} \Phi(z) + L_c^h(\Phi(z)) \qquad (20) \\ L_o(\Phi(z)) &= \frac{1}{2} z^{\mathrm{T}} T^{\mathrm{T}} Q T z + \frac{1}{2} z^{\mathrm{T}} T^{\mathrm{T}} Q\ \Phi^h(z) \\ &\quad + \frac{1}{2} \Phi(z)^{h\mathrm{T}} Q\ \Phi(z) + L_o^h(\Phi(z)). \qquad (21) \end{aligned}$$

Let us also describe σ using Taylor series expansion

$$\sigma_i(z_i) = \sigma_i(0) + \sigma_i^h(z_i). \qquad (22)$$

Substitute this equation for the input-normal realization (6) and (7), and compare it with the expanded form (20) and (21). Then we obtain the following relationship.

$$T^{-1} P Q T = \mathrm{diag}(\sigma_1(0)^2, \sigma_2(0)^2, \ldots, \sigma_n(0)^2)$$

This means that the 1st order terms of the coordinate transformation described by T coincide with the similarity transformation for the input-normal (output-diagonal) realization of the Jacobian linearization (A, B, C) of Σ. The higher order terms have to satisfy

$$z^{\mathrm{T}} T^{\mathrm{T}} P\ \Phi^h(z) + \Phi(z)^{h\mathrm{T}} S\ \Phi(z) + 2 L_c^h(\Phi(z)) = 0 \qquad (23)$$

$$z^{\mathrm{T}} T^{\mathrm{T}} Q\ \Phi^h(z) + \Phi(z)^{h\mathrm{T}} Q\ \Phi(z) + 2 L_o^h(\Phi(z)) = \sum_{i=1}^{n} z_i^2 \sigma_i^h(z_i)^2 \qquad (24)$$

where $S := P^{-1}$. Describe mth order terms as $(\cdot)^{(m)}$, we can write mth order terms of (23) as

$$\begin{aligned} &\left\{ z^{\mathrm{T}} T^{\mathrm{T}} P\ \Phi^h(z) + \Phi(z)^{h\mathrm{T}} S\ \Phi(z) + 2 L_c^h(\Phi(z)) \right\}^{(m)} \\ &= z^{\mathrm{T}} T^{\mathrm{T}} P\ \Phi^h(z)^{(m-1)} + \left\{ \Phi(z)^{h\mathrm{T}} S\ \Phi(z) \right\}^{(m)} \\ &\quad + 2 \left\{ L_c^h(\Phi(z)) \right\}^{(m)} = 0. \qquad (25) \end{aligned}$$

It is clear for equations with order m that first,second and third terms consist of at most $(m-1)$th, $(m-1)$th and $(m-2)$th order terms of $\Phi(z)$ respectively. Similar structure is found in (24). Thus, by calculating each coefficient of the Taylor series terms of the above equations, we can obtain a series of equations which can be solved recursively as in the Hamilton-Jacobi equations case. Furthermore, obtained series of equations are linear in the unknown parameters. However, since $\sigma_i(\cdot)$'s are still unknown functions, equations which contain σ_i's must be excluded.

Here the number of unknown parameters, which are coefficients of mth order terms of the expanded $\Phi(z)$, N^c is given by

$$N^c = \frac{(n+m-1)!n}{(n-1)!m!}. \qquad (26)$$

Since the equations to determine them consist of coefficients of $(m+1)$th order terms of (23) and (24), by excluding equations containing σ_i's, we can obtain the number of them N^e as

$$N^e = \frac{2(n+m)!}{(n-1)!(m+1)!} - n. \qquad (27)$$

In fact, these number parameters satisfy

$$N^c \geq N^e\ , \quad (n \geq 2,\ m \geq 2) \qquad (28)$$

where the equation holds for only $n = 2$. This means that this procedure is always executable and the coordinate transformation can be approximated by a polynomial. Then the transformation converting the input-normal form (6) and (7) into the balanced realization (8) and (9) can be easily executed by (10).

C. Procedure

We can summarize the whole procedure to obtain the reduced order model as follows.

1) Compute the controllability and observability Gramians P and Q of the Jacobian linearization (A, B, C) of Σ by solving (17) and (16).
2) Compute a similarity transformation T to diagonalize PQ.
3) Compute the higher order terms of the controllability and observability functions L_c and L_o by solving the Hamilton-Jacobi equations (4) and (5).
4) Compute the higher order terms of the coordinate transformation for the input-normal form in (6) and (7).
5) Compute the balanced realization by the coordinate transformation obtained above, and decide the order k of the reduced order model.
6) Compute the reduced order model by truncation as in (11).

IV. Numerical Example

A. Description of the plant

In this section, we apply the proposed computation procedure to a double pendulum depicted in Figure 1.

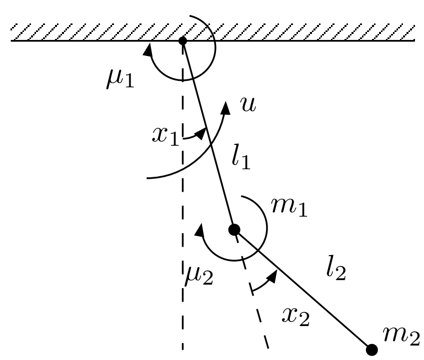

Fig. 1. The double pendulum

Here m_i denotes the mass located at the end of the i-th link, l_i denotes the length of the i-th link, μ_i denotes the friction coefficient of the i-th link, and x_i denotes the angle of the i-th link. We select the physical parameters as $l_1 = l_2 = 1$, $m_1 = m_2 = 1$, $\mu_1 = \mu_2 = 1$, $g = 9.8$ with g the gravity coefficient. The dynamics of this apparatus can be described by an input-affine nonlinear system model (1) with 4 dimensional state-space

$$x = (x_1, x_2, x_3, x_4) := (x_1, x_2, \dot{x}_1, \dot{x}_2).$$

The input u denotes the torque applied to the first link at the fist joint and the output y denotes the horizontal and the vertical coordinates of the position of the mass m_2. The potential energy $P(x)$ and the kinetic energy $K(x)$ for this system are described by

$$P(x) = -m_1 g l_1 \cos x_1 - m_2 g l_1 \cos x_1 - m_2 g l_2 \cos(x_1 + x_2)$$
$$K(x) = (\dot{x}_1, \dot{x}_2) M(x) \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix}$$
$$M(x) =$$
$$\begin{pmatrix} m_1 l_1^2 + m_2 l_1^2 + m_2 l_2^2 + 2 m_2 l_1 l_2 \cos x_2 & m_2 l_2^2 + m_2 l_1 l_2 \cos x_2 \\ m_2 l_2^2 + m_2 l_1 l_2 \cos x_2 & m_2 l_2^2 \end{pmatrix}$$

where $M(x)$ denotes the inertia matrix. Then the dynamics of this apparatus is obtained by the Lagrange's method as follows

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L(x)}{\partial(\dot{x}_1, \dot{x}_2)}^{\mathrm{T}} - \frac{\partial L(x)}{\partial(x_1, x_2)}^{\mathrm{T}} = \begin{pmatrix} u - \mu_1 \dot{x}_1 \\ -\mu_2 \dot{x}_2 \end{pmatrix} \tag{29}$$

with the Lagrangian $L(x) := K(x) - P(x)$. This equation reduces to the system (1) with

$$f(x) = \begin{pmatrix} x_3 \\ x_4 \\ M^{-1} \left(\frac{\partial(K-P)}{\partial(x_1,x_2)}^{\mathrm{T}} - \dot{M} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} - \begin{pmatrix} \mu_1 \dot{x}_1 \\ \mu_2 \dot{x}_2 \end{pmatrix} \right) \end{pmatrix}$$
$$g(x) = \begin{pmatrix} 0 \\ 0 \\ M^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix}.$$

See [13] for the detail of the model.

Applying the proposed procedure and computing the coordinate transformation up to 4-th order terms of Taylor series expansion we can obtain the following Hankel singular value functions.

$$\begin{aligned} \sigma_1(z_1)^2 &= 1.98 \times 10^{-1} + 4.14 \times 10^{-4} z_1^2 + o(|z_1|^3) \\ \sigma_2(z_2)^2 &= 1.72 \times 10^{-1} + 3.28 \times 10^{-4} z_2^2 + o(|z_2|^3) \\ \sigma_3(z_3)^2 &= 5.83 \times 10^{-5} + 1.51 \times 10^{-4} z_3^2 + o(|z_3|^3) \\ \sigma_4(z_4)^2 &= 9.37 \times 10^{-6} + 9.22 \times 10^{-6} z_4^2 + o(|z_4|^3) \end{aligned}$$

Figure 2 plots the Hankel singular value functions $\sigma_i(s)$'s. As noted in the figure, the solid (blue) line denotes $\sigma_1(s)$,

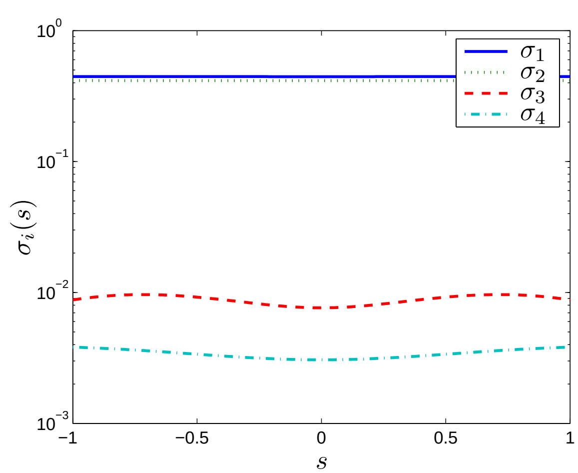

Fig. 2. Hankel singular value functions

the dotted (green) one denotes $\sigma_2(s)$, the dashed (red) one denotes $\sigma_3(s)$ and the dashed and dotted (cyan) one denotes $\sigma_4(s)$, respectively. This figure shows that, in this region, the values of $\sigma_1(s)$ and $\sigma_2(s)$ are much larger than those of $\sigma_3(s)$ and $\sigma_4(s)$. Therefore, we can see that an appropriate dimension of the reduced order model is 2. Once we determine the reduced order, we can obtain obtain the reduced order model directly by applying the procedure proposed in the previous section.

B. Simulations

We have executed some simulations of the original and reduce models to evaluate the effectiveness of the proposed algorithm. Here the time responses for impulsive inputs are depicted in the figures. Figure 3 describes the time response of the horizontal movement and Figure 4 describes that of the vertical one. In the figure, the solid (blue) line denotes the response of the original system, the dashed (red) one denotes that of the linearized reduced order model, and the dashed and dotted (green) one denotes that of the nonlinear reduced order model.

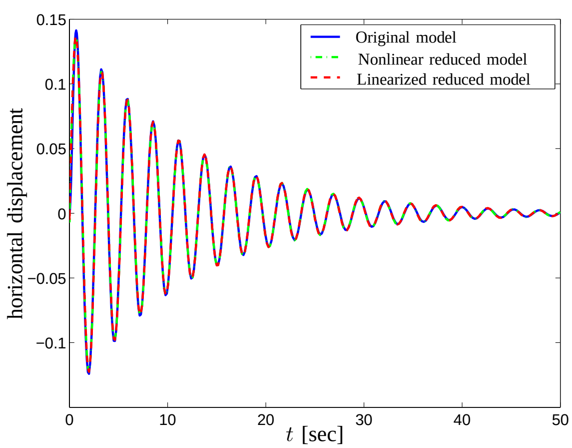

Fig. 3. The horizontal displacement with impulsive input

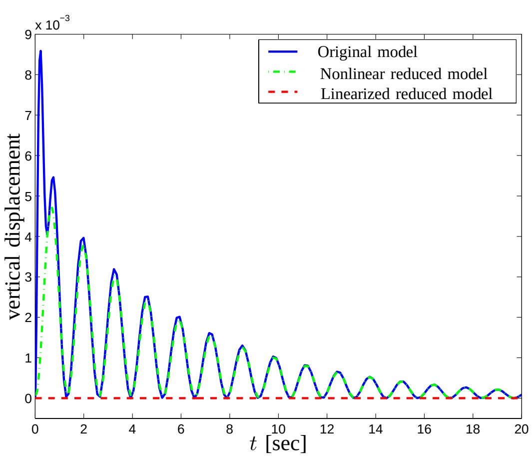

Fig. 4. The vertical displacement with impulsive input

In Figure 3, all trajectories are identical which indicates that both linear and nonlinear reduced order models can approximate the behavior of the original model well. However, in Figure 4, one can observe that the trajectory of the linear reduced order model is quite different from the original whereas the trajectory of the nonlinear reduced order model is almost identical with that of the original system. This is due to the fact that the linearization of the vertical displacement of the mass m_2 is 0 since it consists of a cosine function of the state. These simulations demonstrate the effectiveness of the nonlinear balanced truncation.

We also show the time responses for sinusoidal input $u(t) = \sin 2.5t$. The frequency of this input is close to the natural frequency of the Jacobian linearization of the original model. As in the previous case, Figure 5 describes the times response of the horizontal movement and Figure 6 describes that of the vertical one. In the figure, the denotation of each line is the same as in the impulsive case. Because of the frequency of the input, resonance occurs. We can see that the nonlinear reduced order model can represent this phenomenon in both directions. though the linearized reduced order model can not do in the vertical direction. Those results also exhibits the advantage of the proposed method.

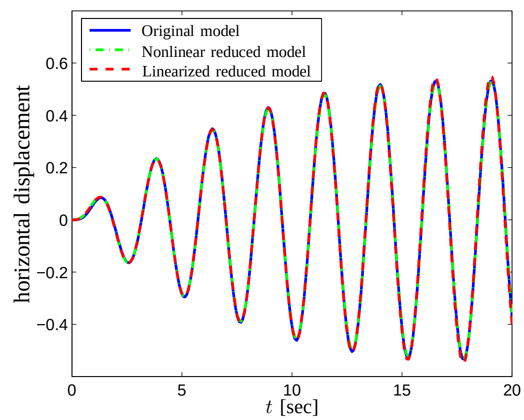

Fig. 5. The horizontal displacement with sinusoidal input

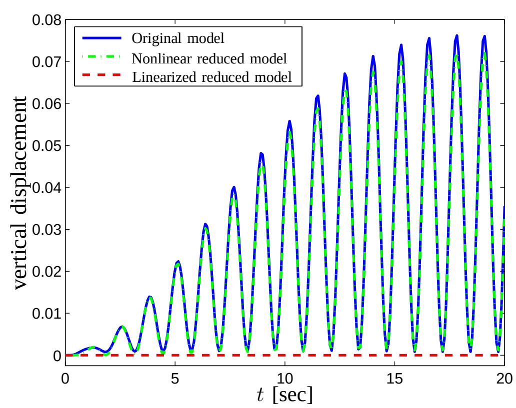

Fig. 6. The vertical displacement with sinusoidal input

On the other hand, this simulation shows a rather small movement with respect to the angle x_1 of the first link. Trajectory of the nonlinear reduced order model for a much bigger movement with respect to x_1 does not fit the trajectory of the original system well. This is because the approximation order (4 in this case) is too small for such a movement and we need to obtain a higher order approximation in this case, although we need more computational effort for higher order approximations.

V. Conclusion

In this paper, a computational algorithm for nonlinear balanced truncation based on Taylor series expansion is proposed. We have shown that the coordinate transformations to obtain the balanced realization can be computed, if the state space realization (f, g, h) is analytic. This algorithm requires a recursive computation with respect to the order in which we need to solve linear equations for unknown parameters in each step. Furthermore, the proposed method is applied to a double pendulum system. Some numerical simulations have demonstrated the effectiveness of the proposed algorithm which cannot be achieved by linear balanced truncation.

Appendix

This appendix proves inequality (28) which is one of necessary conditions to compute the coefficients of Taylor series expansion of a coordinate transformation $x = \Phi(z)$.

This proof is divided into following three cases.

(a) Case $n = 2$
(b) Case $m = 2$
(c) Case $m \geq 3,\ n \geq 3$

(a). Substitute $n = 1$ for (26) and (27). Then we obtain

$$\begin{aligned} N^c &= \frac{2(m+1)!}{m!} = 2(m+1), \\ N^e &= \frac{2(m+2)!}{(m+1)!} - 2 \\ &= 2(m+2) - 2 = 2(m+1). \end{aligned}$$

Thus $N^c = N^e$.

(b). Similarly, by substituting $m = 2$ for (26) and (27), N^c and N^e are as follows

$$\begin{aligned} N^c &= \frac{(n+1)!n}{2(n-1)!} = \frac{1}{2}n^2(n+1) \\ N^e &= \frac{2(n+2)!}{(n-1)!\ 3!} - n = \frac{1}{3}n(n^2+3n-1). \end{aligned}$$

Difference between N^c and N^e is

$$N^c - N^e = \frac{n}{6}(n-2)(n-1).$$

Since the right hand side is positive for all $n > 2$, the inequality $N^c > N^e$ is shown.

(c). Finally, consider the case $n > 3$ and $m > 3$. By calculating difference between N^c and N^e, we obtain

$$\begin{aligned} N^c - N^e &= \frac{(n+m-1)!n}{(n-1)!m!} - \frac{2(n+m)!}{(n-1)!(m+1)!} + n \\ &= \frac{(n+m-1)!(m-1)}{(n-1)!(m+1)!}\left\{n - \frac{2m}{m-1}\right\} + n. \end{aligned}$$

Since $2m/(m-1) \leq 3$ for $m \geq 3$, following inequality is shown.

$$n - \frac{2m}{m-1} \geq 0,\ (n \geq 3).$$

Thus the difference is positive, that is $N^c - N^e$ in this case.

It follows from those three cases that $N^c \geq N^e$ for $n \geq 2$ and $m \geq 2$. This completes the proof.

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