Balanced realization and model reduction of port-Hamiltonian systems

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***Abstract*— This paper is concerned with nonlinear model reduction for electro-mechanical systems described by port-Hamiltonian formulae. A novel weighted balacend realization and model reduction procedure is proposed which preserves port-Hamiltonian structure as well as stability, reachability and observability of the original system. This implies that one can utilize the intrinsic physical properties such as physical energy and the corresponding dissipativity for the reduced order model. Further, the proposed method reduces the computational effort in solving partial differential equations for nonlinear balanced realization. A numerical simulation shows how the proposed method works.**

I. INTRODUCTION

Model order reduction and related techniques have been an attractive topic for many researchers working on different subjects both in engineering and in mathematics, see e.g. [12], [19]. There are several different approaches: Krylov subspace method [19], balanced truncation [9], proper orthogonal decomposition and so on [7]. As for constructing their nonlinear extension, balanced realization approach seems most promising among them. Since nonlinear balanced realization was introduced in [14], many results related to this topic were reported such as balanced truncation for unstable nonlinear systems [16], computational issues [11], [6], minimality consideration [15], global balancing [20], balanced realization based on nonlinear singular value analysis [1], [3], and (frequency) weighted balanced truncation [17].

On the other hand, the class of electro-mechanical systems is one of the most important class of nonlinear systems which can be controlled effectively. There are many control strategies developed for this class of systems. For instance, the port-Hamiltonian modeling and the control techniques for this class of systems are developed by many authors, e.g. [18], [4]. Therefore it is quite a natural requirement to preserve the Hamiltonian structure when we perform model order reduction, since we can utilize the intrinsic physical properties such as energy dissipativity of the original system for the reduced order model.

This paper is devoted to this problem, that is, to obtain a balanced realization algorithm preserving the port-Hamiltonian systems structure. As preliminary results, the authors have proposed a balanced truncation using storage functions [8] where we mainly concentrate on the required and available storage functions of port-Hamiltonian systems. There is another paper pointing out, from the computational point of view, that the controllability function coincides with the Hamiltonian function in a special case [10]. The present paper extends these ideas and show that the weighted controllability and observability functions [17] coincide with the Hamiltonian function under a certain assumption. Another investigation proves that if either the controllability or observability function is identical to the Hamiltonian function, then the Hamiltonian structure is preserved under the corresponding balanced truncation. This implies that the reduced order model obtained by using the weighted energy functions preserves the port-Hamiltonian structure of the original model. We propose a novel weighted balanced truncation procedure for port-Hamiltonian systems based on this idea. Furthermore, a numerical simulation exhibits how the proposed algorithm works.

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II. MODEL ORDER REDUCTION FOR NONLINEAR SYSTEMS

This section refers to some preliminary results on balanced truncation, weighted balanced truncation and coprime factorizations for nonlinear systems.

A. Balanced truncation for nonlinear systems

This section refers to preliminary results on nonlinear balanced realization in [2]. Consider an input-affine, time invariant, asymptotically stable nonlinear system

$$\Sigma : \left\{ \begin{array}{rcl} \dot{x} & = & f(x) + g(x)\ u \\ y & = & h(x) \end{array} \right. \tag{1}$$

with $f(0) = 0$ where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and $y(t) \in \mathbb{R}^r$. Its controllability function $L_c(x)$ and observability function $L_o(x)$ are defined by

$$L_c(\xi) := \inf_{\substack{u \in L_2^- \\ x(-\infty)=0, x(0)=\xi}} \frac{1}{2}\|u\|^2_{L_2^m} \tag{2}$$

$$L_o(\xi) := \frac{1}{2}\|y\|_{L_2^r},\ x(0) = \xi,\ u(t) \equiv 0. \tag{3}$$

In the linear case,

$$L_c(x) = \frac{1}{2}x^{\mathrm{T}}P^{-1}x,\ L_o(x) = \frac{1}{2}x^{\mathrm{T}}Qx$$

hold with the controllability and observability Gramians P and Q. The functions $L_c(x)$ and $L_o(x)$ fulfill the following Hamilton-Jacobi equations

$$\frac{\partial L_c(x)}{\partial x}f(x) + \frac{1}{2}\frac{\partial L_c(x)}{g}(x)g(x)^{\mathrm{T}}\frac{\partial L_c}{\partial x}^{\mathrm{T}} = 0 \tag{4}$$

$$\frac{\partial L_o(x)}{\partial x}f(x) + \frac{1}{2}h(x)^{\mathrm{T}}h(x) = 0 \tag{5}$$

where $x = 0$ of $\dot{x} = -f - gg^{\mathrm{T}}(\partial L_c(x)/\partial x)^{\mathrm{T}}$ is is locally asymptotically stable.

Here we review nonlinear balanced realization.

Theorem 1: [2] Suppose that $L_c(x)$ and $L_o(x)$ exist and that Hankel singular values of the Jacobian linearization of Σ are nonzero and distinct. Then there exist a neighborhood X of the origin and a coordinate transformation $x = \Phi(z)$ on X converting the system into the following form

$$\begin{aligned} L_c(\Phi(z)) &= \frac{1}{2}\sum_{i=1}^{n} \frac{z_i^2}{\sigma_i(z_i)} \\ L_o(\Phi(z)) &= \frac{1}{2}\sum_{i=1}^{n} z_i^2 \sigma_i(z_i) \end{aligned}$$

with $\min\{\sigma_i(s), \sigma_i(-s)\} > \max\{\sigma_{i+1}(s), \sigma_{i+1}(-s)\}$.
Here $\sigma_i(z_i)$'s are Hankel singular value functions of Σ. This realization is called nonlinear balanced realization which has the following properties.

$$z_i = 0 \;\Leftrightarrow\; \frac{\partial L_c(\Phi(z))}{\partial z_i} = 0 \;\Leftrightarrow\; \frac{\partial L_o(\Phi(z))}{\partial z_i} = 0 \quad (6)$$

$$\sigma_i(z_i)^2 = \frac{L_o(\Phi(0,\dots,0,z_i,0,\dots,0))}{L_c(\Phi(0,\dots,0,z_i,0,\dots,0))}$$

The Hankel singular value functions $\sigma_i(z_i)$'s represent the importance of the state variables z_i's with respect to the input-output behavior of the system. Therefore we can obtain a reduced order model by truncating important states. This technique is called balanced truncation. If $\min\{\sigma_k(s), \sigma_k(-s)\} \gg \max\{\sigma_{k+1}(s), \sigma_{k+1}(-s)\}$ holds, then the coordinates to be truncated is $z^{\mathrm{r}} := (z_1, \dots, z_k)$. Let Σ^{r} denote a reduced order model with the state z^{r}.

Theorem 2: [2] The controllability and observability functions $L_c^{\mathrm{r}}(z^{\mathrm{r}})$ and $L_o^{\mathrm{r}}(z^{\mathrm{r}})$ of Σ^{r} satisfy

$$\begin{aligned} L_c^{\mathrm{r}}(z^{\mathrm{r}}) &= L_c(\Phi(z^{\mathrm{r}},0)) \\ L_o^{\mathrm{r}}(z^{\mathrm{r}}) &= L_o(\Phi(z^{\mathrm{r}},0)). \end{aligned}$$

Furthermore, the Hankel singular values $\sigma_i^{\mathrm{r}}(z_i^{\mathrm{r}})$ of Σ^{r} satisfy

$$\sigma_i^{\mathrm{r}}(z_i^{\mathrm{r}}) = \sigma_i(z_i), \quad i = 1, \dots, k.$$

These theorems imply that several important properties of Σ are preserved such as controllability, observability and Lyapunov stability. Local asymptotic stability is also preserved.

III. Balanced realization of port-Hamiltonian systems

Here we show that a weighted controllability and observability functions of port-Hamiltonian systems can be obtained without solving the partial differential equations (4) and (5). Let us consider a port-Hamiltonian system

$$\Sigma : \begin{cases} \dot{x} = (J(x) - R(x))\frac{\partial H(x)}{\partial x}^{\mathrm{T}} + g(x)\; u \\ y = h(x) \end{cases} \quad (7)$$

Here the matrix $J(x) = -J(x)^{\mathrm{T}}$ is skew-symmetric, and $R(x) = R(x)^{\mathrm{T}} \geq 0$ is symmetric and positive semi-definite. $J(x)$ represents the conservative property of the system and $R(x)$ describes the dissipative elements such as friction of mechanical systems or registers in electric circuits. This system can describe many physical systems such as electric circuits, mechanical systems with constraints and a class of distributed parameter systems as well as simple mechanical systems. There are many results on controlling this class of system, see e.g. [18], [13], [4], [5]. Therefore it is quite natural to investigate how to preserve the port-Hamiltonian structure in reducing the order of the model.

The controllability function $L_c(x)$ is defined by Equation (2). Instead of using this functions, here we define a weighted controllability function as follows

$$L_c^w(\xi) := \inf_{\substack{u \in L_2^- \\ x(-\infty)=0, x(0)=\xi}} \int_{-\infty}^{0} u(t)^{\mathrm{T}} D_c(x)\; u(t)\; \mathrm{d}t \quad (8)$$

with a symmetric positive definite matrix $D_c(x)$. Then we can prove the following property.

Lemma 1: Consider the port-Hamiltonian system Σ in Equation (7) with the weighted controllability function $L_c^w(x)$ defined in Equation (8). Suppose that the system satisfies the assumptions in Theorem 1 and that the matrix $D_c(x)$ satisfies

$$g(x)\; D_c(x)\; g(x)^{\mathrm{T}} = R(x).$$

Then the following relationship holds.

$$L_c^w(x) = H(x). \quad (9)$$

Proof: As in the non-weighted case, the weighted controllability function $L_c^w(x)$ satisfies the following Hamilton-Jacobi equation

$$\begin{aligned} 0 &= \frac{\partial L_c^w(x)}{\partial x}(J(x) - R(x))\frac{\partial H(x)}{\partial x}^{\mathrm{T}} \\ &\quad + \frac{1}{2}\frac{\partial L_c^w(x)}{\partial x} g(x)\; 2D_c(x)\; g(x)^{\mathrm{T}} \frac{\partial L_c^w(x)}{\partial x}^{\mathrm{T}} \\ &= \frac{\partial L_c^w(x)}{\partial x}(J(x) - R(x))\frac{\partial H(x)}{\partial x}^{\mathrm{T}} \\ &\quad + \frac{\partial L_c^w(x)}{\partial x} R(x) \frac{\partial L_c^w(x)}{\partial x}^{\mathrm{T}} \\ &= \frac{\partial H(x)}{\partial x}(J(x) - R(x))\frac{\partial (H(x) - L_c^w(x))}{\partial x}^{\mathrm{T}}. \end{aligned}$$

This equation holds if the relationship (9) holds. Due to the uniqueness of the stabilizing solution of the Hamilton-Jacobi equation, we can prove that this is the unique solution which completes the proof. ■

This lemma implies that the Hamiltonian function $H(x)$ can be regarded as a weighted controllability function of the port-Hamiltonian system Σ if

$$R(x) \in \mathrm{Im}\; g(x). \quad (10)$$

Therefore, if we can replace the controllability function $L_c(x)$ with the weighted one $L_c^w(x)$, then the balanced realization can be obtained without solving the Hamilton-Jacobi equation (4).

The dual result with respect to the observability function can be obtained as follows. Let us consider the port-Hamiltonian system (7) with the following artificial output function

$$y = g_0(x)^{\mathrm{T}} \frac{\partial H(x)}{\partial x}^{\mathrm{T}} \tag{11}$$

which coincides with the passive output if $g_o(x) = g(x)$ holds. Furthermore, we define a weighted observability function $L_o^w(x)$ as follows.

$$L_o^w(\xi) := \int_0^\infty y(t)^{\mathrm{T}} D_o(x)\ y(t)\ \mathrm{d}t,\ \ x(0)=\xi,\ \ u(t)\equiv 0. \tag{12}$$

Then we can prove the following lemma.

Lemma 2: Consider the port-Hamiltonian system Σ in Equation (7) with the output function (11) and the weighted observability function $L_o^w(x)$ defined in Equation (12). Suppose that the system satisfies the assumptions in Theorem 1 and that the matrix $D_o(x)$ satisfies

$$g_o(x)\ D_o(x)\ g_o(x)^{\mathrm{T}} = R(x).$$

Then the following relationship holds.

$$L_o^w(x) = H(x). \tag{13}$$

Proof: As in the non-weighted case, the weighted controllability function $L_o^w(x)$ satisfies the following Lyapunov equation

$$\begin{aligned} 0 &= \frac{\partial L_o^w(x)}{\partial x}(J(x)-R(x))\frac{\partial H(x)}{\partial x}^{\mathrm{T}} \\ &\quad + \frac{1}{2}\frac{\partial H(x)}{\partial x} g_o(x)\ 2D_o(x)\ g_o(x)^{\mathrm{T}}\frac{\partial H(x)}{\partial x}^{\mathrm{T}} \\ &= \frac{\partial L_c^w(x)}{\partial x}(J(x)-R(x))\frac{\partial H(x)}{\partial x}^{\mathrm{T}} \\ &\quad + \frac{\partial H(x)}{\partial x}R(x)\frac{\partial H(x)}{\partial x}^{\mathrm{T}} \\ &= \frac{\partial (L_c^w(x)-H(x))}{\partial x}(J(x)-R(x))\frac{\partial H(x)}{\partial x}^{\mathrm{T}}. \end{aligned}$$

This equation holds if the relationship (13) holds. Due to the uniqueness of the stabilizing solution of the Lyapunov equation, we can prove that this is the unique solution which completes the proof. ■

As in the controllability case, this lemma implies that the Hamiltonian function $H(x)$ can be regarded as a weighted observability function of the port-Hamiltonian system Σ if

$$R(x) \in \mathrm{Im}\ g_o(x). \tag{14}$$

The simplest choice of $g_o(x)$ is

$$g_o(x) := R(x)^{\frac{1}{2}}$$

with $D_o(x) = I$.

Thus the Hamiltonian function $H(x)$ can be regarded as a weighted observability function of the port-Hamiltonian system Σ. The weighted observability function can be obtained without solving the Lyapunov equation (5). Also, it is noted that to use both lemmas at once is useless, that is, balancing with the weighted controllability and observability functions $L_c^w(x)$ and $L_o^w(x)$ as defined in Equations (8) and (12), since any state-realization is balanced with respect to these two identical functions.

IV. Model order reduction

This section investigates the model order reduction based on the balanced realization using either the weighted controllability function $L_c^w(x)$ or the weighted observability function $L_o^w(x)$.

In the previous section, if the condition (10) or (14) holds, then there exists either a weighted controllability function $L_c^w(x)$ or a weighted observability function $L_o^w(x)$ that coincides with the Hamiltonian function $H(x)$.

If one of the energy functions is a Hamiltonian function $H(x)$, then the Hamiltonian structure is preserved under the corresponding model order reduction.

Theorem 3: Consider the port-Hamiltonian system Σ in Equation (7). Suppose that either the controllability function or the observability function coincides with the Hamiltonian function $H(x)$. Then the corresponding reduced order model via balanced truncation is a port-Hamiltonian system.

Proof: Suppose that the port-Hamiltonian system Σ is balanced with respect to the Hamiltonian function $H(x)$. Note that the balanced system is described by a port-Hamiltonian system because the port-Hamiltonian structure is invariant under any coordinate transformation [4]. Suppose moreover that the corresponding singular value functions $\sigma_i(\cdot)$'s satisfy

$$\min\{\sigma_k(s), \sigma_k(-s)\} \gg \max\{\sigma_{k+1}(s), \sigma_{k+1}(-s)\}$$

with a certain k and truncate a reduced order model with the dimension k. Dividing the coordinate $x = (x^r, x^b)$ we obtain a system described by

$$\begin{pmatrix} \dot{x}^r \\ \dot{x}^b \end{pmatrix} = \begin{pmatrix} J_{11}(x) & J_{12}(x) \\ J_{21}(x) & J_{22}(x) \end{pmatrix} \begin{pmatrix} \frac{\partial H(x)}{\partial x^r}^{\mathrm{T}} \\ \frac{\partial H(x)}{\partial x^b}^{\mathrm{T}} \end{pmatrix} + \begin{pmatrix} g_1(x) \\ g_2(x) \end{pmatrix} u$$

Therefor the balanced truncation procedure given in Theorem 2 yields the following reduced order model.

$$\dot{x}^r = J_{11}(x^r,0)\frac{\partial H(x^r,0)}{\partial x^r}^{\mathrm{T}} + g_1(x^r,0)\ u$$

since

$$\left.\frac{\partial H(x^r,x^b)}{\partial x^b}^{\mathrm{T}}\right|_{x^b=0} = 0.$$

Then the obtained reduced order model is a port-Hamiltonian system because the matrix $J_{11}(x^r,0)$ has to be negative semi-definite. This completes the proof. ■

Corollary 1: Consider the port-Hamiltonian system Σ in Equation (7). Suppose that the condition (10) or (14) holds. Then there exists a weighted controllability function $L_c^w(x)$ or a weighted observability function $L_o^w(x)$ such that the reduced order model via the corresponding balanced truncation is a port-Hamiltonian system.

Proof: Application of Lemma 1 and Lemma 2 to Theorem 3 directly implies the corollary. ■

In order to obtain a reduced order model in a port-Hamiltonian form, we should adopt the following procedures.

If the original port-Hamiltonian system satisfies the condition (10), then it is natural to use the weighted controllability function $L_c^w(x)$ instead of the original one $L_c(x)$, since Lemma 1 implies that $L_c^w(x)$ can be obtained without solving the Hamilton-Jacobi equation (4). In this case the output function $y = h(x)$ can be arbitrarily chosen according to the control objective.

If the system Σ does not satisfy the condition (10), then we should adopt the Hamiltonian function H as an approximation of the weighted controllability function $L_c^w(x)$, or as an weighted observability function $L_o^w(x)$ with the artificial output function given in Equation (11) satisfying the condition (14).

V. Numerical example

In this section, we apply the proposed model order reduction procedure to a double pendulum system depicted in Figure 1.

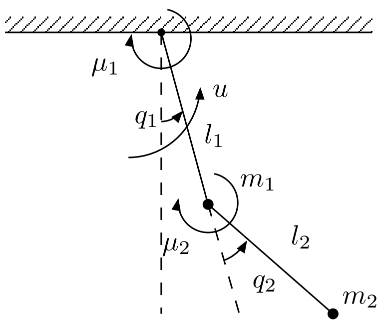

Fig. 1. The double pendulum

Here m_i denotes the mass located at the end of the i-th link, l_i denotes the length of the i-th link, μ_i denotes the friction coefficient of the i-th link, and x_i denotes the angle of the i-th link. We select the physical parameters as $l_1 = l_2 = 1$, $m_1 = m_2 = 1$, $\mu_1 10$, $\mu_2 = 1$, $g_0 = 9.8$ with g_0 the gravity coefficient. The configuration state is given by $q = (q_1, q_2)$ where q_1 and q_2 denote the angle of the links 1 and 2. The input u denotes the torque applied to the first link at the first joint and the output y denotes the horizontal and the vertical coordinates of the position of the mass m_2. The potential energy $P(q)$ and the kinetic energy $K(q, \dot{q})$ for this system are described by

$$P(q) = -m_1 g_0 l_1 \cos q_1 - m_2 g_0 l_1 \cos q_1 - m_2 g_0 l_2 \cos(q_1 + q_2)$$
$$K(q, \dot{q}) = \dot{q}^{\mathrm{T}} M(q)\ \dot{q}$$
$$M(q) =$$
$$\begin{pmatrix} m_1 l_1^2 + m_2 l_1^2 + m_2 l_2^2 + 2 m_2 l_1 l_2 \cos q_2 & m_2 l_2^2 + m_2 l_1 l_2 \cos q_2 \\ m_2 l_2^2 + m_2 l_1 l_2 \cos q_2 & m_2 l_2^2 \end{pmatrix}$$

where $M(q)$ denotes the inertia matrix. The generalized momentum can be defined by $p := M(q)\dot{q}$. Then the dynamics of this apparatus can be described by a port-Hamiltonian system model (7) with

$$\begin{aligned} H(q, p) &= P(q) + K(q, M(q)^{-1} p) \\ J(x) &= \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \\ R(x) &= \mathrm{diag}(0, 0, \mu_1, \mu_2) \\ g(x) &= (0, 0, 1, 0)^{\mathrm{T}} . \end{aligned}$$

For this apparatus, the condition (10) holds if and only if the friction coefficient $\mu_2 = 0$. Here we perform a numerical simulation for the case $\mu_2 \neq 0$ to observe the effectiveness of the proposed method when the Hamiltonian function $H(x)$ is adopted as an approximation of the weighted controllability function $L_c^w(x)$.

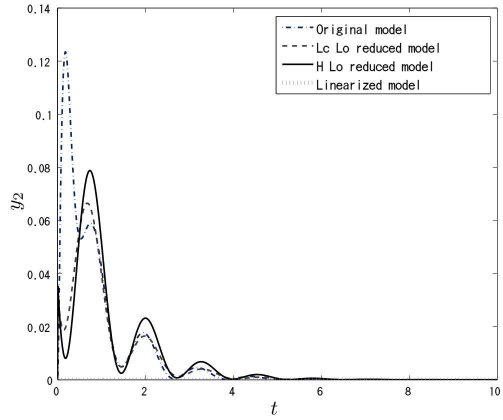

Fig. 2. Response of the vertical displacement of the mass 2

The responses for an impulsive input are depicted in Figure 2. The dashed and dotted line denotes the response of the original system, the dashed line denotes that of the reduced order model without preserving the port-Hamiltonian structure, and the solid line denotes that of the reduced order

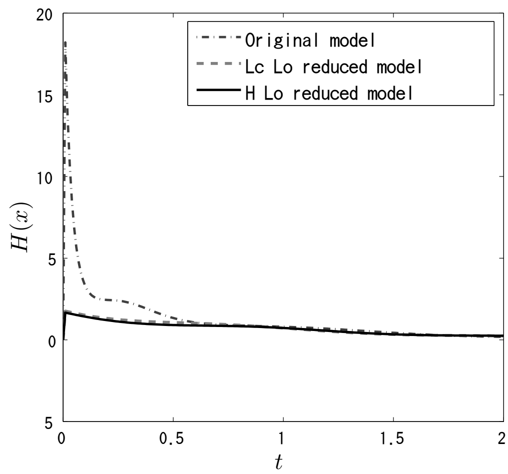

Fig. 3. Response of the Hamiltonian function

port-Hamiltonian model. Figure 3 depicts the corresponding response of the Hamiltonian function. The Hamiltonian function of the reduced order model captures the principal behavior of that of the original model.

VI. Conclusion

This paper has proposed a model order reduction procedure based on nonlinear balanced truncation which preserves the port-Hamiltonian structure of the original model. This method allows us to preserve the energy variable which will be useful in controlling the reduced order model. Also we do not need to solve a Hamilton-Jacobi equation which appears in conventional nonlinear balanced realization procedure.

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