THE BETHE EQUATION AT $q = 0$, THE MÖBIUS INVERSION FORMULA, AND WEIGHT MULTIPLICITIES: III. THE $X_N^{(r)}$ CASE

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Abstract. It is shown that the numbers of off-diagonal solutions to the $U_q(X_N^{(r)})$ Bethe equation at $q = 0$ coincide with the coefficients in the recently introduced canonical power series solution of the Q-system. Conjecturally the canonical solutions are characters of the KR (Kirillov-Reshetikhin) modules. This implies that the numbers of off-diagonal solutions agree with the weight multiplicities, which is interpreted as a formal completeness of the $U_q(X_N^{(r)})$ Bethe ansatz at $q = 0$.

1. Introduction

Enumerating the solutions to the Bethe equation began with the invention of the Bethe ansatz [Be], where Bethe himself obtained a counting formula for sl_2-invariant Heisenberg chain. His calculation is based on the string hypothesis and has been generalized to higher spins [K1], sl_n [K2] and a general classical simple Lie algebra X_n [KR]. These works concern the rational Bethe equation [OW], or in other words, $U_q(X_n^{(1)})$ Bethe equation at $q = 1$.

On the other hand, a systematic count at $q = 0$ started rather recently [KN1, KN2]. The two approaches are contrastive in many respects. To explain them, recall the general setting where integrable Hamiltonians associated with $U_q(X_n^{(1)})$ act on a finite dimensional module called the quantum space. At $q = 1$, the Hamiltonians are invariant and the Bethe vectors are singular with respect to the classical subalgebra X_n, while for $q \neq 1$, such aspects are no longer valid in general. Consequently, by completeness at $q = 1$ (resp. $q = 0$) we mean that the number of solutions to the Bethe equation coincides with the multiplicity of irreducible X_n modules (resp. weight multiplicities) in the quantum space.

In this paper we study the Bethe equation associated with the quantum affine algebra $U_q(X_N^{(r)})$ [RW] at $q = 0$. By extending the analyses of the nontwisted case [KN1, KN2], an explicit formula $R(\nu, N)$ is derived for the number of off-diagonal solutions of the string center equation. Moreover we relate the result to the Q-*system* for $U_q(X_N^{(r)})$ introduced in [KR, K3, HKOTT]. It is a (yet conjectural in general) family of character identities for the KR modules (Definition 2.1). Our main finding is that $R(\nu, N)$ is identified with the coefficients in the canonical solution of the Q-system obtained in [KNT]. Under the Kirillov-Reshetikhin conjecture [KR] (cf. Conjecture 3.4), it leads to a character formula for tensor products of KR modules, which may be viewed as a formal completeness at $q = 0$.

The outline of the paper is as follows. In Section 2 we study the $U_q(X_N^{(r)})$ Bethe equation at $q = 0$. For a generic string solution, the string centers satisfy the key equation (2.18), which we call the string center equation (SCE). There is a one-to-one correspondence between the generic string solutions to the Bethe equation and the generic solutions to the SCE (Theorem 2.10). We then enumerate the off-diagonal solutions of the SCE, and obtain the formula $R(\nu, N)$ in Theorem 2.13. In Section 3 we recall the Q-system for $U_q(X_N^{(r)})$. It corresponds to a special case (called KR-type) of a more general system considered in [KNT]. There, power series solutions are studied, and the notion of the canonical solution is introduced unifying the ideas in [K1, K2, HKOTY, KN2]. For the Q-system in question, we find that the coefficients in the canonical solution are described by $R(\nu, N)$, the number of off-diagonal solutions of the SCE obtained in Section 3 (Theorem 3.3). A consequence of this fact is stated also in the light of the Kirillov-Reshetikhin conjecture [KR, C, KNT]. We note that the canonical solution of the Q-system is expressed also as a ratio of two power series [KNT], which matches the enumeration at $q = 1$ [KR] for the nontwisted cases.

In this paper we omit most of the proofs and calculations, which are parallel with those in [KN1, KN2, KNT].

2. Bethe equation at $q = 0$

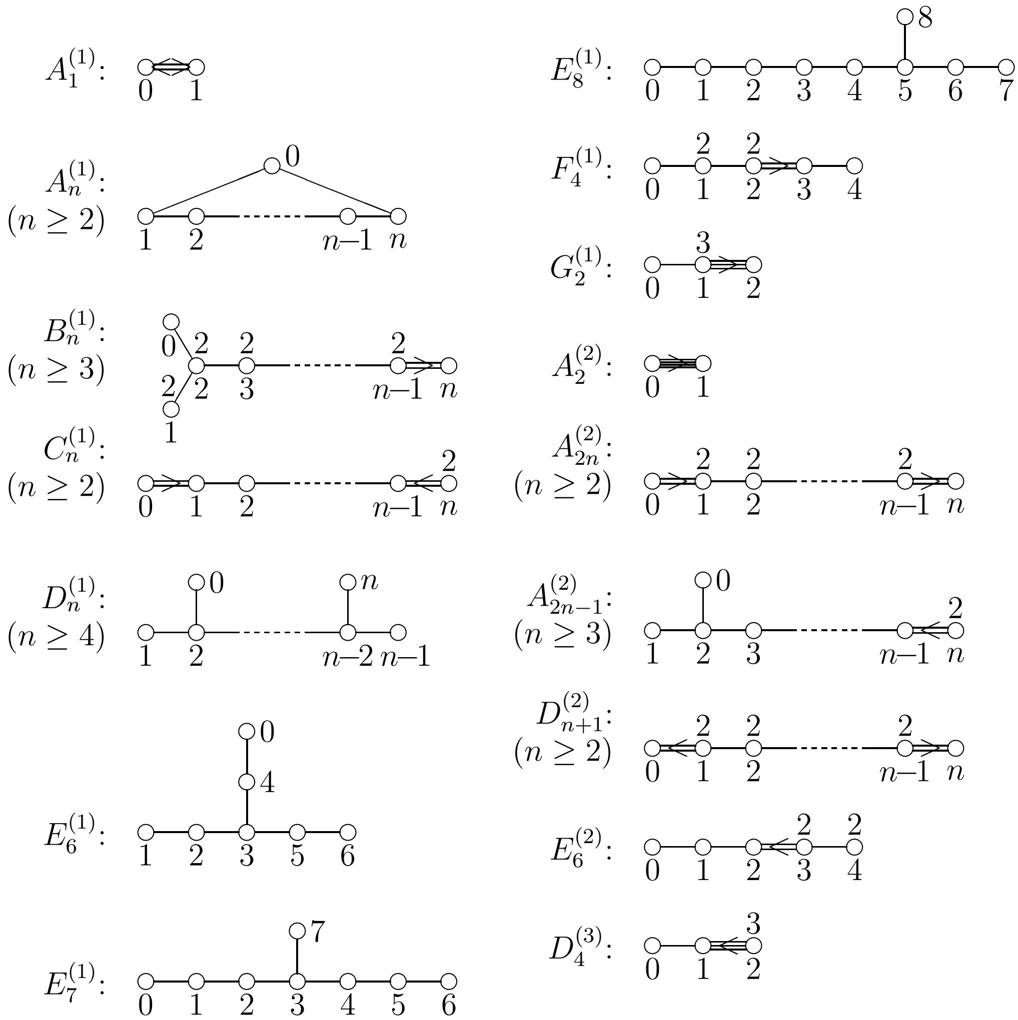
2.1. **Preliminary.** Let $\mathfrak{g} = X_N$ be a finite-dimensional complex simple Lie algebra of rank N. We fix a Dynkin diagram automorphism σ of $\mathfrak{g}$ of order $r = 1, 2, 3$. The affine Lie algebras of type $X_N^{(r)} = A_n^{(1)} (n \geq 1), B_n^{(1)} (n \geq 3), C_n^{(1)} (n \geq 2), D_n^{(1)} (n \geq 4), E_n^{(1)} (n = 6, 7, 8), F_4^{(1)}, G_2^{(1)}, A_{2n}^{(2)} (n \geq 1), A_{2n-1}^{(2)} (n \geq 2), D_{n+1}^{(2)} (n \geq 2), E_6^{(2)}$ and $D_4^{(3)}$ are realized as the canonical central extension of the loop algebras based on the pair $(\mathfrak{g}, \sigma)$. Let $\mathfrak{g}_0$ be the finite-dimensional σ-invariant subalgebra of $\mathfrak{g}$; namely,

$\mathfrak{g}$	X_n	A_{2n}	A_{2n-1}	D_{n+1}	E_6	D_4
r	1	2	2	2	2	3
$\mathfrak{g}_0$	X_n	B_n	C_n	B_n	F_4	G_2

Let $A' = (A'_{ij})$ $(i, j \in I)$ and $A = (A_{ij})$ $(i, j \in I_\sigma)$ be the Cartan matrices of $\mathfrak{g}$ and $\mathfrak{g}_0$, respectively, where I_σ is the set of σ-orbits of I. We define the numbers d'_i, d_i, ϵ'_i, ϵ_i $(i \in I)$ as follows: d'_i $(i \in I)$ are coprime positive integers such that $(d'_i A'_{ij})$ is symmetric; d_i $(i \in I_\sigma)$ are coprime positive integers such that $(d_i A_{ij})$ is symmetric, and we set $d_i = d_{\pi(i)}$ $(i \in I)$, where $\pi : I \to I_\sigma$ is the canonical projection. $\epsilon'_i = r$ if $\sigma(i) = i$, and 1 otherwise; $\epsilon_i = 2$ if $A'_{i\sigma(i)} < 0$, and 1 otherwise. Let $\kappa_0 = 2$ if $X_N^{(r)} = A_{2n}^{(2)}$, and 1 otherwise. By the definition one has $d'_i = d_i$ and $\epsilon'_i = 1$ if $r = 1$; $d'_i = 1$ if $r > 1$; $\epsilon_i = 1$ if $X_N^{(r)} \neq A_{2n}^{(2)}$.

In this paper we let $\{1, 2, \ldots, N\}$ and $\{1, 2, \ldots, n\}$ label the sets I and I_σ, respectively, and enumerate the nodes of the Dynkin diagram of $X_N^{(r)}$ by $I_\sigma \cup \{0\}$ as specified in Table 1. The diagrams (and the enumeration of the nodes for $r > 1$) coincide with TABLE Aff1-3 in [Kac], except the $A_{2n}^{(2)}$ case. We fix an injection $\iota : I_\sigma \to I$ such that $\pi \circ \iota = \mathrm{id}_{I_\sigma}$ and $A_{ab} < 0 \Leftrightarrow A'_{\iota(a)\iota(b)} < 0$ for any $a, b \in I_\sigma$. To be specific, assume that the labeling of the nodes for the Dynkin diagram of $\mathfrak{g}$ are given by dropping the 0-th ones from $X_N^{(1)}$ case in Table 1. Then we simply set $\iota(a) = a$ and regard ι as the embedding of the subset $\{1, \ldots, n\} \hookrightarrow \{1, \ldots, N\}$. The symbols d'_a, ϵ'_a and A'_{ab} for $a, b \in I_\sigma = \{1, \ldots, n\}$ should

TABLE 1. Dynkin diagrams for $X_N^{(r)}$. The enumeration of the nodes with $I_\sigma \cup \{0\} = \{0, 1, \dots, n\}$ is specified under or the right side of the nodes. In addition, the numbers d_a ($a \in I_\sigma$) are attached *above* the nodes if and only if $d_a \neq 1$.



be interpreted accordingly. One can check

$$\kappa_0 \epsilon'_a d'_a = \epsilon_a d_a,$$
$$\sum_{s=1}^{r} A'_{a\sigma^s(b)} = \frac{\epsilon'_a}{\epsilon_a} A_{ab}.$$

We use the notation:

$$H = \{(a, m) \mid a \in I_\sigma, m \in \mathbb{Z}_{\geq 1}\}. \tag{2.1}$$

Let $U_q(X_N^{(r)})$ be the quantum affine algebra. The irreducible finite-dimensional $U_q(X_N^{(r)})$-modules are parameterized by N-tuples of polynomials $(P_i(u))_{i \in I}$ (*Drinfeld polynomials*) with unit constant terms [CP1, CP2]. They satisfy the relation $P_{\sigma(i)}(u) = P_i(\omega^{\epsilon'_i} u)$, where $\omega = \exp(2\pi\sqrt{-1}/r)$. Thus it is enough to specify $(P_b(u))_{b\in I_\sigma}$. Following [KNT] we introduce

Definition 2.1. For each $(a,m) \in H$ and $\zeta \in \mathbb{C}^\times$, let $W^{(a)}_m(\zeta)$ be the finite-dimensional irreducible $U_q(X^{(r)}_N)$-module whose Drinfeld polynomials $P_b(u)$ $(b = 1,\dots,n)$ are specified as follows: $P_b(u) = 1$ for $b \neq a$, and

$$P_a(u) = \prod_{k=1}^{m}(1 - \zeta q^{\epsilon_a d_a(m+2-2k)}u).$$

We call $W^{(a)}_m(\zeta)$ a *KR (Kirillov-Reshetikhin) module.*

2.2. **The $U_q(X^{(r)}_N)$ Bethe equation.** Let

$$\mathcal{N} = \{\, N = (N^{(a)}_m)_{(a,m)\in H} \mid N^{(a)}_m \in \mathbb{Z}_{\geq 0}, \sum_{(a,m)\in H} N^{(a)}_m < \infty \,\}.$$

Given $\nu = (\nu^{(a)}_m) \in \mathcal{N}$, we define a tensor product module:

$$W^\nu = \bigotimes_{(a,m)\in H} (W^{(a)}_m(\zeta^{(a)}_m))^{\otimes \nu^{(a)}_m}, \tag{2.2}$$

where $\zeta^{(a)}_m \in \mathbb{C}^\times$. In the context of solvable lattice models [B], one can regard W^ν as the quantum space on which the commuting family of transfer matrices act. Reshetikhin and Wiegmann [RW] wrote down the $U_q(X^{(r)}_N)$ Bethe equation and conjectured its relevance to the spectrum of those transfer matrices. In our formulation, it is the simultaneous equation on the complex variables $x^{(a)}_i$ $(i \in \{1,2,\dots,M_a\},\ a \in I_\sigma)$ having the form:

(2.3)

$$\prod_{s=1}^{r}\prod_{m=1}^{\infty}\left(\frac{\omega^s (x^{(a)}_i)^{\frac{1}{\epsilon'_a}} q^{m\kappa_0 d'_a \delta_{a,\sigma^s(a)}} - 1}{\omega^s (x^{(a)}_i)^{\frac{1}{\epsilon'_a}} - q^{m\kappa_0 d'_a \delta_{a,\sigma^s(a)}}}\right)^{\nu^{(a)}_m} = -\prod_{s=1}^{r}\prod_{b\in I_\sigma}\prod_{j=1}^{M_b}\frac{\omega^s (x^{(a)}_i)^{\frac{1}{\epsilon'_a}} q^{\kappa_0 d'_a A'_{a\sigma^s(b)}} - (x^{(b)}_j)^{\frac{1}{\epsilon'_b}}}{\omega^s (x^{(a)}_i)^{\frac{1}{\epsilon'_a}} - (x^{(b)}_j)^{\frac{1}{\epsilon'_b}} q^{\kappa_0 d'_a A'_{a\sigma^s(b)}}}.$$

For the nontwisted case $r = 1$, this reduces to eq.(2.3) in [KN2]. The both sides are actually rational functions of $(x^{(a)}_i)$. In the sequel we consider a polynomial version of (2.3) specified as follows:

$$F^{(a)}_{i+} G^{(a)}_{i-} = F^{(a)}_{i-} G^{(a)}_{i+}, \tag{2.4}$$

$$F^{(a)}_{i+} = \prod_{k=1}^{\infty}(x^{(a)}_i q^{k\kappa_0\epsilon'_a d'_a} - 1)^{\nu^{(a)}_k},$$

$$F^{(a)}_{i-} = \prod_{k=1}^{\infty}(x^{(a)}_i - q^{k\kappa_0\epsilon'_a d'_a})^{\nu^{(a)}_k},$$

$$G^{(a)}_{i+} = \prod_{b=1}^{\tilde n}\prod_{j=1}^{M_b}((x^{(a)}_i)^{\frac{\epsilon'_{ab}}{\epsilon'_a}} q^{\kappa_0\epsilon'_{ab} d'_a A'_{ab}} - (x^{(b)}_j)^{\frac{\epsilon'_{ab}}{\epsilon'_b}}),$$

$$G^{(a)}_{i-} = \prod_{b=1}^{\tilde n}\prod_{j=1}^{M_b}((x^{(a)}_i)^{\frac{\epsilon'_{ab}}{\epsilon'_a}} - (x^{(b)}_j)^{\frac{\epsilon'_{ab}}{\epsilon'_b}} q^{\kappa_0\epsilon'_{ab} d'_a A'_{ab}}),$$

where $\epsilon'_{ab}=\max(\epsilon'_a,\epsilon'_b)$, and $\tilde{n}=n$ except for $\tilde{n}=n+1$ for $A^{(2)}_{2n}$. When $X^{(r)}_N=A^{(2)}_{2n}$, we have set $x^{(n+1)}_j=-x^{(n)}_j$ and $M_{n+1}=M_n$.

Remark 2.2. Let $\mathcal{P}^{(a)}_m(u)$ denote the a-th Drinfeld polynomial of the KR module $W^{(a)}_m(1)$. Then we have

$$\frac{F^{(a)}_{i-}}{F^{(a)}_{i+}}=\prod_{(a,m)\in H}\left(q^{\epsilon_a d_a m}\frac{\mathcal{P}^{(a)}_m(q^{-2\epsilon_a d_a}x^{(a)}_i)}{\mathcal{P}^{(a)}_m(x^{(a)}_i)}\right)^{\nu^{(a)}_m}.$$

In view of this, we expect without proof that the solutions of (2.3) determine the spectrum of transfer matrices acting on (2.2) with the choice $\zeta^{(a)}_m=1$.

We consider a class of solutions $(x^{(a)}_i)$ of (2.4) such that $x^{(a)}_i=x^{(a)}_i(q)$ is meromorphic function of q around $q=0$. For a meromorphic function $f(q)$ around $q=0$, let $\mathrm{ord}(f)$ be the order of the leading power of the Laurent expansion of $f(q)$ around $q=0$, i.e.,

$$f(q)=q^{\mathrm{ord}(f)}(f^0+f^1q+\cdots),\qquad f^0\neq 0,$$

and let $\tilde{f}(q):=f^0+f^1q+\cdots$ be the normalized series. When $f(q)$ is identically zero, we set $\mathrm{ord}(f)=\infty$. For each $N=(N^{(a)}_m)\in\mathcal{N}$, we set

$$H'=H'(N):=\{\ (a,m)\in H\mid N^{(a)}_m>0\ \},\tag{2.5}$$

where H is defined in (2.1). We have $|H'|<\infty$.

Definition 2.3. Let $(M_a)^n_{a=1}$ be the one in the Bethe equation (2.4), and let $N=(N^{(a)}_m)\in\mathcal{N}$ satisfy $\sum_{m=1}^{\infty}mN^{(a)}_m=M_a$. A meromorphic solution $(x^{(a)}_i)$ of (2.4) around $q=0$ is called a *string solution of pattern* N if

(i) $\mathrm{ord}(F^{(a)}_{i+}G^{(a)}_{i-})<\infty$ for any (a,i).

(ii) $(x^{(a)}_i)$ can be arranged as $(x^{(a)}_{m\alpha i})$ with

$$(a,m)\in H',\quad \alpha=1,\dots,N^{(a)}_m,\quad i=1,\dots,m$$

such that

(a) $d^{(a)}_{m\alpha i}:=\mathrm{ord}(x^{(a)}_{m\alpha i})=(m+1-2i)\kappa_0\epsilon'_a d'_a$.

(b) $z^{(a)}_{m\alpha}:=x^{(a)0}_{m\alpha 1}=x^{(a)0}_{m\alpha 2}=\cdots=x^{(a)0}_{m\alpha m}\ (\neq 0)$, where $x^{(a)0}_{m\alpha i}$ is the coefficient of the leading power of $x^{(a)}_{m\alpha i}$.

For each (a,m,α), $(x^{(a)}_{m\alpha i})^m_{i=1}$ is called an *m-string of color a*, and $z^{(a)}_{m\alpha}$ is called the *string center* of the m-string $(x^{(a)}_{m\alpha i})^m_{i=1}$. Thus, $N^{(a)}_m$ is the number of the m-strings of color a.

For a string solution $x^{(a)}_{m\alpha i}(q)=q^{d^{(a)}_{m\alpha i}}\tilde{x}^{(a)}_{m\alpha i}(q)$ of pattern N, the Bethe equation (2.4) reads

$$F^{(a)}_{m\alpha i+}G^{(a)}_{m\alpha i-}=F^{(a)}_{m\alpha i-}G^{(a)}_{m\alpha i+},\tag{2.6}$$

$$F^{(a)}_{m\alpha i+} = \prod_{k=1}^{\infty} (\tilde{x}^{(a)}_{m\alpha i} q^{d^{(a)}_{m\alpha i} + k\kappa_0 \epsilon'_a d'_a} - 1)^{\nu^{(a)}_k}, \tag{2.7}$$

$$F^{(a)}_{m\alpha i-} = \prod_{k=1}^{\infty} (\tilde{x}^{(a)}_{m\alpha i} q^{d^{(a)}_{m\alpha i}} - q^{k\kappa_0 \epsilon'_a d'_a})^{\nu^{(a)}_k}, \tag{2.8}$$

$$G^{(a)}_{m\alpha i+} = \prod_{b=1}^{\tilde{n}} \prod_{k=1}^{\infty} \prod_{\beta=1}^{N^{(b)}_k} \prod_{j=1}^{k} ((\tilde{x}^{(a)}_{m\alpha i})^{\frac{\epsilon'_{ab}}{\epsilon'_a}} q^{\frac{\epsilon'_{ab}}{\epsilon'_a} d^{(a)}_{m\alpha i} + \kappa_0 \epsilon'_{ab} d'_a A'_{ab}} - (\tilde{x}^{(b)}_{k\beta j})^{\frac{\epsilon'_{ab}}{\epsilon'_b}} q^{\frac{\epsilon'_{ab}}{\epsilon'_b} d^{(b)}_{k\beta j}}), \tag{2.9}$$

$$G^{(a)}_{m\alpha i-} = \prod_{b=1}^{\tilde{n}} \prod_{k=1}^{\infty} \prod_{\beta=1}^{N^{(b)}_k} \prod_{j=1}^{k} ((\tilde{x}^{(a)}_{m\alpha i})^{\frac{\epsilon'_{ab}}{\epsilon'_a}} q^{\frac{\epsilon'_{ab}}{\epsilon'_a} d^{(a)}_{m\alpha i}} - (\tilde{x}^{(b)}_{k\beta j})^{\frac{\epsilon'_{ab}}{\epsilon'_b}} q^{\frac{\epsilon'_{ab}}{\epsilon'_b} d^{(b)}_{k\beta j} + \kappa_0 \epsilon'_{ab} d'_a A'_{ab}}), \tag{2.10}$$

where for $X^{(r)}_N = A^{(2)}_{2n}$, we have set $\tilde{x}^{(n+1)}_{k\beta j} = -\tilde{x}^{(n)}_{k\beta j}$, $d^{(n+1)}_{k\beta j} = d^{(n)}_{k\beta j}$ and $N^{(n+1)}_k = N^{(n)}_k$. According to the procedure similar to [KN2], we can take the $q \to 0$ limit of (2.6) and obtain a key equation:

$$1 = (-1)^m \prod_{i=1}^{m} \frac{F^{(a)0}_{m\alpha i+} G^{(a)0}_{m\alpha i-}}{F^{(a)0}_{m\alpha i-} G^{(a)0}_{m\alpha i+}}. \tag{2.11}$$

In order to estimate the order of the Bethe equation (2.6), we introduce

$$\begin{aligned}
\xi^{(a)}_{m\alpha i+} &= \kappa_0 \epsilon'_a d'_a \sum_{k=1}^{\infty} \nu^{(a)}_k \min(m+1-2i+k, 0), \\
\xi^{(a)}_{m\alpha i-} &= \kappa_0 \epsilon'_a d'_a \sum_{k=1}^{\infty} \nu^{(a)}_k \min(m+1-2i, k), \\
\eta^{(a)}_{m\alpha i+} &= \kappa_0 \sum_{b=1}^{\tilde{n}} \sum_{k=1}^{\infty} \sum_{\beta=1}^{N^{(b)}_k} \sum_{j=1}^{k} \epsilon'_{ab} \min(d'_a(m+1-2i+A'_{ab}), d'_b(k+1-2j)), \\
\eta^{(a)}_{m\alpha i-} &= \kappa_0 \sum_{b=1}^{\tilde{n}} \sum_{k=1}^{\infty} \sum_{\beta=1}^{N^{(b)}_k} \sum_{j=1}^{k} \epsilon'_{ab} \min(d'_a(m+1-2i), d'_b(k+1-2j+A'_{ba})).
\end{aligned} \tag{2.12}$$

Definition 2.4. A string solution $(x^{(a)}_{m\alpha i})$ to (2.6) is called *generic* if

$$\begin{aligned}
&\mathrm{ord}(F^{(a)}_{m\alpha i\pm}) = \xi^{(a)}_{m\alpha i\pm}, \\
&\mathrm{ord}(G^{(a)}_{m\alpha i+}) = \eta^{(a)}_{m\alpha i+} + \zeta^{(a)}_{m\alpha i}, \qquad \mathrm{ord}(G^{(a)}_{m\alpha i-}) = \eta^{(a)}_{m\alpha i-} + \zeta^{(a)}_{m\alpha i+1},
\end{aligned} \tag{2.13}$$

where $\zeta^{(a)}_{m\alpha i} := \mathrm{ord}(\tilde{x}^{(a)}_{m\alpha i} - \tilde{x}^{(a)}_{m\alpha i-1})$ for $2 \le i \le m$, and $\zeta^{(a)}_{m\alpha 1} = \zeta^{(a)}_{m\alpha, m+1} = 0$.

Given a quantum space data $\nu \in \mathcal{N}$ and a string pattern $N \in \mathcal{N}$, we set

$$\gamma^{(a)}_m = \gamma^{(a)}_m(\nu) = \sum_{k=1}^{\infty} \min(m,k)\nu^{(a)}_k, \tag{2.14}$$

$$P^{(a)}_m = P^{(a)}_m(\nu, N) = \gamma^{(a)}_m - \sum_{(b,k)\in H} \frac{A_{ab}}{\epsilon_a d'_b} \min(d'_a m, d'_b k) N^{(b)}_k, \tag{2.15}$$

$$\hat{P}^{(a)}_m = \hat{P}^{(a)}_m(\nu, N) = \gamma^{(a)}_m - \sum_{(b,k)\in H} \frac{A'_{ab}}{d'_b} \min(d'_a m, d'_b k) N^{(b)}_k. \tag{2.16}$$

The number $\hat{P}^{(a)}_m$ will appear only in the RHS of (2.18).

Lemma 2.5. *We have*

$$\begin{aligned}
&(\xi^{(a)}_{m\alpha i+} + \eta^{(a)}_{m\alpha i-}) - (\xi^{(a)}_{m\alpha i-} + \eta^{(a)}_{m\alpha i+}) \\
&\quad = \begin{cases} -\kappa_0 \epsilon'_a d'_a (P^{(a)}_{m+1-2i} + N^{(a)}_{m+1-2i}) - \kappa_0 \Delta^{(a)}_{m+1-2i} & 1 \le i < \frac{m+1}{2} \\ 0 & i = \frac{m+1}{2} \\ \kappa_0 \epsilon'_a d'_a (P^{(a)}_{2i-m-1} + N^{(a)}_{2i-m-1}) + \kappa_0 \Delta^{(a)}_{2i-m-1} & \frac{m+1}{2} < i \le m, \end{cases}
\end{aligned}$$

where $\Delta^{(a)}_j = 0$ except for the following nontwisted cases: If there is a' such that $d_a > d_{a'} = 1$ and $A_{aa'} \neq 0$, then

$$\Delta^{(a)}_j = \begin{cases} -N^{(a')}_{2j} & d_a = 2 \\ -(N^{(a')}_{3j-1} + N^{(a')}_{3j} + N^{(a')}_{3j+1}) & d_a = 3. \end{cases}$$

For a generic string solution, one can determine the order $\zeta^{(a)}_{m\alpha i}$ from (2.6), (2.13) and Lemma 2.5. Requiring that the resulting $\zeta^{(a)}_{m\alpha i}$ should be positive and finite (cf. Definition 2.3), one has

Proposition 2.6. *A necessary condition for the existence of a generic string solution of pattern N is*

$$\sum_{k=1}^{\min(i-1,m+1-i)} \left\{ d'_a (P^{(a)}_{m+1-2k} + N^{(a)}_{m+1-2k}) + \Delta^{(a)}_{m+1-2k} \right\} > 0, \tag{2.17}$$

for $(a,m) \in H', 1 \le \alpha \le N^{(a)}_m, 2 \le i \le m$.

For a generic string solution, (2.11) becomes an equation for the string centers $(z^{(a)}_{m\alpha})$. We call it the string center equation (SCE).

Proposition 2.7. *Let $(x^{(a)}_{m\alpha i})$ be a generic string solution of pattern N. Then its string centers $(z^{(a)}_{m\alpha})$ satisfy the following equations ($(a,m) \in H',\ 1 \le \alpha \le N^{(a)}_m$):*

$$\prod_{(b,k)\in H'} \prod_{\beta=1}^{N^{(b)}_k} (z^{(b)}_{k\beta})^{A_{am\alpha,bk\beta}} = (-1)^{\hat{P}^{(a)}_m + N^{(a)}_m + 1}, \tag{2.18}$$

$$A_{am\alpha,bk\beta} = \delta_{ab}\delta_{mk}\delta_{\alpha\beta}(P^{(a)}_m + N^{(a)}_m) + \frac{A_{ba}}{\epsilon_b d'_a} \min(d'_a m, d'_b k) - \delta_{ab}\delta_{mk}. \tag{2.19}$$

Note that all the quantities in (2.15), (2.16) and (2.19) are integers. As in [KN2], Proposition 2.7 is derived by explicitly evaluating the ratio (2.11) by

Lemma 2.8. *For $a \in \{1, 2, \dots, n\}$ and $b \in \{1, 2, \dots, \tilde{n}\}$, we have*

$$\prod_{i=1}^{m} F^{(a)0}_{m\alpha i\epsilon} = \begin{cases} (-1)^{\gamma^{(a)}_m} f_{am\alpha} & \epsilon = + \\ (z^{(a)}_{m\alpha})^{\gamma^{(a)}_m} f_{am\alpha} & \epsilon = -, \end{cases}$$

$$\prod_{i=1}^{m}\prod_{j=1}^{k}((\tilde{x}^{(a)}_{m\alpha i})^{\frac{\epsilon'_{ab}}{\epsilon'_a}} q^{\frac{\epsilon'_{ab}}{\epsilon'_a} d^{(a)}_{m\alpha i} + \frac{1}{2}(1+\epsilon)\kappa_0 \epsilon'_{ab} d'_a A'_{ab}} - (\tilde{x}^{(b)}_{k\beta j})^{\frac{\epsilon'_{ab}}{\epsilon'_b}} q^{\frac{\epsilon'_{ab}}{\epsilon'_b} d^{(b)}_{k\beta j} + \frac{1}{2}(1-\epsilon)\kappa_0 \epsilon'_{ab} d'_a A'_{ab}})^0$$

$$= \begin{cases} (-(z^{(b)}_{k\beta})^{\frac{\epsilon'_{ab}}{\epsilon'_b}})^{A'_{ab}\min(d'_a m, d'_b k)/d'_b - \delta_{ab}\delta_{mk}} g^{bk\beta}_{amk} & \epsilon = 1 \\ (z^{(a)}_{m\alpha})^{\frac{\epsilon'_{ab}}{\epsilon'_a} A'_{ab}\min(d'_a m, d'_b k)/d'_b - \delta_{ab}\delta_{mk}} (-1)^{(m-1)\delta_{ab}\delta_{mk}\delta_{\alpha\beta}} g^{bk\beta}_{amk} & \epsilon = -1, \end{cases}$$

for some $f_{am\alpha}$ and $g^{bk\beta}_{am\alpha}$, where we have set $z^{(n+1)}_{m\alpha} := -z^{(n)}_{m\alpha}$.

The quantities $f_{am\alpha}$ and $g^{bk\beta}_{am\alpha}$ depend on the string centers $(z^{(a)}_{m\alpha})$, whose explicit formulae are available in [KN2] for nontwisted case. However we do not need them here. A string solution is generic if and only if $f_{am\alpha} \neq 0$ and $g^{bk\beta}_{am\alpha} \neq 0$ for any $a \in \{1, \dots, n\}, b \in \{1, \dots, \tilde{n}\}, m, k \in \mathbb{Z}_{\geq 1}, 1 \leq \alpha \leq N^{(a)}_m, 1 \leq \beta \leq N^{(b)}_k$. These conditions are equivalent to

(2.20)

$z^{(a)}_{m\alpha} \neq 1$ if there is $k \geq 1$ such that $\nu^{(a)}_k > 0$ and $k \in \langle m \rangle$,

$(z^{(a)}_{m\alpha})^{\frac{\epsilon'_{ab}}{\epsilon'_a}} \neq (z^{(b)}_{k\beta})^{\frac{\epsilon'_{ab}}{\epsilon'_b}}$ if $(a, m, \alpha) \neq (b, k, \beta)$ and $d'_a A'_{ab} \in \{ i d'_a - j d'_b \mid i \in \langle m \rangle, j \in \langle k \rangle \}$,

where $\langle m \rangle = \{m-1, m-3, \dots, -m+1\}$. Apart from the exceptional case $(a, m, \alpha) = (b, k, \beta)$, the condition (2.20) says that the two terms in each factor in (2.7) – (2.10) possess different leading terms whenever their orders coincide.

Definition 2.9. A solution to the SCE (2.18) is called *generic* if it satisfies (2.20).

Let A be the matrix with the entry $A_{am\alpha,bk\beta}$ in (2.19). The main theorem in this subsection is

Theorem 2.10. *Suppose that $N \in \mathcal{N}$ satisfies the conditions (2.17) and* $\det A \neq 0$. *Then, there is a one-to-one correspondence between generic string solutions of pattern N to the Bethe equation (2.6) and generic solutions to the SCE (2.18) of pattern N.*

Remark 2.11. Given the Bethe equation (2.3), the choice of $F^{(a)}_{i\pm}$ and $G^{(a)}_{i\pm}$ in (2.4) is not the unique one. For example one may restrict the b-product in $G^{(a)}_{i\pm}$ to those satisfying $A'_{ab} \neq 0$. Such an ambiguity influences Definition 2.3 (i), (2.7) – (2.10), (2.12), (2.20), hence Definition 2.9. However, the ratio in (2.11) is left unchanged, and all the statements in Lemma 2.5, Propositions 2.6, 2.7 and Theorem 2.10 remain valid.

2.3. **Counting of off-diagonal solutions to SCE .** For $k \in \mathbb{C}$ and $j \in \mathbb{Z}$, we define the binomial coefficient by

$$\binom{k}{j} = \frac{\Gamma(k+1)}{\Gamma(k-j+1)\Gamma(j+1)}.$$

For each ν, $N \in \mathcal{N}$, we define the number $R(\nu, N)$ by

$$R(\nu, N) = \left(\det_{(a,m),(b,k) \in H'} F_{am,bk} \right) \prod_{(a,m) \in H'} \frac{1}{N_m^{(a)}} \binom{P_m^{(a)} + N_m^{(a)} - 1}{N_m^{(a)} - 1}, \tag{2.21}$$

$$F_{am,bk} = \sum_{\beta=1}^{N_k^{(b)}} A_{am\alpha,bk\beta} = \delta_{ab}\delta_{mk} P_m^{(a)} + \frac{A_{ba}}{\epsilon_b d'_a} \min(d'_a m, d'_b k) N_k^{(b)}, \tag{2.22}$$

for $N \neq 0$. Here $H' = H'(N)$ and $P_m^{(a)} = P_m^{(a)}(\nu, N)$ are given by (2.5) and (2.15). For $N = 0$, we set $R(\nu, 0) = 1$ irrespective of ν. It is easy to see that $R(\nu, N)$ is an integer.

Definition 2.12. A solution $(z_{m\alpha}^{(a)})$ to the SCE is called *off-diagonal* (*diagonal*) if $z_{m\alpha}^{(a)} = z_{m\beta}^{(a)}$ only for $\alpha = \beta$ (otherwise).

Our main result in this subsection is

Theorem 2.13. *Suppose $P_m^{(a)}(\nu, N) \geq 0$ for any $(a, m) \in H'$. Then the number of off-diagonal solutions to the SCE (2.18) of pattern N divided by $\prod_{(a,m) \in H'} N_m^{(a)}!$ is equal to $R(\nu, N)$.*

The proof is due to the inclusion-exclusion principle and an explicit evaluation of the Möbius inversion formula similar to [KN1, KN2].

3. $R(\nu, N)$ and Q-system

So much for the Bethe equation, we now turn to the Q-system. For $a, b \in I_\sigma$ and $m, k \in \mathbb{Z}$, set

$$G_{am,bk} = \begin{cases} -\frac{1}{\epsilon_b} A_{ba} \delta_{m,k} & r > 1 \\ -A_{ba}(\delta_{m,2k-1} + 2\delta_{m,2k} + \delta_{m,2k+1}) & d_b/d_a = 2 \\ -A_{ba}(\delta_{m,3k-2} + 2\delta_{m,3k-1} + 3\delta_{m,3k} & d_b/d_a = 3 \\ \qquad +2\delta_{m,3k+1} + \delta_{m,3k+2}) & \\ -A_{ab}\delta_{d_a m, d_b k} & \text{otherwise.} \end{cases}$$

Let α_a and Λ_a ($a \in I_\sigma$) be the simple roots and the fundamental weights of $\mathfrak{g}_0$. We set

$$x_a = e^{\epsilon_a \Lambda_a}, \qquad y_a = e^{-\alpha_a},$$

which are related as

$$y_a = \prod_{b=1}^{n} x_b^{-A_{ba}/\epsilon_b}. \tag{3.1}$$

Definition 3.1. The system of equations ($Q_0^{(a)}(y) = 1$)

$$(Q_m^{(a)}(y))^2 = Q_{m+1}^{(a)}(y) Q_{m-1}^{(a)}(y) + y_a^m (Q_m^{(a)}(y))^2 \prod_{(b,k) \in H} (Q_m^{(b)}(y))^{G_{am,bk}} \tag{3.2}$$

for a family $(Q_m^{(a)}(y))_{(a,m) \in H}$ of power series of $y = (y_a)_{a=1}^n$ with unit constant terms is called the Q-system.

The factor y_a^m in the RHS is absorbed away if (3.2) is written in terms of the combination $x_a^m Q_m^{(a)}(y)$. The resulting form of the Q-system has originally appeared in [KR] $(A_n^{(1)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)})$, [K3] $(E_{6,7,8}^{(1)}, F_4^{(1)}, G_2^{(1)})$ and [HKOTT] (twisted case). Definition 3.1 corresponds to an *infinite* Q-system in the terminology of [KNT]. Its solution is not unique in general. Following [KNT] we introduce

Definition 3.2. A solution of (3.2) is *canonical* if the limit $\lim_{m\to\infty} Q_m^{(a)}(y)$ exists in the ring $\mathbb{C}[[y]]$ of formal power series of $y = (y_a)_{a=1}^n$ with the standard topology.

Theorem 3.3. *([KNT]) There exists a unique canonical solution $(\mathbf{Q}_m^{(a)}(y))_{(a,m)\in H}$ of the Q-system (3.2). Moreover, for any $\nu \in \mathcal{N}$, it admits the formula:*

$$\prod_{(a,m)\in H} (\mathbf{Q}_m^{(a)}(y))^{\nu_m^{(a)}} = R^\nu(y),$$

where the power series $R^\nu(y)$ is defined by

$$R^\nu(y) = \sum_{N\in\mathcal{N}} R(\nu, N) \prod_{a=1}^{n} y_a^{\sum_{m=0}^{\infty} m N_m^{(a)}}$$

in terms of the integer $R(\nu, N)$ in (2.21).

In the proof of the theorem [KNT], the expression $R(\nu, N)$ emerges from a general argument on the Q-system, which is independent of the Bethe equation. Our main finding in this paper is that it coincides with the number of off-diagonal solutions to the SCE obtained in Theorem 2.13.

Let us state the consequence of this fact in the light of the Kirillov-Reshetikhin conjecture. Let $\mathrm{ch}_m^{(a)}(x)$ denote the Laurent polynomial of $x = (x_a)_{a=1}^n$ representing the $\mathfrak{g}_0$-character of the KR module $W_m^{(a)}(\zeta)$. Then, $\mathcal{Q}_m^{(a)}(y) := x_a^{-m}\mathrm{ch}_m^{(a)}(x)|_{x=x(y)}$, where $x(y)$ is the inverse map of (3.1), is a polynomial of $y = (y_a)_{a=1}^n$ with the unit constant term. We call $\mathcal{Q}_m^{(a)}(y)$ the *normalized* $\mathfrak{g}_0$*-character* of $W_m^{(a)}(\zeta)$. The normalized character of the $\mathfrak{g}_0$-module W^ν in (2.2) is given by

$$\mathcal{Q}^\nu(y) = \prod_{(a,m)\in H} (\mathcal{Q}_m^{(a)}(y))^{\nu_m^{(a)}}.$$

The Kirillov-Reshetikhin conjecture [KR] is formulated in [KNT] as

Conjecture 3.4. *$\mathcal{Q}_m^{(a)}(y) = \mathbf{Q}_m^{(a)}(y)$ for any $(a, m) \in H$.*

Combining Theorem 3.3 and Conjecture 3.4, we relate the weight multiplicity in the tensor product of KR modules to the number of off-diagonal solutions to the SCE:

Corollary 3.5 (Formal completeness of the Bethe ansatz at $q = 0$)**.** *Under Conjecture 3.4 one has*

$$\mathcal{Q}^\nu(y) = R^\nu(y).$$

Conjecture 3.4 implies that $(\prod_{(a,m)\in H} x_a^{m\nu_m^{(a)}}) R^\nu(y(x))$ is a Laurent polynomial invariant under the Weyl group of $\mathfrak{g}_0$. In fact canonical solutions have also been obtained as linear combinations of characters of irreducible finite dimensional $\mathfrak{g}_0$-modules for $X_N^{(r)} = A_n^{(1)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)}$ [KR, HKOTY], and for $X_N^{(r)} = A_{2n-1}^{(2)}, A_{2n}^{(2)}, D_{n+1}^{(2)}, D_4^{(3)}$ [HKOTT]. For the current status of Conjecture 3.4, see section 5.7 of [KNT].

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