THE CANONICAL SOLUTIONS OF THE Q-SYSTEMS AND THE KIRILLOV-RESHETIKHIN CONJECTURE

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Abstract. We study a class of systems of functional equations closely related to various kinds of integrable statistical and quantum mechanical models. We call them the finite and infinite Q-systems according to the number of functions and equations. The finite Q-systems appear as the thermal equilibrium conditions (the Sutherland-Wu equation) for certain statistical mechanical systems. Some infinite Q-systems appear as the relations of the normalized characters of the KR modules of the Yangians and the quantum affine algebras. We give two types of power series formulae for the unique solution (resp. the unique canonical solution) for a finite (resp. infinite) Q-system. As an application, we reformulate the Kirillov-Reshetikhin conjecture on the multiplicities formula of the KR modules in terms of the canonical solutions of Q-systems.

1. Introduction

In the series of works [K1, K2, KR], Kirillov and Reshetikhin studied the formal counting problem (the *formal completeness*) of the Bethe vectors of the XXX-type integrable spin chains, and they empirically reached a remarkable conjectural formula on the characters of a certain family of finite-dimensional modules of the Yangian $Y(\mathfrak{g})$. Let us formulate it in the following way.

Conjecture 1.1. *Let $\mathfrak{g}$ be a complex simple Lie algebra of rank n. We set $y = (y_a)_{a=1}^n$, $y_a = e^{-\alpha_a}$ for the simple roots α_a of $\mathfrak{g}$. Let $\mathcal{Q}_m^{(a)}(y)$ be the normalized $\mathfrak{g}$-character of the KR module $W_m^{(a)}(u)$ ($a = 1, \dots, n$; $m = 1, 2, \dots$; $u \in \mathbb{C}$) of the Yangian $Y(\mathfrak{g})$; and $\mathcal{Q}^\nu(y) := \prod_{(a,m)}(\mathcal{Q}_m^{(a)}(y))^{\nu_m^{(a)}}$. Then, the formula*

$$\mathcal{Q}^\nu(y) \prod_{\alpha\in\Delta_+}(1-e^{-\alpha}) = \sum_{N=(N_m^{(a)})} \prod_{(a,m)} \binom{P_m^{(a)}(\nu,N) + N_m^{(a)}}{N_m^{(a)}} (y_a)^{mN_m^{(a)}}, \tag{1.1}$$

$$P_m^{(a)}(\nu,N) = \sum_{k=1}^{\infty} \nu_k^{(a)} \min(k,m) - \sum_{(b,k)} N_k^{(b)} d_a A_{ab} \min(\frac{m}{d_b}, \frac{k}{d_a}) \tag{1.2}$$

holds. Here, $A = (A_{ab})$ is the Cartan matrix of $\mathfrak{g}$, d_a are coprime positive integers such that $(d_a A_{ab})$ is symmetric, Δ_+ is the set of all the positive roots of $\mathfrak{g}$, and $\binom{a}{b} = \Gamma(a+1)/\Gamma(a-b+1)\Gamma(b+1)$.

Remark 1.2. Due to the Weyl character formula, the series in the RHS of (1.1) should be a *polynomial* of y, and its coefficients are identified with the multiplicities of the $\mathfrak{g}$-irreducible components of the tensor product $\bigotimes_{(a,m)} W^{(a)}_m(u^{(a)}_m)^{\otimes \nu^{(a)}_m}$, where $u^{(a)}_m$ are arbitrary.

Remark 1.3. There are actually two versions of Conjecture 1.1. The above one is the version in [HKOTY] which followed [K1, K2]. In the version in [KR], the binomial coefficients $\binom{a}{b}$ are set to be 0 if $a < b$; furthermore, the equality is claimed, not for the entire series in the both hand sides of (1.1), but only for their coefficients of the powers y^M "in the fundamental Weyl chamber"; namely, $M = (M_a)_{a=1}^n$ satisfies

$$\sum_{(a,m)} \nu^{(a)}_m m\Lambda_a - \sum_{a=1}^n M_a\alpha_a \in P_+, \tag{1.3}$$

where Λ_a are the fundamental weights and P_+ is the set of the dominant integral weights of $\mathfrak{g}$. So far, it is not proved that the two conjectures are equivalent. The both conjectures are naturally translated into the ones for the untwisted quantum affine algebras, which are extendable to the twisted quantum affine algebras [HKOTT]. In this paper, we refer all these conjectures as the Kirillov-Reshetikhin conjecture. More comments and the current status of the conjecture will be given in Section 5.7.

In [KR, K3], it was claimed that the $\mathcal{Q}^{(a)}_m(y)$'s satisfy a system of equations

$$\begin{aligned}(\mathcal{Q}^{(a)}_m(y))^2 &= \mathcal{Q}^{(a)}_{m-1}(y)\mathcal{Q}^{(a)}_{m+1}(y)\\ &\qquad + (y_a)^m(\mathcal{Q}^{(a)}_m(y))^2 \prod_{(b,k)} (\mathcal{Q}^{(b)}_k(y))^{G_{am,bk}}.\end{aligned} \tag{1.4}$$

Here, $\mathcal{Q}^{(a)}_0(y) = 1$, and $G_{am,bk}$ are the integers defined as

$$G_{am,bk} = \begin{cases} -A_{ba}(\delta_{m,2k-1} + 2\delta_{m,2k} + \delta_{m,2k+1}) & d_b/d_a = 2\\ -A_{ba}(\delta_{m,3k-2} + 2\delta_{m,3k-1} + 3\delta_{m,3k} & d_b/d_a = 3\\ \qquad +2\delta_{m,3k+1} + \delta_{m,3k+2}) & \\ -A_{ab}\delta_{d_am,d_bk} & \text{otherwise.}\end{cases} \tag{1.5}$$

See (4.22) for the original form of (1.4) in [KR, K3]. The relations (1.4) and (4.22) are often called the *Q-system*. The importance of the role of the Q-system to the formula (1.1) was recognized in [K1, K2, KR], and more explicitly exhibited in [HKOTY, KN2]. In this paper we proceed one step further in this direction; we study the equation (1.4) in a more general point of view, and give a characterization of the special power series solution in (1.1). For this purpose, we introduce *finite and infinite Q-systems*, where the former (resp. the latter) is a finite (resp. infinite) system of equations for a finite (resp. infinite) family of power series of the variable with finite (resp. infinite) components. The equation (1.4), which is an infinite system of equations with the variable with finite components, is regarded as an infinite Q-system with the specialization of the variable (a *specialized Q-system*). We show that every finite Q-system has a unique solution which has the same type of the power series formula as (1.1) (Theorem 2.4). In contrast, infinite Q-systems and their specializations, in general, admit more than one solutions. However, every infinite Q-system, or its specialization, has a unique *canonical* solution (Theorems 3.7 and 4.2), whose definition is given in Definition 3.5. The formula (1.1) turns out to be exactly the power series formula for the canonical solution of (1.4) (Theorem 4.3 and Proposition 4.9). Therefore, one can rephrase Conjecture 1.1 in a more intrinsic way as follows (Conjecture 5.5): *The family* $(\mathcal{Q}^{(a)}_m(y))$ *of the normalized* $\mathfrak{g}$-*characters of the KR modules is characterized as the canonical solution of (1.4).* This is the main statement of the paper.

Interestingly, the finite Q-systems also appear in other types of integrable statistical mechanical systems. Namely, they appear as the thermal equilibrium condition (the Sutherland-Wu equation) for the Calogero-Sutherland model [S], as well as the one for the ideal gas of the Haldane exclusion statistics [W]. The property of the solution of the finite Q-systems are studied in [A, AI, IA] from the point of view of the quasi-hypergeometric functions. We expect that the study of the Q-system and its variations and extensions will be useful for the representation theory of the quantum groups, and for the understanding of the nature of the integrable models as well.

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2. Finite Q-systems

A considerable part of the results in this section can be found in the work by Aomoto and Iguchi [A, IA]. We present here a more direct approach. More detailed remarks will be given in Section 2.4.

2.1. **Finite Q-systems.**

Throughout Section 2, let H denote a finite index set. Let $w = (w_i)_{i\in H}$ and $v = (v_i)_{i\in H}$ be complex multivariables, and let $G = (G_{ij})_{i,j\in H}$ be a given complex square matrix of size $|H|$. We consider a holomorphic map $\mathcal{D} \to \mathbb{C}^H$, $v \mapsto w(v)$ with

$$w_i(v) = v_i \prod_{j\in H} (1 - v_j)^{-G_{ij}}, \tag{2.1}$$

where $\mathcal{D}$ is some neighborhood of $v = 0$ in $\mathbb{C}^H$. The Jacobian $(\partial w/\partial v)(v)$ is 1 at $v = 0$, so that the map $w(v)$ is bijective around $v = w = 0$. Let $v(w)$ be the inverse map around $v = w = 0$. Inverting (2.1), we obtain the following functional equation for $v_i(w)$'s:

$$v_i(w) = w_i \prod_{j\in H} (1 - v_j(w))^{G_{ij}}. \tag{2.2}$$

By introducing new functions

$$Q_i(w) = 1 - v_i(w), \tag{2.3}$$

the equation (2.2) is written as

$$Q_i(w) + w_i \prod_{j\in H} (Q_j(w))^{G_{ij}} = 1. \tag{2.4}$$

From now on, we regard (2.4) as a system of equations for a family $(Q_i(w))_{i\in H}$ of power series of $w = (w_i)_{i\in H}$ with the unit constant terms (*i.e.*, the constant terms are 1). Here, for any power series $f(w)$ with the unit constant term and any complex number α, we mean by $(f(w))^\alpha \in \mathbb{C}[[w]]$ the αth power of $f(w)$ with the unit constant term. We can easily reverse the procedure from (2.1) to (2.4), and we have

Proposition 2.1. *The power series expansion of $Q_i(w)$ in (2.3) gives the unique family $(Q_i(w))_{i\in H}$ of power series of w with the unit constant terms which satisfies (2.4).*

Definition 2.2. The following system of equations for a family $(Q_i(w))_{i\in H}$ of power series of w with the unit constant terms is called a *(finite) Q-system*: For each $i \in H$,

$$\prod_{j\in H} (Q_j(w))^{D_{ij}} + w_i \prod_{j\in H} (Q_j(w))^{G_{ij}} = 1, \tag{2.5}$$

where $D = (D_{ij})_{i,j\in H}$ and $G = (G_{ij})_{i,j\in H}$ are arbitrary complex matrices with $\det D \neq 0$. The equation (2.4), which is the special case of (2.5) with $D = I$ (I: the identity matrix), is called a *standard* Q-system.

It is easy to see that there is a one-to-one correspondence between the solutions of the Q-system (2.5) and the solutions of the standard Q-system

$$Q'_i(w) + w_i \prod_{j\in H} (Q'_j(w))^{G'_{ij}} = 1, \quad G' = GD^{-1}, \tag{2.6}$$

where the correspondence is given by

$$Q'_i(w) = \prod_{j\in H} (Q_j(w))^{D_{ij}}, \tag{2.7}$$

$$Q_i(w) = \prod_{j\in H} (Q'_j(w))^{(D^{-1})_{ij}}. \tag{2.8}$$

Therefore, from Proposition 2.1, we immediately have

Theorem 2.3. *There exists a unique solution of the Q-system (2.5), which is given by (2.8), where $(Q'_i(w))_{i\in H}$ is the unique solution of the standard Q-system (2.6).*

2.2. **Power series formulae.** In what follows, we use the binomial coefficient in the following sense: For $a \in \mathbb{C}$ and $b \in \mathbb{Z}_{\geq 0}$,

$$\binom{a}{b} = \frac{\Gamma(a+1)}{\Gamma(a-b+1)\Gamma(b+1)}, \tag{2.9}$$

where the RHS means the limit value for the singularities. We set $\mathcal{N} := (\mathbb{Z}_{\geq 0})^H$. For D, G in (2.5) and $\nu = (\nu_i)_{i \in H} \in \mathbb{C}^H$, we define two power series of w,

$$K^\nu_{D,G}(w) = \sum_{N \in \mathcal{N}} K(D,G;\nu,N) w^N, \quad w^N = \prod_{i \in H} w_i^{N_i}, \tag{2.10}$$

$$R^\nu_{D,G}(w) = \sum_{N \in \mathcal{N}} R(D,G;\nu,N) w^N \tag{2.11}$$

with the coefficients

$$K(D,G;\nu,N) = \prod_{i \in H(N)} \binom{P_i + N_i}{N_i}, \tag{2.12}$$

$$R(D,G;\nu,N) = \Bigl(\det_{H(N)} F_{ij}\Bigr) \prod_{i \in H(N)} \frac{1}{N_i} \binom{P_i + N_i - 1}{N_i - 1}, \tag{2.13}$$

where we set $H(N) = \{\, i \in H \mid N_i \neq 0 \,\}$ for each $N \in \mathcal{N}$,

$$P_i = P_i(D,G;\nu,N) := -\sum_{j \in H} \nu_j (D^{-1})_{ji} - \sum_{j \in H} N_j (GD^{-1})_{ji}, \tag{2.14}$$

$$F_{ij} = F_{ij}(D,G;\nu,N) := \delta_{ij} P_j + (GD^{-1})_{ij} N_j, \tag{2.15}$$

and $\det_{H(N)}$ is a shorthand notation for $\det_{i,j \in H(N)}$. In (2.12) and (2.13), $\det_\emptyset$ and $\prod_\emptyset$ mean 1; namely, $K^\nu_{D,G}(w)$ and $R^\nu_{D,G}(w)$ are power series with the unit constant terms. It is easy to check that the both series converge for $|w_i| < |\gamma_i^{\gamma_i}/(\gamma_i+1)^{\gamma_i+1}|$, where $\gamma_i = -(GD^{-1})_{ii}$ and $z^z = \exp(z \log z)$ with the principal branch $-\pi < \mathrm{Im}(\log z) \leq \pi$ chosen.

Now we state our main results in this section.

Theorem 2.4 (Power series formulae)**.** *Let $(Q_i(w))_{i \in H}$ be the unique solution of (2.5). For $\nu \in \mathbb{C}^H$, let $Q^\nu_{D,G}(w) := \prod_{i \in H} (Q_i(w))^{\nu_i}$. Then,*

$$Q^\nu_{D,G}(w) = K^\nu_{D,G}(w) / K^0_{D,G}(w), \tag{2.16}$$

$$Q^\nu_{D,G}(w) = R^\nu_{D,G}(w). \tag{2.17}$$

The power series formulae for $Q_i(w)$ are obtained as special cases of (2.16) and (2.17) by setting $\nu = (\nu_j)_{j \in H}$ as $\nu_j = \delta_{ij}$.

One may recognize that the first formula (2.16) is analogous to the formula (1.1), where the denominator $K^0_{D,G}(w)$ in (2.16) corresponds to the Weyl denominator in the LHS of (1.1). As mentioned in Section 1, the formula (1.1) is interpreted as the formal completeness of the *XXX-type* Bethe vectors. In the same sense, the second formula (2.17) is analogous to the formal completeness of the *XXZ-type* Bethe vectors in [KN1, KN2]. See Section 2.4 for more remarks.

Example 2.5. Let $|H| = 1$. Then, (2.5) is an equation for a single power series $Q(w)$,

$$(Q(w))^D + w(Q(w))^G = 1, \tag{2.18}$$

where $D \neq 0$ and G are complex numbers, and the series (2.11) reads as

$$R^{\nu}_{D,G}(w) = \frac{\nu}{D}\sum_{N=0}^{\infty} \frac{\Gamma((\nu + NG)/D)(-w)^N}{\Gamma((\nu + NG)/D - N + 1)N!}. \tag{2.19}$$

The equation (2.18) and the power series formula (2.19) are well known and have a very long history since Lambert (e.g. [B, pp. 306–307]).

Example 2.6. Consider the case $G = O$ in (2.5),

$$\prod_{j\in H} (Q_j(w))^{D_{ij}} + w_i = 1. \tag{2.20}$$

This is easily solved as

$$Q_i(w) = \prod_{j\in H} (1 - w_j)^{(D^{-1})_{ij}}, \tag{2.21}$$

and, therefore,

$$Q^{\nu}_{D,O}(w) = \prod_{i\in H} (1 - w_i)^{\sum_{j\in H} \nu_j (D^{-1})_{ji}} = \prod_{i\in H} (1 - w_i)^{-P_i(D,O;\nu,N)}, \tag{2.22}$$

where $N \in \mathcal{N}$ is arbitrary. Using the binomial theorem

$$(1 - x)^{-\beta-1} = \sum_{N=0}^{\infty} \binom{\beta + N}{N} x^N, \tag{2.23}$$

one can directly check that

$$Q^{\nu}_{D,O}(w) = \sum_{N\in\mathcal{N}} \prod_{i\in H(N)} \binom{P_i - 1 + N_i}{N_i} w_i^{N_i} = R^{\nu}_{D,O}(w), \tag{2.24}$$

$$Q^{\nu}_{D,O}(w) = \frac{\prod_{j\in H} (1 - w_i)^{\sum_{j\in H} \nu_j (D^{-1})_{ji} - 1}}{\prod_{j\in H} (1 - w_i)^{-1}} = \frac{K^{\nu}_{D,O}(w)}{K^{0}_{D,O}(w)}. \tag{2.25}$$

2.3. **Proof of Theorem 2.4 and basic formulae.** Theorem 2.4 is regarded as a particularly nice example of the multivariable Lagrange inversion formula (e.g. [G]) where all the explicit calculations can be carried through. Here, we present the most direct calculation based on the multivariable residue formula (the *Jacobi formula* in [G, Theorem 3]).

We first remark that

Lemma 2.7. *Let $G' = GD^{-1}$. For each $\nu \in \mathbb{C}^H$, let $\nu' \in \mathbb{C}^H$ with $\nu'_i = \sum_{j\in H} \nu_j (D^{-1})_{ji}$. Then,*

$$Q^{\nu}_{D,G}(w) = Q^{\nu'}_{I,G'}(w), \tag{2.26}$$

$$K^{\nu}_{D,G}(w) = K^{\nu'}_{I,G'}(w), \quad R^{\nu}_{D,G}(w) = R^{\nu'}_{I,G'}(w). \tag{2.27}$$

Proof. The equality (2.26) is due to Theorem 2.3. The ones (2.27) follow from the fact $P_i(D,G;\nu,N) = P_i(I,G';\nu',N)$. □

By Lemma 2.7, we have only to prove Theorem 2.4 for the standard case $D = I$. Recall that (Proposition 2.1) $Q^{\nu}_{I,G}(w) = \prod_{i\in H}(1-v_i(w))^{\nu_i}$, where $v = v(w)$ is the inverse map of (2.1). Thus, Theorem 2.4 follows from

Proposition 2.8 (Basic formulae)**.** *Let $v = v(w)$ be the inverse map of (2.1). Then, the power series expansions*

$$\det_H\Bigl(\frac{w_j}{v_i}\frac{\partial v_i}{\partial w_j}(w)\Bigr) \prod_{i\in H}(1-v_i(w))^{\nu_i-1} = K^{\nu}_{I,G}(w), \tag{2.28}$$

$$\prod_{i\in H}(1-v_i(w))^{\nu_i} = R^{\nu}_{I,G}(w) \tag{2.29}$$

hold around $w = 0$.

Proof. The first formula (2.28). We evaluate the coefficient for w^N in the LHS of (2.28) as follows:

$$\begin{aligned}
&\operatorname*{Res}_{w=0} \frac{\partial v}{\partial w}(w) \prod_{i\in H}\Bigl\{(1-v_i(w))^{\nu_i-1}(v_i(w))^{-1}(w_i)^{1-N_i-1}\Bigr\}dw\\
&= \operatorname*{Res}_{v=0} \prod_{i\in H}\Bigl\{(1-v_i)^{\nu_i-1}(v_i)^{-1}\Bigl(v_i\prod_{j\in H}(1-v_j)^{-G_{ij}}\Bigr)^{-N_i}\Bigr\}dv\\
&= \operatorname*{Res}_{v=0} \prod_{i\in H}\Bigl\{(1-v_i)^{-P_i(I,G;\nu,N)-1}(v_i)^{-N_i-1}\Bigr\}dv\\
&= \prod_{i\in H}\binom{P_i(I,G;\nu,N)+N_i}{N_i} = K(I,G;\nu,N),
\end{aligned}$$

where we used (2.23) to get the last line. Thus, (2.28) is proved.

The second formula (2.29). By a simple calculation, we have

$$\begin{aligned}
\det_H\Bigl(\frac{v_j}{w_i}\frac{\partial w_i}{\partial v_j}(v)\Bigr)\prod_{i\in H}(1-v_i) &= \det_H\Bigl(\delta_{ij} + (-\delta_{ij}+G_{ij})v_i\Bigr)\\
&= \sum_{J\subset H} d_J \prod_{i\in J} v_i,
\end{aligned} \tag{2.30}$$

where $d_J := \det_J(-\delta_{ij}+G_{ij})$, and the sum is taken over all the subsets J of H. Therefore, the LHS of (2.29) is written as

$$\det_H\Bigl(\frac{w_j}{v_i}\frac{\partial v_i}{\partial w_j}(w)\Bigr)\sum_{J\subset H} d_J \prod_{i\in H}\Bigl\{(1-v_i(w))^{\nu_i-1}v_i(w)^{\theta(i\in J)}\Bigr\}. \tag{2.31}$$

By a similar residue calculation as above, the coefficient for w^N of (2.31) is evaluated as ($\theta(\text{true}) = 1$ and $\theta(\text{false}) = 0$)

$$\begin{aligned}
&\sum_{J\subset H} d_J \operatorname*{Res}_{v=0} \prod_{i\in H} \Bigl\{ (1-v_i)^{-P_i(I,G;\nu,N)-1} (v_i)^{-N_i+\theta(i\in J)-1} \Bigr\} dv \\
&= \sum_{J\subset H(N)} d_J \prod_{i\in H(N)} \binom{P_i(I,G;\nu,N)+N_i-\theta(i\in J)}{N_i-\theta(i\in J)} \\
&= \Bigl(\sum_{J\subset H(N)} d_J \prod_{i\in J} N_i \prod_{i\in H(N)\setminus J} (P_i+N_i) \Bigr) \prod_{i\in H(N)} \frac{1}{N_i} \binom{P_i+N_i-1}{N_i-1} \\
&= \det_{H(N)} \Bigl(\delta_{ij}(P_j+N_j) + (-\delta_{ij}+G_{ij})N_j \Bigr) \prod_{i\in H(N)} \frac{1}{N_i} \binom{P_i+N_i-1}{N_i-1} \\
&= R(I,G;\nu,N).
\end{aligned}$$

Thus, (2.29) is proved. □

This completes the proof of Theorem 2.4.

Example 2.9. We say that the map $w(v)$ in (2.1) is *lower-triangular* if the matrix G_{ij} is strictly lower-triangular with respect to a certain total order $\prec$ in H (*i.e.*, $G_{ij} = 0$ for $i \preceq j$). Let $w(v)$ be a lower-triangular map. Then,

$$\det_H \Bigl(\frac{v_j}{w_i} \frac{\partial w_i}{\partial v_j}(v) \Bigr) = \det_H \Bigl(\delta_{ij} + \frac{G_{ij} v_j}{1-v_j} \Bigr) = 1. \tag{2.32}$$

Thus, the formula (2.28) is simplified as

$$\prod_{i\in H} (1-v_i(w))^{\nu_i - 1} = K^{\nu}_{I,G}(w). \tag{2.33}$$

This type of formulae has appeared in [K1, K2, HKOTY].

Let us isolate the case $\nu = 0$ from (2.28), together with the formula (2.30), for the later use:

Corollary 2.10 (Denominator formulae)**.**

$$K^0_{I,G}(w) = \det_H \Bigl(\frac{w_j}{v_i} \frac{\partial v_i}{\partial w_j}(w) \Bigr) \prod_{i\in H} (1-v_i(w))^{-1}, \tag{2.34}$$

$$K^0_{I,G}(w) = \Bigl\{ \det_H \Bigl(\delta_{ij}(1-v_i(w)) + G_{ij} v_i(w) \Bigr) \Bigr\}^{-1}. \tag{2.35}$$

From (2.35) and the first formula of Theorem 2.4, we obtain

Corollary 2.11.

$$Q^{\nu}_{I,G}(w) = \sum_{J\subset H} g_J K^{\nu+\delta_J}_{I,G}(w), \tag{2.36}$$

$$g_J := \sum_{\substack{J'\subset H \\ |J'|=|J|}} \operatorname{sgn} \begin{pmatrix} J\overline{J} \\ J'\overline{J'} \end{pmatrix} \det_{i\in J, j\in J'} \bigl(\delta_{ij} - G_{ij} \bigr) \det_{i\in \overline{J}, j\in \overline{J'}} G_{ij}, \tag{2.37}$$

where $\delta_J = (\theta_i)_{i\in H}$, $\theta_i = 1$ *if* $i \in J$ *and* 0 *otherwise, and* $\overline{J} = H \setminus J$.

From Corollary 2.11, one can easily reproduce the second formula of Theorem 2.4. We leave it as an exercise for the reader.

2.4. **Remarks on related works.** *i) The formal completeness of the Bethe vectors.* In [K1, K2, HKOTY, KN1, KN2, KNT], the formal completeness of the XXX/XXZ-type Bethe vectors are studied. In the course of their analysis, several power series formulae in this section appeared in specialized/implicit forms. For example, Lemma 1 in [K1] is a special case of (2.33), Theorem 4.7 in [KN2] is a special case of Proposition 2.8, *etc.* From the current point of view, however, the relation between these power series formulae and the underlying *finite* Q-systems was not clearly recognized therein. As a result, these power series formulae and the *infinite* Q-systems were somewhat abruptly combined in the limiting procedure to obtain the power series formula for the *infinite* Q-systems. We are going to straighten out this logical entangle, and make the logical structure more transparent by Theorem 2.4 and the forthcoming Theorems 3.10, 4.3, Proposition 4.9, and Conjecture 5.5.

ii) The ideal gas with the Haldane statistics and the Sutherland-Wu equation. The series $K^{\nu}_{D,G}(w)$ has an interpretation of the grand partition function of the ideal gas with the Haldane exclusion statistics [W]. The finite Q-system appeared in [W] as the thermal equilibrium condition for the distribution functions of the same system. See also [IA] for another interpretation. The one variable case (2.18) also appeared in [S] as the thermal equilibrium condition for the distribution function of the Calogero-Sutherland model. As an application of our second formula in Theorem 2.4, we can quickly reproduce the "cluster expansion formula" in [I, Eq. (129)], which was originally calculated by the Lagrange inversion formula, as follows:

(2.38)
$$\begin{aligned}\log Q_i(w) &= \Big[\frac{\partial}{\partial \nu_i} R^{\nu}_{I,G}(w)\Big]_{\nu=0}\\ &= \sum_{N\in\mathcal{N}} \det_{\substack{H(N)\\ j,k\neq i}} F_{jk}(I,G;0,N) \prod_{j\in H(N)} \frac{1}{N_j}\binom{P_j(I,G;0,N)+N_j-1}{N_j-1} w^N,\end{aligned}$$

where $\{Q_i(w)\}_{i\in H}$ is the solution of (2.4). The Sutherland-Wu equation also plays an important role for the conformal field theory spectra. (See [BS] and the references therein.)

iii) Quasi-hypergeometric functions. The series $K^{\nu}_{D,G}(w)$ is a special example of the quasi-hypergeometric functions by Aomoto and Iguchi [AI]; when G'_{ij} are all integers, it reduces to a general hypergeometric function of Barnes-Mellin type. A quasi-hypergeometric function satisfies a system of fractional differential equations and a system of difference-differential equations [AI]. It also admits an integral representation [A]. In particular, the integral representation for $K^{\nu}_{I,G}(w)$ reduces to a simple form ([A, Eq. (2.30)], [IA, Eq. (89)]); in our notation,

$$K^{\nu}_{I,G}(w) = \frac{1}{(2\pi\sqrt{-1})^{|H|}} \int \Big\{ \prod_{i\in H} t_i^{\nu_i - 1} f_i(w,t)^{-1} \Big\} dt, \tag{2.39}$$

$$f_i(w,t) := t_i - 1 + w_i \prod_{j\in H} t_j^{G_{ij}}, \tag{2.40}$$

where the integration is along a circle around $t_i = 1$ starting from $t_i = 0$ for each t_i. We see that $f_i(w,t) = 0$ is the standard Q-system (2.4). The integral (2.39) is easily evaluated by the Cauchy theorem as [A, eq. (2.32)]

$$K^{\nu}_{I,G}(w) = Q^{\nu}_{I,G}(w) / \det_H(\delta_{ij} Q_i(w) + G_{ij}(1 - Q_j(w))), \tag{2.41}$$

where $\{Q_i(w)\}_{i\in H}$ is the solution of (2.4). The formula (2.41) reproduces a version of the Lagrange inversion formula (the Good formula [G, Theorem 2]), and it is equivalent to the formulae (2.16), (2.30), and (2.34).

3. Infinite Q-systems

3.1. Infinite Q-systems. Throughout Section 3, let H be a countable infinite index set. We fix an increasing sequence of *finite* subsets of H, $H_1 \subset H_2 \subset \cdots \subset H$ such that $\varinjlim H_L = H$. The result below does not depend on the choice of the sequence $\{H_L\}_{L=1}^{\infty}$. A natural choice is $H = \mathbb{N}$ and $H_L = \{1, \dots, L\}$. However, we introduce this generality to accommodate the situation we encounter in Section 4 (cf. (4.1)).

Let $w = (w_i)_{i\in H}$ be a multivariable with infinitely many components. For each $L \in \mathbb{N}$, let $w_L = (w_i)_{i\in H_L}$ be the submultivariable of w. The field $\mathbb{C}[[w_L]]$ of the power series of w_L over $\mathbb{C}$ is equipped with the standard $\mathfrak{X}_L$-adic topology, where $\mathfrak{X}_L$ is the ideal of $\mathbb{C}[[w_L]]$ generated by w_i's ($i \in H_L$). For $L < L'$, there is a natural projection $p_{LL'} : \mathbb{C}[[w_{L'}]] \to \mathbb{C}[[w_L]]$ such that $p_{LL'}(w_i) = w_i$ if $i \in H_L$ and 0 if $i \in H_{L'} \setminus H_L$. A *power series* $f(w)$ *of* w is an element of the projective limit $\mathbb{C}[[w]] = \varprojlim \mathbb{C}[[w_L]]$ of the projective system

$$\mathbb{C}[[w_1]] \leftarrow \mathbb{C}[[w_2]] \leftarrow \mathbb{C}[[w_3]] \leftarrow \cdots \tag{3.1}$$

with the induced topology. Let p_L be the canonical projection $p_L : \mathbb{C}[[w]] \to \mathbb{C}[[w_L]]$, and $f_L(w_L)$ be the Lth projection image of $f(w) \in \mathbb{C}[[w]]$; namely, $f_L(w_L) = p_L(f(w))$ and $f(w) = (f_L(w_L))_{L=1}^{\infty}$.

Here are some basic properties of power series which we use below:

(i) We also present a power series $f(w)$ as a formal sum

$$f(w) = \sum_{N\in\mathcal{N}} a_N w^N, \quad a_N \in \mathbb{C}, \tag{3.2}$$

$$\mathcal{N} = \{ N = (N_i)_{i\in H} \mid N_i \in \mathbb{Z}_{\geq 0}, \text{ all but finitely many } N_i \text{ are zero} \}, \tag{3.3}$$

(the definition of $\mathcal{N}$ is reset here for the infinite index set H) whose Lth projection image is

$$f_L(w_L) = \sum_{N \in \mathcal{N}_L} a_N w^N, \tag{3.4}$$

$$\mathcal{N}_L = \{ N \in \mathcal{N} \mid N_i = 0 \text{ for } i \notin H_L \}. \tag{3.5}$$

(ii) For any power series $f(w)$ with the unit constant term and any complex number α, the αth power $(f(w))^\alpha := ((f_L(w_L))^\alpha)_{L=1}^\infty \in \mathbb{C}[[w]]$ is uniquely defined and has the unit constant term again.

(iii) Let $f_i(w)$ $(i \in H)$ be a family of power series and $f_{i,L}(w_L)$ be their Lth projections. If their infinite product exists irrespective of the order of the product, we write it as $\prod_{i \in H} f_i(w)$. $\prod_{i\in H} f_i(w)$ exists if and only if $\prod_{i \in H} f_{i,L}(w_L)$ exists for each L; furthermore, if they exist, the latter is the Lth projection of the former.

Definition 3.1. The following system of equations for a family $(Q_i(w))_{i \in H}$ of power series of w with the unit constant terms is called an *(infinite) Q-system*: For each $i \in H$,

$$\prod_{j \in H} (Q_j(w))^{D_{ij}} + w_i \prod_{j \in H} (Q_j(w))^{G_{ij}} = 1. \tag{3.6}$$

Here, $D = (D_{ij})_{i,j \in H}$ and $G = (G_{ij})_{i,j\in H}$ are arbitrary infinite-size complex matrices satisfying the following two conditions:

(D) The matrix D is invertible, *i.e.*, there exists a matrix D^{-1} such that $DD^{-1} = D^{-1}D = I$.

(G') The matrix product $G' = GD^{-1}$ is well-defined.

When $D = I$, the equation (3.6) is called a *standard* Q-system.

Remark 3.2. The condition (G') is rephrased as "for each i and k, all but finitely many $G_{ij}(D^{-1})_{jk}$ $(j \in H)$ are zero". Similarly, the condition (D) implies that, for each i and k, all but finitely many $D_{ij}(D^{-1})_{jk}$, $(D^{-1})_{ij}D_{jk}$ $(j \in H)$ are zero. For the standard case, (D) is trivially satisfied, and (G') is satisfied for any complex matrix G.

Unlike the finite Q-systems, the uniqueness of the solution does not hold for the infinite Q-systems, in general. For instance, the following example admits infinitely many solutions.

Example 3.3. Let $H = \mathbb{Z}$, and consider a Q-system,

$$\frac{Q_{i-1}(w)Q_{i+1}(w)}{(Q_i(w))^2} + w_i = 1, \tag{3.7}$$

where $Q_0(w) = 1$. This can be easily solved as

$$Q_i(w) = (Q_1(w))^i \prod_{j=1}^{i-1} (1 - w_j)^{i-j}, \tag{3.8}$$

where $Q_1(w)$ is an arbitrary series of w with the unit constant term.

3.2. **Canonical solution.**

3.2.1. *Solution of standard Q-system.* First, we consider the standard case

$$Q_i(w) + w_i \prod_{j\in H} (Q_j(w))^{G_{ij}} = 1. \tag{3.9}$$

Let $Q_{i,L}(w_L) := p_L(Q_i(w))$ be the Lth projection image of $Q_i(w)$. Then, (3.9) is equivalent to a series of equations ($L = 1, 2, \dots$),

$$Q_{i,L}(w_L) + p_L(w_i) \prod_{j\in H} (Q_{j,L}(w_L))^{G_{ij}} = 1, \tag{3.10}$$

which are further equivalent to

$$Q_{i,L}(w_L) = 1 \quad i \notin H_L, \tag{3.11}$$

$$Q_{i,L}(w_L) + w_i \prod_{j\in H_L} (Q_{j,L}(w_L))^{G_{ij}} = 1 \quad i \in H_L. \tag{3.12}$$

Namely, a standard infinite Q-system is an infinite series of standard finite Q-systems which is compatible with the projections (3.1). By Proposition 2.1, (3.12) uniquely determines $Q_{i,L}(w_L)$ for $i \in H_L$. Furthermore, so determined $(Q_{i,L}(w_L))_{L=1}^{\infty}$ belongs to $\mathbb{C}[[w]]$, again because of the uniqueness of the solution of (3.12). Therefore,

Proposition 3.4. *There exists a unique solution $(Q_i(w))_{i\in H}$ of the standard Q-system (3.9), whose Lth projections $Q_{i,L}(w_L) := p_L(Q_i(w))$ are determined by (3.11) and (3.12).*

3.2.2. *Canonical solution.* As we have seen in Example 3.3, the uniqueness property does not hold for a general infinite Q-system (3.6). This is because, unlike the standard case, the Lth projection of (3.6) is not necessarily a finite Q-system. The non-uniqueness property also implies that, unlike the finite case, (3.6) does not always reduce to the standard one

$$Q'_i(w) + w_i \prod_{j\in H} (Q'_j(w))^{G'_{ij}} = 1, \quad G' = GD^{-1}. \tag{3.13}$$

In fact, the relations (2.7) and (2.8) are no longer equivalent due to the infinite products therein. However, the construction of a solution of a general Q-system from a standard one in Theorem 2.3 still works. We call the so obtained solution as the *canonical solution.* Let us give a more intrinsic definition, however.

Definition 3.5. We say that a solution $(Q_i(w))_{i\in H}$ of the Q-system (3.6) is *canonical* if it satisfies the following condition:

(Inversion property): For any $i \in H$,

$$\prod_{j\in H} \Big\{ \prod_{k\in H} (Q_k(w))^{(D^{-1})_{ij} D_{jk}} \Big\} = Q_i(w). \tag{3.14}$$

Remark 3.6. The condition (3.14) is not trivial, because, in general, one cannot freely exchange the order of the infinite double product therein.

Theorem 3.7. *There exists a unique canonical solution of the Q-system (3.6), which is given by*

$$Q_i(w) = \prod_{j\in H} (Q'_j(w))^{(D^{-1})_{ij}}, \tag{3.15}$$

where $(Q'_i(w))_{i\in H}$ is the unique solution of the standard Q-system (3.13).

Proof. First, we remark that the infinite product (3.15) exists, because its Lth projection image reduces to the finite product

$$Q_{i,L}(w_L) = \prod_{j\in H_L} (Q'_{j,L}(w_L))^{(D^{-1})_{ij}} \tag{3.16}$$

due to (3.11). Let us show that the family $(Q_i(w))_{i\in H}$ in (3.15) is a solution of the Q-system (3.6). With the substitution of (3.16), the Lth projection image of the first term in the LHS of (3.6) is

$$\begin{aligned} \prod_{j\in H} (Q_{j,L}(w_L))^{D_{ij}} &= \prod_{j\in H} \Big\{ \prod_{k\in H_L} (Q'_{k,L}(w_L))^{D_{ij}(D^{-1})_{jk}} \Big\} \\ &= \begin{cases} Q'_{i,L}(w_L) & i \in H_L \\ 1 = Q'_{i,L}(w_L) & i \notin H_L. \end{cases} \end{aligned} \tag{3.17}$$

In the second equality above, we exchanged the order of the products. It is allowed because the double product is a finite one (cf. Remark 3.2). The second term in the LHS of (3.6) can be calculated in a similar way as follows:

$$\begin{aligned} \prod_{j\in H} (Q_{j,L}(w_L))^{G_{ij}} &= \prod_{j\in H} \Big\{ \prod_{k\in H_L} (Q'_{k,L}(w_L))^{G_{ij}(D^{-1})_{jk}} \Big\} \\ &= \prod_{k\in H_L} (Q'_{k,L}(w_L))^{G'_{ik}}. \end{aligned} \tag{3.18}$$

From (3.17) and (3.18), we conclude that (3.6) reduces to (3.13). Furthermore, by (3.17), we have

$$\prod_{j\in H} (Q_j(w))^{D_{ij}} = Q'_i(w). \tag{3.19}$$

Then, substituting (3.19) in (3.15), we obtain (3.14). Therefore, $(Q_i(w))_{i\in H}$ is a canonical solution of (3.6). Next, we show the uniqueness. Suppose that $(Q_i(w))_{i\in H}$ is a canonical solution of (3.6). We define $Q'_i(w)$ as

$$Q'_i(w) = \prod_{j\in H} (Q_j(w))^{D_{ij}}. \tag{3.20}$$

Then, by the inversion property (3.14), we have

$$Q_i(w) = \prod_{j\in H} (Q'_j(w))^{(D^{-1})_{ij}}. \tag{3.21}$$

Also, by (3.6),

$$Q'_{i,L}(w_L) = 1, \quad i \notin H_L. \tag{3.22}$$

With (3.21) and (3.22), the same calculation as (3.18) shows that $(Q'_i(w))_{i\in H}$ is the (unique) solution of (3.13). Therefore, by (3.21), $Q_i(w)$ is unique. □

Example 3.8. Let us find the canonical solution of the Q-system (3.7) in Example 3.3. We have

$$D_{ij} = -2\delta_{ij} + \delta_{i,j-1} + \delta_{i,j+1}, \quad (D^{-1})_{ij} = -\min(i,j). \tag{3.23}$$

Let $H_L = \{1,\dots,L\}$. By (3.20) and (3.22), the Lth projection of the LHS of (3.14) equals to

$$\begin{aligned}
&\prod_{j=1}^{L}\Big\{\prod_{k=j-1}^{j+1}(Q_{k,L}(w_L))^{(D^{-1})_{ij}D_{jk}}\Big\}\\
&=\Big(\prod_{k=1}^{L}\Big\{\prod_{j=k-1}^{k+1}(Q_{k,L}(w_L))^{(D^{-1})_{ij}D_{jk}}\Big\}\Big)\\
&\quad\times (Q_{L+1,L}(w_L))^{(D^{-1})_{iL}D_{L,L+1}}(Q_{L,L}(w_L))^{-(D^{-1})_{i,L+1}D_{L+1,L}}\\
&=\Big(\prod_{k=1}^{L}(Q_{k,L}(w_L))^{\delta_{ik}}\Big)(Q_{L+1,L}(w_L))^{-\min(i,L)}(Q_{L,L}(w_L))^{\min(i,L+1)}
\end{aligned} \tag{3.24}$$

Therefore, the condition (3.14) reads as

$$Q_{i,L}(w_L) = \begin{cases} Q_{i,L}(w_L)(Q_{L,L}(w_L)/Q_{L+1,L}(w_L))^i & i \le L \\ Q_{L,L}(w_L)(Q_{L,L}(w_L)/Q_{L+1,L}(w_L))^L & i \ge L+1. \end{cases} \tag{3.25}$$

This is equivalent to

$$Q_{i,L}(w_L) = Q_{L,L}(w_L), \quad i \ge L+1. \tag{3.26}$$

Using (3.8) and (3.26), one can easily obtain

$$Q_1(w) = \prod_{j=1}^{\infty}(1-w_j)^{-1}. \tag{3.27}$$

Therefore, the canonical solution of (3.7) is given by

$$Q_i(w) = \prod_{j=1}^{\infty}(1-w_j)^{-\min(i,j)}. \tag{3.28}$$

3.3. **Power series formula.** Let $(Q_i(w))_{i\in H}$ be the canonical solution of (3.6), and $(Q'_i(w))_{i\in H}$ be the unique solution of the standard Q-system (3.13). For the matrix D in (3.6), let $\boldsymbol{\nu}(D)$ be the set of all $\nu = (\nu_i)_{i\in H}$ such that $\nu_i \in \mathbb{C}$ and, for each i, the sum $\sum_{j\in H} \nu_j (D^{-1})_{ji}$ exists (*i.e.,* all but finitely many $\nu_j(D^{-1})_{ji}$ ($j \in H$) are zero). For each $\nu \in \boldsymbol{\nu}(D)$, we define

$$Q^{\nu}_{D,G}(w) := \prod_{i\in H} (Q_i(w))^{\nu_i} = \prod_{i\in H} \Big\{ \prod_{j\in H} (Q'_j(w))^{\nu_i (D^{-1})_{ij}} \Big\}. \tag{3.29}$$

The last infinite product exists, because its Lth projection image reduces to a finite product due to (3.11) and the definition of $\boldsymbol{\nu}(D)$. For each $\nu \in \boldsymbol{\nu}(D)$, let $\nu' = (\nu'_i) \in \boldsymbol{\nu}(I)$, $\nu'_i = \sum_{j\in H} \nu_j (D^{-1})_{ji}$. Then, by (3.29), we have

$$Q^{\nu}_{D,G}(w) = Q^{\nu'}_{I,G'}(w), \quad G' = GD^{-1}. \tag{3.30}$$

It follows from (3.11) and (3.30) that

Lemma 3.9.

$$p_L(Q^{\nu}_{D,G}(w)) = Q^{\nu'_L}_{I_L,G'_L}(w_L), \tag{3.31}$$

where the RHS is for the solution of the finite Q-system with the finite index set H_L, and $I_L = (\delta_{ij})_{i,j\in H_L}$, $G'_L = (G'_{ij})_{i,j\in H_L}$, $\nu'_L = (\nu'_i)_{i\in H_L}$ are the H_L-truncations of I, G', ν', respectively.

For D, G in (3.6) and $\nu \in \boldsymbol{\nu}(D)$, we define the power series $K^{\nu}_{D,G}(w)$ and $R^{\nu}_{D,G}(w)$ by the superficially identical formulae (2.10)–(2.15) with D, G, ν, N, *etc.*, therein being replaced by the ones for the infinite index set H.

Theorem 3.10 (Power series formulae)**.** *For the canonical solution $(Q_i(w))_{i\in H}$ of (3.6) and $\nu \in \boldsymbol{\nu}(D)$, let $Q^{\nu}_{D,G}(w)$ be the series in (3.29). Then,*

$$Q^{\nu}_{D,G}(w) = K^{\nu}_{D,G}(w)/K^{0}_{D,G}(w) = R^{\nu}_{D,G}(w). \tag{3.32}$$

Proof. By Theorem 2.4 and Lemma 3.9, it is enough to show that

$$p_L(K^{\nu}_{D,G}(w)) = K^{\nu'_L}_{I_L,G'_L}(w_L), \quad p_L(R^{\nu}_{D,G}(w)) = R^{\nu'_L}_{I_L,G'_L}(w_L). \tag{3.33}$$

By (3.2)–(3.5), (3.33) further reduces to the following equality:

$$P_i(D,G;\nu,N) = P_i(I_L,G'_L;\nu'_L,N_L), \quad N \in \mathcal{N}_L,\ i \in H_L, \tag{3.34}$$

where $N_L = (N_i)_{i\in H_L}$ is the H_L-truncation of N. □

4. Q-systems of KR type

In this section, we introduce a class of infinite Q-systems which we call the Q-systems of KR type. This is a preliminary step towards the reformulation of Conjecture 1.1.

4.1. **Specialized Q-systems.** Throughout the section, we take the countable index set as

$$H = \{1, \dots, n\} \times \mathbb{N} \tag{4.1}$$

for a given natural number n. We choose the increasing sequence $H_1 \subset H_2 \subset \cdots \subset H$ with $\varinjlim H_L = H$ as $H_L = \{1, \dots, n\} \times \{1, \dots, L\}$. Let $y = (y_a)_{a=1}^n$ be a multivariable with n components.

Definition 4.1. The following system of equations for a family $(\mathcal{Q}_m^{(a)}(y))_{(a,m)\in H}$ of power series of y with the unit constant terms is called a *specialized (infinite) Q-system*: For each $(a, m) \in H$,

$$\prod_{(b,k)\in H} (\mathcal{Q}_k^{(b)}(y))^{D_{am,bk}} + (y_a)^m \prod_{(b,k)\in H} (\mathcal{Q}_k^{(b)}(y))^{G_{am,bk}} = 1, \tag{4.2}$$

where the infinite-size complex matrices $D = (D_{am,bk})_{(a,m),(b,k)\in H}$ and $G = (G_{am,bk})_{(a,m),(b,k)\in H}$ satisfy the same conditions (D) and (G') as in Definition 3.1. A solution of (4.2) is called *canonical* if it satisfies the condition

$$\prod_{(b,k)\in H} \left\{ \prod_{(c,j)\in H} (\mathcal{Q}_j^{(c)}(y))^{(D^{-1})_{am,bk} D_{bk,cj}} \right\} = \mathcal{Q}_m^{(a)}(y). \tag{4.3}$$

Let $\mathbb{C}[[y]]$ be the field of power series of y with the standard topology, J_L be the ideal of $\mathbb{C}[[y]]$ generated by $(y_a)^{L+1}$'s ($a = 1, \dots, n$), and $\mathbb{C}[[y]]_L$ be the quotient $\mathbb{C}[[y]]/J_L$. We can identify $\mathbb{C}[[y]]$ with the projective limit of the projective system,

$$\mathbb{C}[[y]]_1 \leftarrow \mathbb{C}[[y]]_2 \leftarrow \mathbb{C}[[y]]_3 \leftarrow \cdots . \tag{4.4}$$

Let $w = (w_m^{(a)})_{(a,m)\in H}$ be a multivariable, and let $w(y)$ be the map with

$$w_m^{(a)}(y) = (y_a)^m. \tag{4.5}$$

The map (4.5) induces the maps ψ_L and ψ such that

$$\begin{array}{ccc} \mathbb{C}[[w_L]] & \leftarrow & \mathbb{C}[[w]] \\ \psi_L \downarrow & & \psi \downarrow \\ \mathbb{C}[[y]]_L & \leftarrow & \mathbb{C}[[y]]. \end{array} \tag{4.6}$$

We call the image $\psi(f(w)) \in \mathbb{C}[[y]]$ the *specialization of* $f(w)$, and write it as $f(w(y))$. Explicitly, for $f(w)$ in (3.2),

$$f(w(y)) = \sum_{M_1,\dots,M_n=0}^{\infty} \Bigg(\sum_{\substack{N\in\mathcal{N} \\ \sum_{m=1}^{\infty} m N_m^{(a)} = M_a}} a_N \Bigg) \prod_{a=1}^{n} (y_a)^{M_a}. \tag{4.7}$$

Theorem 4.2. *There exists a unique canonical solution of the specialized Q-system (4.2), which is given by the specialization* $\mathcal{Q}_m^{(a)}(y) = Q_m^{(a)}(w(y))$ *of* *the canonical solution* $(Q_m^{(a)}(w))_{(a,m)\in H}$ *of the following* Q*-system:*

$$\prod_{(b,k)\in H} (Q_k^{(b)}(w))^{D_{am,bk}} + w_m^{(a)} \prod_{(b,k)\in H} (Q_k^{(b)}(w))^{G_{am,bk}} = 1. \tag{4.8}$$

Proof. Since the map ψ is continuous, it preserves the infinite product. Therefore, the specialization of the canonical solution of (4.8) gives a canonical solution of (4.2). Let us show the uniqueness. By repeating the same proof for Theorem 3.7, the uniqueness is reduced to the one for the standard case $D = I$. Let us write (4.2) for $D = I$ as $(L = 1, 2, \dots)$

$$\mathcal{Q}_m^{(a)}(y) \equiv 1 \mod J_L \quad (a,m) \notin H_L, \tag{4.9}$$

$$\mathcal{Q}_m^{(a)}(y) + (y_a)^m \prod_{(b,k)\in H_L} (\mathcal{Q}_k^{(b)}(y))^{G_{am,bk}} \equiv 1 \mod J_L \quad (a,m) \in H_L. \tag{4.10}$$

These equations uniquely determine $\mathcal{Q}_m^{(a)}(y) \bmod J_L$. Since L is arbitrary, $\mathcal{Q}_m^{(a)}(y)$ is unique. □

By the specialization of Theorem 3.10, we immediately obtain

Theorem 4.3 (Power series formulae)**.** *Let* $(\mathcal{Q}_m^{(a)}(y))_{(a,m)\in H}$ *be the canonical solution of the* Q*-system (4.2). Let* $\mathcal{Q}^{\nu}_{D,G}(y) = \prod_{(a,m)\in H} (\mathcal{Q}_m^{(a)}(w))^{\nu_m^{(a)}}$, $\nu \in \boldsymbol{\nu}(D)$*. Then,*

$$\mathcal{Q}^{\nu}_{D,G}(y) = \mathcal{K}^{\nu}_{D,G}(y)/\mathcal{K}^{0}_{D,G}(y) = \mathcal{R}^{\nu}_{D,G}(y), \tag{4.11}$$

where the series $\mathcal{K}^{\nu}_{D,G}(y) = K^{\nu}_{D,G}(w(y))$ *and* $\mathcal{R}^{\nu}_{D,G}(y) = R^{\nu}_{D,G}(w(y))$ *are the specializations of the series in Theorem 3.10.*

4.2. **Convergence property.** Let us consider the special case of the specialized Q-system (4.2) where the matrix D and its inverse D^{-1} are given by

$$D_{am,bk} = -\delta_{ab}(2\delta_{mk} - \delta_{m,k+1} - \delta_{m,k-1}), \tag{4.12}$$

$$(D^{-1})_{am,bk} = -\delta_{ab} \min(m,k). \tag{4.13}$$

Then, (4.2) is written in the form $(\mathcal{Q}_0^{(a)}(y) = 1)$

$$\begin{aligned}(\mathcal{Q}_m^{(a)}(y))^2 &= \mathcal{Q}_{m-1}^{(a)}(y)\mathcal{Q}_{m+1}^{(a)}(y)\\ &\qquad + (y_a)^m (\mathcal{Q}_m^{(a)}(y))^2 \prod_{(b,k)\in H} (\mathcal{Q}_k^{(b)}(y))^{G_{am,bk}}.\end{aligned} \tag{4.14}$$

Proposition 4.4. *A solution* $(\mathcal{Q}_m^{(a)}(y))$ *of the specialized* Q*-system (4.14) is canonical if and only if it satisfies the following condition:*

(Convergence property):

$$\text{For each } a, \text{ the limit } \lim_{m\to\infty} \mathcal{Q}_m^{(a)}(y) \text{ exists in } \mathbb{C}[[y]]. \tag{4.15}$$

Proof. Let $(\mathcal{Q}^{(a)}_m(y))_{(a,m)\in H}$ be a solution of (4.14). The same calculation as (3.24) in Example 3.8 shows that (4.3) is equivalent to the following equality for each L (cf. (3.26)):

$$\mathcal{Q}^{(a)}_m(y) \equiv \mathcal{Q}^{(a)}_L(y) \mod J_L, \quad m \geq L+1. \tag{4.16}$$

Clearly, the condition (4.15) follows from the condition (4.16). Conversely, assume the condition (4.15). By (4.14), we have

$$\mathcal{Q}^{(a)}_m(y)/\mathcal{Q}^{(a)}_{m-1}(y) \equiv \mathcal{Q}^{(a)}_{m+1}(y)/\mathcal{Q}^{(a)}_m(y) \mod J_L, \quad (m \geq L+1). \tag{4.17}$$

Because of (4.15), the both hand sides of (4.17) are 1 mod J_L. Thus, we have $\mathcal{Q}^{(a)}_m(y) \equiv \mathcal{Q}^{(a)}_{m-1}(y) \mod J_L$ $(m \geq L+1)$. Therefore, (4.16) holds. □

4.3. Q-system of KR type and denominator formula.

Definition 4.5. A specialized Q-system (4.2) is called a *Q-system of KR (Kirillov-Reshetikhin) type* if the matrices D and G further satisfy the following conditions:

(KR-I) The matrix D and its inverse D^{-1} are given by (4.12) and (4.13).

(KR-II) There exists a well-order $\prec$ in H such that $G' = GD^{-1}$ has the form

$$G'_{am,bk} = g_{ab}m \quad \text{for } (a,m) \preceq (b,k), \tag{4.18}$$

where g_{ab} $(a,b=1,\dots,n)$ are integers with $\det_{1\leq a,b\leq n} g_{ab} \neq 0$.

Example 4.6. Let $t_a > 0$ and h_{ab} $(a,b=1,\dots,n)$ be real numbers such that $g_{ab} := h_{ab}t_b$ are integers and $\det h_{ab} \neq 0$. We define a well-order $\prec$ in H as follows: $(a,m) \prec (b,k)$ if $t_b m < t_a k$, or if $t_b m = t_a k$ and $a < b$. Then,

$$G'_{am,bk} = h_{ab} \min(t_b m, t_a k) \tag{4.19}$$

satisfies the condition (KR-II) with $g_{ab} = h_{ab}t_b$.

Let $x = (x_a)_{a=1}^n$ be a multivariable with n components, and $y(x)$ be the map

$$y_a(x) = \prod_{b=1}^n (x_b)^{-g_{ab}}, \tag{4.20}$$

where g_{ab} are the integers in (4.18). We set

$$\mathbf{Q}^{(a)}_m(x) := (x_a)^m \mathcal{Q}^{(a)}_m(y(x)), \tag{4.21}$$

which are Laurent series of x.

Proposition 4.7. *The family $(\mathbf{Q}^{(a)}_m(x))_{(a,m)\in H}$ satisfies a system of equations ($\mathbf{Q}^{(a)}_0(x) = 1$),*

$$(\mathbf{Q}^{(a)}_m(x))^2 = \mathbf{Q}^{(a)}_{m-1}(x)\mathbf{Q}^{(a)}_{m+1}(x) + (\mathbf{Q}^{(a)}_m(x))^2 \prod_{(b,k)\in H} (\mathbf{Q}^{(b)}_k(x))^{G_{am,bk}}. \tag{4.22}$$

Proof. By comparing (4.14) and (4.22), it is enough to prove the equality

$$\sum_{k=1}^{\infty} G_{am,bk}(-k) = g_{ab}m. \tag{4.23}$$

Due to the condition (KR-II), for given (a,m) and b, there is some number L such that $G'_{am,bk} = g_{ab}m$ holds for any $k \geq L$. Then, for $k > L$, we have

$$G_{am,bk} = \sum_{j=1}^{\infty} G'_{am,bj} D_{bj,bk} = g_{ab}m(-2+1+1) = 0. \tag{4.24}$$

Therefore, the LHS of (4.23) is evaluated as

$$\sum_{k=1}^{L}\sum_{j=1}^{\infty} G'_{am,bj} D_{bj,bk}(-k) = (L+1)G'_{am,bL} - LG'_{am,bL+1} = g_{ab}m. \tag{4.25}$$

□

Remark 4.8. The relation (4.22) is the original form of the Q-system in [K2, K3, KR], where the matrix G is taken as (1.5). See also (5.14) and (5.16). Note that, in the second term of the RHS in (4.22), the factor $(\mathcal{Q}^{(a)}_m(y))^2$ is cancelled by the factor in the product for $(b,k)=(a,m)$, because $G_{am,am} = -2$.

Proposition 4.9 (Denominator formula)**.** *Let $(\mathcal{Q}^{(a)}_m(y))_{(a,m)\in H}$ be the canonical solution of the Q-system of KR type (4.14). Let $\mathbf{K}^0_{D,G}(x) := \mathcal{K}^0_{D,G}(y(x))$, where $\mathcal{K}^0_{D,G}(y)$ is the power series in (4.11). Then, the formula*

$$\mathbf{K}^0_{D,G}(x) = \det_{1\leq a,b\leq n}\Bigl(\frac{\partial \mathbf{Q}^{(a)}_1}{\partial x_b}(x)\Bigr) \tag{4.26}$$

holds.

A proof of Proposition 4.9 is given in Appendix A. In Conjecture 5.7, Proposition 4.9 will be used to identify $\mathbf{K}^0_{D,G}(x)$ for some G with the *Weyl denominators* of the simple Lie algebras.

5. Q-systems and the Kirillov-Reshetikhin conjecture

In this section, we reformulate Conjecture 1.1 in terms of the canonical solutions of certain Q-systems of KR type (Conjecture 5.5). Then, we present several character formulae, *all of which are equivalent to Conjecture 5.5.*

5.1. **Quantum affine algebras.** We formulate Conjecture 1.1 in the following setting: Firstly, we translate the conjecture for the KR modules of the *(untwisted) quantum affine algebra* $U_q(X_n^{(1)})$, based on the widely-believed correspondence between the finite-dimensional modules of $Y(X_n)$ and $U_q(X_n^{(1)})$ (for the simply-laced case, see [V]). Secondly, we also include the *twisted* quantum affine algebra case, following [HKOTT].

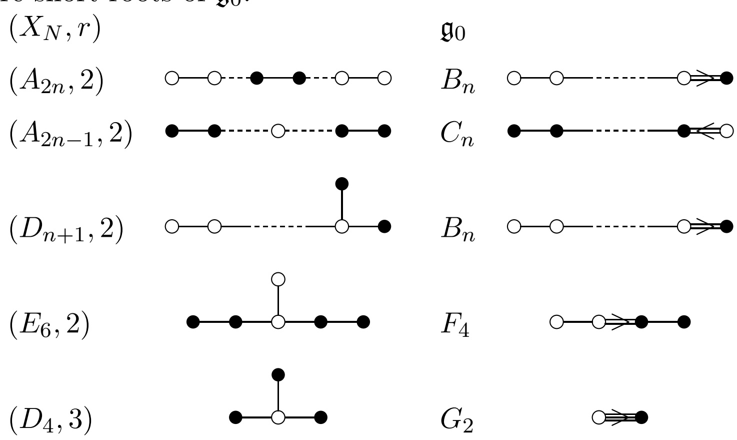
First, we introduce some notations. Let $\mathfrak{g} = X_N$ be a complex simple Lie algebra of rank N. We fix a Dynkin diagram automorphism σ of $\mathfrak{g}$ with $r = \operatorname{ord}\sigma$. Let $\mathfrak{g}_0$ be the σ-invariant subalgebra of $\mathfrak{g}$; namely,

(5.1)

$\mathfrak{g}$	X_n	A_{2n}	A_{2n-1}	D_{n+1}	E_6	D_4
r	1	2	2	2	2	3
$\mathfrak{g}_0$	X_n	B_n	C_n	B_n	F_4	G_2

See Figure 1. Let $A' = (A'_{ij})$ $(i,j \in I)$ and $A = (A_{ij})$ $(i,j \in I_\sigma)$ be the Cartan matrices of $\mathfrak{g}$ and $\mathfrak{g}_0$, respectively, where I_σ is the set of the σ-orbits on I. We define the numbers d'_i, d_i, ϵ'_i, ϵ_i $(i \in I)$ as follows: d'_i $(i \in I)$ are coprime positive integers such that $(d'_i A'_{ij})$ is symmetric; d_i $(i \in I_\sigma)$ are coprime positive integers such that $(d_i A_{ij})$ is symmetric, and we set $d_i = d_{\pi(i)}$ $(i \in I)$, where $\pi : I \to I_\sigma$ is the canonical projection; $\epsilon'_i = r$ if $\sigma(i) = i$, and 1 otherwise; $\epsilon_i = 2$ if $A'_{i\sigma(i)} < 0$, and 1 otherwise. It immediately follows that $d'_i = d_i$ and $\epsilon'_i = 1$ if $r = 1$; $d'_i = 1$ if $r > 1$; $\epsilon_i = 1$ if $X_N^{(r)} \neq A_{2n}^{(2)}$. It is easy to check the following relations: Set $\kappa_0 = 2$ if

FIGURE 1. The Dynkin diagrams of X_N and $\mathfrak{g}_0$ for $r > 1$. The filled circles in X_N correspond to the ones in $\mathfrak{g}_0$ which are short roots of $\mathfrak{g}_0$.



$X_N^{(r)} = A_{2n}^{(2)}$, and 1 otherwise. Then,

$$\kappa_0 d'_i \sum_{s=1}^{r} A'_{i\sigma^s(j)} = d_i A_{\pi(i)\pi(j)}, \tag{5.2}$$

$$\kappa_0 \epsilon'_i d'_i = \epsilon_i d_i. \tag{5.3}$$

For $q \in \mathbb{C}^\times$, we set $q'_i = q^{\kappa_0 d'_i}$, $q_i = q^{d_i}$, and $[n]_q = (q^n - q^{-n})/(q - q^{-1})$.

We use the "second realization" of the quantum affine algebra $U_q = U_q(X_N^{(r)})$ [D2, J] with the generators $X_{ik}^{\pm}$ $(i \in I, k \in \mathbb{Z})$, H_{ik} $(i \in I, k \in \mathbb{Z} \setminus \{0\})$, $K_i^{\pm 1}$ $(i \in I)$, and the central elements $c^{\pm 1/2}$. As far as finite-dimensional U_q-modules are concerned, we can set $c^{\pm 1/2} = 1$. Some of the defining relations in the quotient (the *quantum loop algebra*) $U_q/(c^{\pm 1/2} - 1)$ are presented below to fix notations (here we follow the convention in [CP2, CP3]):

$$X^{\pm}_{\sigma(i)k} = \omega^k X^{\pm}_{ik}, \quad H_{\sigma(i)k} = \omega^k H_{ik}, \quad K^{\pm 1}_{\sigma(i)k} = K^{\pm 1}_{ik}, \tag{5.4}$$

$$K_i X^{\pm}_{jk} K_i^{-1} = q^{\pm \kappa_0 d'_i \sum_{s=1}^{r} A'_{i\sigma^s(j)}} X^{\pm}_{jk}, \tag{5.5}$$

$$[H_{ik}, X^{\pm}_{jl}] = \pm \frac{1}{k} \left(\sum_{s=1}^{r} [k \kappa_0 d'_i A'_{i\sigma^s(j)}]_q \omega^{sk} \right) X^{\pm}_{j,k+l}, \tag{5.6}$$

$$[X^{+}_{ik}, X^{-}_{jl}] = \left(\sum_{s=1}^{r} \delta_{\sigma^s(i),j} \omega^{sl} \right) \frac{\Psi^{+}_{i,k+l} - \Psi^{-}_{i,k+l}}{q_i - q_i^{-1}}, \tag{5.7}$$

where $\omega = \exp(2\pi i/r)$, and $\Psi^{\pm}_{ik}$ $(i \in I, k \in \mathbb{Z})$ are defined by

$$\sum_{k=0}^{\infty} \Psi^{\pm}_{i,\pm k} u^k = K_i^{\pm 1} \exp\left(\pm (q_i - q_i^{-1}) \sum_{l=1}^{\infty} H_{i,\pm l} u^l \right) \tag{5.8}$$

with $\Psi^{\pm}_{ik} = 0$ $(\pm k < 0)$.

Remark 5.1. In [CP3], there are some misprints which are relevant here. Namely, the relation $[H_{ik}, X^{\pm}_{jl}]$ should read (5.6) here; in Proposition 2.2 and Theorem 3.1 (ii), q should read q_i for such i that $\sigma(i) \neq i$ and $a_{i\sigma(i)} \neq 0$ therein. We thank V. Chari for the correspondence concerning these points.

Let $V(\psi^{\pm})$ denote the irreducible U_q-module with a highest weight vector v and the highest weight $\psi^{\pm} = (\psi^{\pm}_{ik})$, namely,

$$X^{+}_{ik} v = 0, \tag{5.9}$$

$$\Psi^{\pm}_{ik} v = \psi^{\pm}_{ik} v, \quad \psi^{\pm}_{ik} \in \mathbb{C}. \tag{5.10}$$

The following theorem gives the classification of the finite-dimensional U_q-modules:

Theorem 5.2 (Theorem 3.3 [CP2], Theorem 3.1 [CP3]). *The $U_q(X_N^{(r)})$-module $V(\psi^\pm)$ is finite-dimensional if and only if there exist N-tuple of polynomials $(P_i(u))_{i\in I}$ with the unit constant terms such that*

$$\sum_{k=0}^{\infty} \psi_{ik}^{+} u^k = \sum_{k=0}^{\infty} \psi_{i,-k}^{-} u^{-k} = {q'_i}^{\epsilon'_i \deg P_i} \frac{P_i({q'_i}^{-2\epsilon'_i} u^{\epsilon'_i})}{P_i(u^{\epsilon'_i})}, \tag{5.11}$$

where the first two terms are the Laurent expansions of the third term about $u = 0$ and $u = \infty$, respectively.

The polynomials $(P_i(u))_{i\in I}$ are called the *Drinfeld polynomials* of $V(\psi^\pm)$. It follows from (5.3), (5.10), and (5.11) that

$$K_i^{\pm 1} v = {q'_i}^{\pm\epsilon'_i \deg P_i} v = q_i^{\pm\epsilon_i \deg P_i} v. \tag{5.12}$$

5.2. **The KR modules.** We take an inclusion $\iota : I_\sigma \hookrightarrow I$ such that $\pi \circ \iota = \mathrm{id}$, and regard I_σ as a subset of I. Let us label the set I_σ with $\{1, \dots, n\}$. The Drinfeld polynomials (5.11) satisfy the relation $P_{\sigma(i)}(u) = P_i(\omega u)$ $(\sigma(i) \neq i)$ by (5.4) and (5.8). Therefore, it is enough to specify the polynomials $P_i(u)$ only for those $i \in \{1, \dots, n\} \subset I$.

We set $H = \{1, \dots, n\} \times \mathbb{N}$ as in (4.1).

Definition 5.3. For each $(a, m) \in H$ and $\zeta \in \mathbb{C}^\times$, let $W_m^{(a)}(\zeta)$ be the finite-dimensional irreducible U_q-module whose Drinfeld polynomials $P_b(u)$ $(b = 1, \dots, n)$ are specified as follows: $P_b(u) = 1$ for $b \neq a$, and

$$P_a(u) = \prod_{k=1}^{m} (1 - \zeta {q'_a}^{\epsilon'_a (m+2-2k)} u). \tag{5.13}$$

We call $W_m^{(a)}(\zeta)$ a *KR (Kirillov-Reshetikhin) module.*

By (5.2) and (5.5), we see that $X_{a0}^{\pm}$ and $K_a^{\pm 1}$ $(a = 1, \dots, n)$ generate the subalgebra $U_q(\mathfrak{g}_0)$. It is well known that all $W_m^{(a)}(\zeta)$ $(\zeta \in \mathbb{C}^\times)$ share the same $U_q(\mathfrak{g}_0)$-module structure. If we set $K_a^{\pm 1} = q_a^{\pm H_a}$ and take the limit $q \to 1$, $X_{a0}^{\pm}$ and H_a $(a = 1, \dots, n)$ generate the Lie algebra $\mathfrak{g}_0$. Accordingly, $W_m^{(a)}(\zeta)$ is equipped with $\mathfrak{g}_0$-module structure. We call its $\mathfrak{g}_0$-character the $\mathfrak{g}_0$-*character of* $W_m^{(a)}(\zeta)$. The $\mathfrak{g}_0$-highest weight of $W_m^{(a)}(\zeta)$, in the same sense as above, is $m\epsilon_a \Lambda_a$ by (5.12) and (5.13).

5.3. **The Kirillov-Reshetikhin conjecture.** We define the matrix $G' = (G'_{am,bk})_{(a,m),(b,k)\in H}$ with the entry

$$G'_{am,bk} = \sum_{s=1}^{r} \frac{d'_b}{\epsilon'_b} A'_{b\sigma^s(a)} \min\Big(\frac{m}{d'_b}, \frac{k}{d'_a}\Big) \tag{5.14}$$

$$= \begin{cases} d_b A_{ba} \min(\frac{m}{d_b}, \frac{k}{d_a}) & r = 1 \\ \frac{1}{\epsilon_b} A_{ba} \min(m, k) & r > 1. \end{cases} \tag{5.15}$$

It follows from (5.15) and Example 4.6 that G' satisfies the condition (KR-II) in Definition 4.5 with $g_{ab} = A_{ba}/\epsilon_b$. Below, we consider the Q-system of KR type with the matrix $G := G'D$, where D is the matrix in (4.12). By using (A.6) of [KN2]), the entry of G is explicitly written as

$$G_{am,bk} = \begin{cases} -\frac{1}{\epsilon_b} A_{ba}\delta_{m,k} & r > 1 \\ -A_{ba}(\delta_{m,2k-1} + 2\delta_{m,2k} + \delta_{m,2k+1}) & d_b/d_a = 2 \\ -A_{ba}(\delta_{m,3k-2} + 2\delta_{m,3k-1} + 3\delta_{m,3k} & d_b/d_a = 3 \\ \qquad\qquad +2\delta_{m,3k+1} + \delta_{m,3k+2}) & \\ -A_{ab}\delta_{d_a m, d_b k} & \text{otherwise.} \end{cases} \tag{5.16}$$

Let α_a and Λ_a $(a = 1, \dots, n)$ be the simple roots and the fundamental weights of $\mathfrak{g}_0$. We set

$$x_a = e^{\epsilon_a \Lambda_a}, \quad y_a = e^{-\alpha_a}. \tag{5.17}$$

Then, they satisfy the relation (4.20) for the above g_{ab}; namely,

$$y_a = \prod_{b=1}^{n} x_b^{-A_{ba}/\epsilon_b}. \tag{5.18}$$

Definition 5.4. Let $\mathbf{Q}_m^{(a)}(x)$ be the Laurent polynomial of $x = (x_a)_{a=1}^n$ representing the $\mathfrak{g}_0$-character of the KR module $W_m^{(a)}(\zeta)$. Then, $\mathcal{Q}_m^{(a)}(y) := (x_a)^{-m}\mathbf{Q}_m^{(a)}(x)|_{x=x(y)}$, where $x(y)$ is the inverse map of (5.18), is a polynomial of $y = (y_a)_{a=1}^n$ with the unit constant term. We call $\mathcal{Q}_m^{(a)}(y)$ the *normalized* $\mathfrak{g}_0$*-character* of $W_m^{(a)}(\zeta)$.

Now we present a reformulation of Conjecture 1.1. This is the main statement of the paper.

Conjecture 5.5. *Let* $\mathcal{Q}_m^{(a)}(y)$ *be the normalized* $\mathfrak{g}_0$*-character of the KR module* $W_m^{(a)}(\zeta)$ *of* $U_q(X_N^{(r)})$. *Then, the family* $(\mathcal{Q}_m^{(a)}(y))_{(a,m)\in H}$ *is characterized as the canonical solution of the Q-system of KR type (4.14) with G given in (5.16).*

Let $\mathcal{Q}^\nu(y) = \prod_{(a,m)}(\mathcal{Q}_m^{(a)}(y))^{\nu_m^{(a)}}$ for $\nu \in \boldsymbol{\nu}(D)$. By Theorem 4.3, Conjecture 5.5 is equivalent to

Conjecture 5.6 ([KN2])**.** *The formulae*

$$\mathcal{Q}^\nu(y) = \mathcal{K}_{D,G}^\nu(y)/\mathcal{K}_{D,G}^0(y) = \mathcal{R}_{D,G}^\nu(y) \tag{5.19}$$

hold, where $\mathcal{K}_{D,G}^\nu(y)$ *and* $\mathcal{R}_{D,G}^\nu(y)$ *are the power series in (4.11) with D in (4.12) and G in (5.16). Therefore,* $\mathcal{R}_{D,G}^\nu(y)$ *is a* polynomial *of y, and its coefficients are identified with the* $\mathfrak{g}_0$*-weight multiplicities of the tensor product* $\bigotimes_{(a,m)\in H} W_m^{(a)}(\zeta_m^{(a)})^{\otimes \nu_m^{(a)}}$, *where* $\zeta_m^{(a)}$ *are arbitrary.*

5.4. **Equivalence to Conjecture 1.1.** Let $\Delta_+^{\mathfrak{g}}$ denote the set of all the positive roots of $\mathfrak{g}$. Originally, Conjecture 5.5 is formulated for $X_N^{(r)} \neq A_{2n}^{(2)}$ as follows (cf. Conjecture 1.1):

Conjecture 5.7 ([K1, K2, HKOTY, HKOTT]). *For $X_N^{(r)} \neq A_{2n}^{(2)}$, the formula*

$$\mathcal{Q}^{\nu}(y) = \frac{\mathcal{K}_{D,G}^{\nu}(y)}{\prod_{\alpha \in \Delta_+^{\mathfrak{g}_0}} (1 - e^{-\alpha})} \tag{5.20}$$

holds, where $\mathcal{K}_{D,G}^{\nu}(y)$ is the power series in (4.11) with D in (4.12) and G in (5.16). Therefore, $\mathcal{K}_{D,G}^{\nu}(y)$ is a polynomial *of y, and its coefficients are identified with the multiplicities of the $\mathfrak{g}_0$-irreducible components of the tensor product $\bigotimes_{(a,m)\in H} W_m^{(a)}(\zeta_m^{(a)})^{\otimes \nu_m^{(a)}}$, where $\zeta_m^{(a)}$ are arbitrary.*

Proof of the equivalence between Conjectures 5.6 and 5.7 for $X_N^{(r)} \neq A_{2n}^{(2)}$. Suppose that Conjecture 5.7 holds. Then, setting $\nu = 0$ in (5.20), we have

$$\mathcal{K}_{D,G}^{0}(y) = \prod_{\alpha \in \Delta_+^{\mathfrak{g}_0}} (1 - e^{-\alpha}). \tag{5.21}$$

Therefore, $\mathcal{Q}^{\nu}(y) = \mathcal{K}_{D,G}^{\nu}(y)/\mathcal{K}_{D,G}^{0}(y)$ holds. Conversely, suppose that the family of the normalized $\mathfrak{g}_0$-characters $(\mathcal{Q}_m^{(a)}(y))_{(a,m)\in H}$ is the canonical solution of (4.14). Then, the equality (5.21) follows from Proposition 4.9 and the lemma below. □

Lemma 5.8. *Let $\mathfrak{g}$ be a complex simple Lie algebra of rank n, and α_a and Λ_a be the simple roots and the fundamental weights of $\mathfrak{g}$. We set $x_a = e^{\Lambda_a}$, $y_a = e^{-\alpha_a/k_a}$, where k_a $(a = 1, \dots, n)$ are 1 or 2. Suppose that $f_a(y)$ $(a = 1, \dots, n)$ are polynomials of y with the unit constant terms such that $\mathbf{f}_a(x) = x_a f_a(y(x))$ are invariant under the action of the Weyl group of $\mathfrak{g}$. Then,*

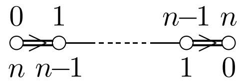
$$\det_{1\le a,b\le n} \Bigl(\frac{\partial \mathbf{f}_a}{\partial x_b}(x) \Bigr) = \prod_{\alpha \in \Delta_+^{\mathfrak{g}}} (1 - e^{-\alpha}). \tag{5.22}$$

Proof. The proof is the same as the one for Lemma 8.6 in [HKOTY]. □

In the case $A_{2n}^{(2)}$, (5.21) does not hold under Conjecture 5.6, because the assumption in Lemma 5.8 is not satisfied by (5.17). We treat the case $A_{2n}^{(2)}$ separately below.

5.5. **The $A_{2n}^{(2)}$ case.**

Figure 2. The Dynkin diagram of $A^{(2)}_{2n}$. The upper and lower labels respect the subalgebra B_n and C_n, respectively.



5.5.1. *The B_n-character.* For $A^{(2)}_{2n}$, $\mathfrak{g}_0 = B_n$. Let $\{1,\dots,n\}$ label I_σ as the upper label in Figure 2. Accordingly, $\epsilon_a = 1$ for $a = 1,\dots,n-1$, and 2 for $a = n$. We continue to set $y_a = e^{-\alpha_a}$ as in Section 5.3. We will show later, in (5.34) and (5.36), that under Conjecture 5.5 the following formula holds instead of the formula (5.21):

$$\mathcal{K}^0_{D,G}(y) = \prod_{a=1}^{n}\Bigl(1 + \prod_{k=a}^{n} y_k\Bigr) \prod_{\alpha\in\Delta_+^{B_n}} (1 - e^{-\alpha}). \tag{5.23}$$

Therefore, Conjecture 5.5 for $X^{(r)}_N = A^{(2)}_{2n}$ is equivalent to

Conjecture 5.9. *For $X^{(r)}_N = A^{(2)}_{2n}$, the formula*

$$\mathcal{Q}^\nu(y) = \frac{\mathcal{K}^\nu_{D,G}(y)\prod_{a=1}^{n}\bigl(1+\prod_{k=a}^{n} y_k\bigr)^{-1}}{\prod_{\alpha\in\Delta_+^{B_n}} (1 - e^{-\alpha})} \tag{5.24}$$

holds for the normalized B_n-characters of the KR-modules.

5.5.2. *The C_n-character.* As is well-known, $U_q(A^{(2)}_{2n})$ has a realization with the "Chevalley generators" $X^\pm_a$ and $K^{\pm1}_a$ $(a = 0,\dots,n)$ (e.g. [CP3, Proposition 1.1]). Among them, $X^\pm_a$ and $K^{\pm1}_a$ $(a = 1,\dots,n)$ are identified with $X^\pm_{a0}$, $K^{\pm1}_a$ in (5.4)–(5.8), and generate the subalgebra $U_q(B_n)$. On the other hand, $X^\pm_a$ and $K^{\pm1}_a$ $(a = 0,\dots,n-1)$ generate the subalgebra $U_{q^2}(C_n)$. See Figure 2. If we set $K_a = q_a^{H_a}$ $(a = 0,\dots,n-1)$, where $q_0 = q^{d_0}$, $d_0 = 4$, then $X^\pm_a$ and H_a $(a = 0,\dots,n-1)$ generate the Lie algebra C_n in the limit $q \to 1$. This provides $W^{(a)}_m(\zeta)$ with the C_n-module structure, by which the *C_n-character of* $W^{(a)}_m(\zeta)$ is defined.

Let $\dot\alpha_a$ and $\dot\Lambda_a$ $(a = 1,\dots,n)$ be the simple roots and the fundamental weights labeled with the lower label in Figure 2. By looking at the same U_q-module as B_n and C_n-modules as above, a linear bijection $\phi : \mathfrak{h}^* \to \dot{\mathfrak{h}}^*$ is induced, where $\mathfrak{h}^*$ and $\dot{\mathfrak{h}}^*$ are the duals of the Cartan subalgebras of B_n and C_n, respectively.

Lemma 5.10. *Under the bijection ϕ, we have the correspondence* $(\dot\Lambda_0 = 0)$*:*

$$\epsilon_a\Lambda_a \mapsto \dot\Lambda_{n-a} - \dot\Lambda_n, \tag{5.25}$$

$$\alpha_a \mapsto \begin{cases} \dot\alpha_{n-a} & a = 1,\dots,n-1 \\ -(\dot\alpha_1 + \cdots + \dot\alpha_{n-1} + \frac{1}{2}\dot\alpha_n) & a = n. \end{cases} \tag{5.26}$$

Proof. It is obtained from the relations among H_i and α_i for $A^{(2)}_{2n}$ [Kac]:

$$0 = c = \sum_{i=0}^{n} a_i^\vee H_i, \quad 0 = \delta = \sum_{i=0}^{n} a_i \alpha_i, \tag{5.27}$$

where $(a_0^\vee, \dots, a_n^\vee) = (2, \dots, 2, 1)$ and $(a_0, \dots, a_n) = (1, 2, \dots, 2)$ for the upper label in Figure 2. □

Let $\mathcal{W}(X_n)$ denote the Weyl group of X_n.

Lemma 5.11. *There is an element $s \in \mathcal{W}(C_n)$ which acts on $\dot{\mathfrak{h}}^*$ as follows:*

$$\phi(\epsilon_a \Lambda_a) \mapsto \dot{\Lambda}_a \quad (a = 1, \dots, n), \tag{5.28}$$

$$\phi(\alpha_a) \mapsto \frac{1}{\epsilon_a} \dot{\alpha}_a \quad (a = 1, \dots, n). \tag{5.29}$$

Proof. We take the standard orthonormal basis ε_a of $\dot{\mathfrak{h}}^*$. Let s be the element such that $s : \varepsilon_a \mapsto -\varepsilon_{n-a+1}$. Then,

$$\dot{\Lambda}_{n-a} - \dot{\Lambda}_n = -(\varepsilon_{n-a+1} + \dots + \varepsilon_n) \mapsto \varepsilon_1 + \dots + \varepsilon_a = \dot{\Lambda}_a, \tag{5.30}$$

$$\dot{\alpha}_{n-a} = \varepsilon_{n-a} - \varepsilon_{n-a+1} \mapsto \varepsilon_a - \varepsilon_{a+1} = \dot{\alpha}_a \quad (a = 1, \dots, n-1), \tag{5.31}$$

$$-(\dot{\alpha}_1 + \dots + \dot{\alpha}_{n-1} + \frac{1}{2}\dot{\alpha}_n) = -\varepsilon_1 \mapsto \varepsilon_n = \frac{1}{2}\dot{\alpha}_n. \tag{5.32}$$

□

According to (5.30)–(5.32), we set

$$x_a = e^{\dot{\Lambda}_a}, \quad y_a = e^{-\dot{\alpha}_a/\epsilon_a}. \tag{5.33}$$

Then, the relation (5.18) is preserved, since ϕ and s above are linear. Lemma 5.11 assures that the following definition is well-defined.

Definition 5.12. Let $\mathbf{Q}^{(a)}_m(x)$ be the Laurent polynomial of $x = (x_a)_{a=1}^n$ representing the C_n-character of the KR module $W^{(a)}_m(\zeta)$. Then, $\mathcal{Q}^{(a)}_m(y) := (x_a)^{-m} \mathbf{Q}^{(a)}_m(x)|_{x=x(y)}$ is a polynomial of $y = (y_a)_{a=1}^n$ with the unit constant term. We call $\mathcal{Q}^{(a)}_m(y)$ the *normalized C_n-character* of $W^{(a)}_m(\zeta)$.

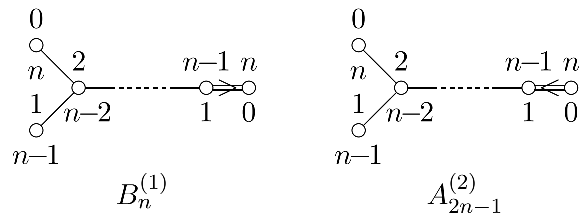
Moreover, by Lemma 5.11 and the $\mathcal{W}(C_n)$-invariance of the C_n-character of $W^{(a)}_m(\zeta)$, we have

Proposition 5.13. *The normalized B_n-character and the normalized C_n-character of $W^{(a)}_m(\zeta)$ of $U_q(A^{(2)}_{2n})$ coincide as polynomials of y.*

Thus, Conjecture 5.5 for the normalized B_n-characters of $A^{(2)}_{2n}$ is applied for the normalized C_n-characters as well. Furthermore, in contrast to the B_n case, Lemma 5.8 is now applicable for (5.33). Therefore, under Conjecture 5.5, we have

$$\mathcal{K}^0_{D,G}(y) = \prod_{\alpha \in \Delta_+^{C_n}} (1 - e^{-\alpha}). \tag{5.34}$$

FIGURE 3. The Dynkin diagrams of $B_n^{(1)}$ and $A_{2n-1}^{(2)}$. The upper and lower labels respect the subalgebra $\mathfrak{g}_0$ and $\dot{\mathfrak{g}}$, respectively.



Hence, we conclude that Conjecture 5.5 for $X_N^{(r)} = A_{2n}^{(2)}$ is also equivalent to

Conjecture 5.14 ([HKOTT]). *For $X_N^{(r)} = A_{2n}^{(2)}$, the formula*

$$\mathcal{Q}^\nu(y) = \frac{\mathcal{K}^\nu_{D,G}(y)}{\prod_{\alpha\in\Delta_+^{C_n}}(1-e^{-\alpha})} \tag{5.35}$$

holds for the normalized C_n-characters of the KR-modules, where y is specified as (5.33).

The following relation is easily derived from the explicit expressions of the Weyl denominators of B_n and C_n (e.g. [FH]):

$$\prod_{\alpha\in\Delta_+^{C_n}}(1-e^{-\alpha}) = \prod_{a=1}^{n}\Big(1+\prod_{k=a}^{n} y_k\Big)\prod_{\alpha\in\Delta_+^{B_n}}(1-e^{-\alpha}), \tag{5.36}$$

where the equality holds under the following identifications: $y_a = e^{-\dot{\alpha}_a/\epsilon_a}$ for the LHS and $y_a = e^{-\alpha_a}$ for the RHS under the label in Figure 2. From (5.34) and (5.36), we obtain (5.23).

5.6. **Characters for the rank n subalgebras.** The procedure to deduce the C_n-characters from the B_n-characters for $A_{2n}^{(2)}$ in Section 5.5 is also applicable to the $\dot{\mathfrak{g}}$-characters for any rank n subalgebra $\dot{\mathfrak{g}} \neq \mathfrak{g}_0$ of $X_N^{(r)}$. (The characters of the lower rank subalgebras are obtained by their specializations.) Let us demonstrate how it works in two examples:

Case I. $X_N^{(r)} = B_n^{(1)}$, $\mathfrak{g}_0 = B_n$, $\dot{\mathfrak{g}} = D_n$.

Case II. $X_N^{(r)} = A_{2n-1}^{(2)}$, $\mathfrak{g}_0 = C_n$, $\dot{\mathfrak{g}} = D_n$.

Let α_a and Λ_a (resp. $\dot{\alpha}_a$ and $\dot{\Lambda}_a$) ($a = 1, \dots, n$) be the simple roots and the fundamental weights of $\mathfrak{g}_0$ (resp. $\dot{\mathfrak{g}}$) labeled with the upper (resp. lower) label in Figure 3. As in Section 5.5, a linear bijection $\phi : \mathfrak{h}^* \to \dot{\mathfrak{h}}^*$ is induced, where $\mathfrak{h}^*$ and $\dot{\mathfrak{h}}^*$ are the duals of the Cartan subalgebras of $\mathfrak{g}_0$ and $\dot{\mathfrak{g}}$, respectively.

Doing a similar calculation to Lemmas 5.10 and 5.11, we have

Lemma 5.15. *Under the bijection ϕ, we have the correspondence ($\dot{\Lambda}_0 = 0$):*
Case I.

$$\Lambda_a \mapsto \begin{cases} \dot{\Lambda}_{n-a} - \dot{\Lambda}_n & a = 1, n \\ \dot{\Lambda}_{n-a} - 2\dot{\Lambda}_n & a = 2, \dots, n-1, \end{cases} \tag{5.37}$$

$$\alpha_a \mapsto \begin{cases} \dot{\alpha}_{n-a} & a = 1, \dots, n-1 \\ -\frac{1}{2}(2\dot{\alpha}_1 + \cdots + 2\dot{\alpha}_{n-2} + \dot{\alpha}_{n-1} + \dot{\alpha}_n) & a = n. \end{cases} \tag{5.38}$$

Case II.

$$\Lambda_a \mapsto \begin{cases} \dot{\Lambda}_{n-1} - \dot{\Lambda}_n & a = 1 \\ \dot{\Lambda}_{n-a} - 2\dot{\Lambda}_n & a = 2, \dots, n, \end{cases} \tag{5.39}$$

$$\alpha_a \mapsto \begin{cases} \dot{\alpha}_{n-a} & a = 1, \dots, n-1 \\ -(2\dot{\alpha}_1 + \cdots + 2\dot{\alpha}_{n-2} + \dot{\alpha}_{n-1} + \dot{\alpha}_n) & a = n. \end{cases} \tag{5.40}$$

Lemma 5.16. *There is an element $s \in \mathcal{W}(D_n)$ which acts on $\dot{\mathfrak{h}}^*$ as follows:*
Case I.

$$\phi(\Lambda_a) \mapsto \begin{cases} \dot{\Lambda}_a & a = 1, \dots, n-2, n \\ \dot{\Lambda}_{n-1} + \dot{\Lambda}_n & a = n-1, \end{cases} \tag{5.41}$$

$$\phi(\alpha_a) \mapsto \begin{cases} \dot{\alpha}_a & a = 1, \dots, n-1 \\ \frac{1}{2}(-\dot{\alpha}_{n-1} + \dot{\alpha}_n) & a = n. \end{cases} \tag{5.42}$$

Case II.

$$\phi(\Lambda_a) \mapsto \begin{cases} \dot{\Lambda}_a & a = 1, \dots, n-2 \\ \dot{\Lambda}_{n-1} + \dot{\Lambda}_n & a = n-1 \\ 2\dot{\Lambda}_n & a = n, \end{cases} \tag{5.43}$$

$$\phi(\alpha_a) \mapsto \begin{cases} \dot{\alpha}_a & a = 1, \dots, n-1 \\ -\dot{\alpha}_{n-1} + \dot{\alpha}_n & a = n. \end{cases} \tag{5.44}$$

Accordingly, we set
Case I.

$$x_a = e^{\dot{\Lambda}_a} \ (a = 1, \dots, n-2, n), \ e^{\dot{\Lambda}_{n-1} + \dot{\Lambda}_n} \ (a = n-1), \tag{5.45}$$

$$y_a = e^{-\dot{\alpha}_a} \ (a = 1, \dots, n-1), \ e^{(\dot{\alpha}_{n-1} - \dot{\alpha}_n)/2} \ (a = n). \tag{5.46}$$

Case II.

$$x_a = e^{\dot{\Lambda}_a} \ (a = 1, \dots, n-2), \ e^{\dot{\Lambda}_{n-1} + \dot{\Lambda}_n} \ (a = n-1), \ e^{2\dot{\Lambda}_n} \ (a = n), \tag{5.47}$$

$$y_a = e^{-\dot{\alpha}_a} \ (a = 1, \dots, n-1), \ e^{\dot{\alpha}_{n-1} - \dot{\alpha}_n} \ (a = n). \tag{5.48}$$

Then, the relation (5.18) is preserved. Define the $\dot{\mathfrak{g}}$-characters of $W^{(a)}_m(\zeta)$ in the same way as Definition 5.12. Then, the normalized $\mathfrak{g}_0$-character and the normalized $\dot{\mathfrak{g}}$-character of $W^{(a)}_m(\zeta)$ coincide as polynomials of y. Thus, Conjecture 5.5 for the normalized $\mathfrak{g}_0$-characters is applied for the normalized

$\dot{\mathfrak{g}}$-characters as well. So far, the situation is parallel to the C_n case for $A^{(2)}_{2n}$. From now on, the situation is parallel to the B_n case for $A^{(2)}_{2n}$. The following relations are easily derived from the explicit expressions of the Weyl denominators for B_n, C_n, D_n:

$$\prod_{\alpha\in\Delta_+^{B_n}}(1-e^{-\alpha}) = \prod_{a=1}^{n}\Bigl(1-\prod_{k=a}^{n}y_k\Bigr)\prod_{\alpha\in\Delta_+^{D_n}}(1-e^{-\alpha}), \tag{5.49}$$

$$\prod_{\alpha\in\Delta_+^{C_n}}(1-e^{-\alpha}) = \prod_{a=1}^{n}\Bigl(1-y_n^{-1}\prod_{k=a}^{n}y_k^2\Bigr)\prod_{\alpha\in\Delta_+^{D_n}}(1-e^{-\alpha}), \tag{5.50}$$

where the equalities hold under the following identifications: (5.17) for the LHSs, (5.46) for the RHS of (5.49), (5.48) for the RHS of (5.50) under the label in Figure 3. We conclude that Conjecture 5.5 for $B^{(1)}_n$ and $A^{(2)}_{2n-1}$ is equivalent to

Conjecture 5.17. *(i) For $B^{(1)}_n$, the formula*

$$\mathcal{Q}^{\nu}(y) = \frac{\mathcal{K}^{\nu}_{D,G}(y)\prod_{a=1}^{n}\bigl(1-\prod_{k=a}^{n}y_k\bigr)^{-1}}{\prod_{\alpha\in\Delta_+^{D_n}}(1-e^{-\alpha})} \tag{5.51}$$

holds for the D_n-characters of the KR-modules, where y is specified as (5.46).

(ii) For $A^{(2)}_{2n-1}$, the formula

$$\mathcal{Q}^{\nu}(y) = \frac{\mathcal{K}^{\nu}_{D,G}(y)\prod_{a=1}^{n}\bigl(1-y_n^{-1}\prod_{k=a}^{n}y_k^2\bigr)^{-1}}{\prod_{\alpha\in\Delta_+^{D_n}}(1-e^{-\alpha})} \tag{5.52}$$

holds for the D_n-characters of the KR-modules, where y is specified as (5.48).

The manifest polynomial expressions of the numerators in the RHSs of (5.24), (5.51), and (5.52) for $\mathcal{Q}^{(a)}_m(y)$ are available in [HKOTT] with some other examples.

5.7. **Related works.** Below we list the related works on Conjectures 1.1 and 5.5–5.7 mostly chronologically. However, the list is by no means complete. The series $\mathcal{K}^{\nu}_{D,G}(y)$ in (5.20) admits a natural q-analogue called the *fermionic formula*. This is another fascinating subject, but we do not cover it here. See [BS, HKOTY, HKOTT] and reference therein. It is convenient to refer the formula (5.20) with the binomial coefficient (2.9) as *type I*, and the ones with the binomial coefficient in Remark 1.3 as *type II*. (In the context of the XXX-type integrable spin chains, $N^{(a)}_m$ and $P^{(a)}_m$ represent the numbers of *m-strings* and *m-holes* of color a, respectively. Therefore one must demand $P^{(a)}_m \geq 0$, which implies that the relevant formulae are necessarily of type II.) The manifest expression of the decomposition of $\mathbf{Q}^{(a)}_m$

such as

$$\mathbf{Q}_1^{(2)} = \chi(\Lambda_2) + \chi(\Lambda_5) \tag{5.53}$$

is referred as *type III*, where $\chi(\lambda)$ is the character of the irreducible X_n-module $V(\lambda)$ with highest weight λ. Since there is no essential distinction between these conjectured formulae for $Y(X_n)$ and $U_q(X_n^{(1)})$, we simply refer the both cases as X_n below. At this moment, however, the proofs should be separately given for nonsimply-laced case [V].

0 [Be]. Bethe solved the XXX spin chain of length N by inventing what is later known as the Bethe ansatz and the string hypothesis. As a check of the completeness of his eigenvectors for the XXX Hamiltonian, he proved, in our terminology, the type II formula of $\mathcal{Q}^\nu(y)$ with $\nu_m^{(1)} = N\delta_{m1}$ for A_1. See [F, FT] for a readable exposition in the framework of the quantum inverse scattering method.

1 [K1, K2]. Kirillov proposed and proved the type I formula of the irreducible modules $V(m\Lambda_a)$ for A_1 [K1] and A_n [K2]. The idea of the use of the generating function and the Q-system, which is extended in the present paper, originates in this work.

2 [KKR]. Kerov *et al.* proposed and proved the type II formula for A_n by the combinatorial method, where the bijection between the Littlewood-Richardson tableaux and the rigged configurations was constructed.

3 [D1]. Drinfeld claimed that $V(m\Lambda_a)$ can be lifted to a $Y(X_n)$-module, if the Kac label for α_a in $X_n^{(1)}$ is 1. These modules are often called the *evaluation modules*, and identified with some KR-modules. A method of proof is given in [C] for $U_q(X_n^{(1)})$. Therefore, the type III formula $\mathbf{Q}_m^{(a)} = \chi(m\Lambda_a)$ holds for those a. Some of the corresponding R-matrices for the classical algebras, $X_n = A_n$, B_n, C_n, D_n, were obtained earlier in [KRS, R] by the *reproduction scheme* (also known as the *fusion procedure*) in the context of the algebraic Bethe ansatz method.

4 [OW]. Ogievetsky and Wiegmann proposed the type III formula of $\mathbf{Q}_1^{(a)}$ for some a for the exceptional algebras from the reproduction scheme.

5 [KR, K3]. Kirillov and Reshetikhin formulated the type II formula for any simple Lie algebra X_n. For that purpose, they vaguely introduced a family of $Y(X_n)$-modules, which we identify with the KR modules here. They proposed the type II formula for any X_n, and the Q-system and the type III formula for $X_n = B_n$, C_n, D_n. The Q-system for exceptional algebras X_n was also proposed in [K3]. Due to the long-term absence of the proofs of the announced results by the authors, we regard these statements as conjectures at our discretion in this paper. See Remark 5.18 for further remark.

Remark 5.18. Let $X_n = B_n, C_n, D_n$. Let $\mathbf{Q}_m^{(a)}$ and $\mathcal{Q}_m^{(a)}$ be the X_n-character and the normalized X_n-character of the "KR module" proposed in [KR]. Then, one can organize the conjectures in [KR] as follows:

(i) All $\mathbf{Q}_m^{(a)}$'s are given by the type III formula in [KR].

(ii) The family $(\mathbf{Q}_m^{(a)})_{(a,m)\in H}$ satisfies the Q-system (4.22) for X_n, and $\mathbf{Q}_1^{(a)}$'s $(a = 1, \dots, n)$ are given by the type III formula in [KR]. (Note that the Q-system (4.22), or equivalently (1.4), recursively determines all $\mathbf{Q}_m^{(a)}$'s from the initial data $(\mathbf{Q}_1^{(a)})_{a=1}^n$.)

(iii) Any power $\mathcal{Q}^\nu$ is given by the type II formula.

As stated in [KR], one can certainly show the equivalence between (i) and (ii) without referring to the KR modules themselves. See [HKOTY]. One can also confirm the equivalence between (i) and a weak version of (iii):

(iii') All $\mathcal{Q}_m^{(a)}$'s are given by the type II formula.

See [Kl] and Appendix A in [HKOTY]. The family $(\mathcal{Q}_m^{(a)})_{(a,m)\in H}$ given by (i) satisfies the convergence property (4.15). Thus, (i), (ii), and (iii') are all equivalent to

(iv) The family $(\mathcal{Q}_m^{(a)})_{(a,m)\in H}$ is the canonical solution of the Q-system (1.4).

Therefore, as shown in Section 5.4 (also [KN2]), they are also equivalent to

(v) Any power $\mathcal{Q}^\nu$ is given by the type I formula (1.1).

This is why we call Conjecture 1.1 the Kirillov-Reshetikhin conjecture. The equivalence between (iii) and the others has not been proved yet as we mentioned in Remark 1.3.

6 [CP1, CP2]. Chari and Pressley proved the type III formula of $\mathbf{Q}_1^{(a)}$ in most cases for $Y(X_n)$ [CP1], and for $U_q(X_n^{(1)})$ [CP2], where the list is complete except for E_7 and E_8.

7 [Ku]. The type III formula of $\mathbf{Q}_m^{(a)}$ was proposed for some a for the exceptional algebras.

8 [Kl]. Kleber analyzed a combinatorial structure of the type II formula for the simply-laced algebras. In particular, it was proved that the type III formula of $\mathbf{Q}_m^{(a)}$ and the corresponding type II formula are equivalent for A_n and D_n.

9 [HKOTY, HKOTT]. Hatayama *et al.* gave a characterization of the type I formula as the solution of the Q-system which are $\mathbb{C}$-linear combinations of the X_n-characters with the property equivalent to the convergence property (4.15). Using it, the equivalence of the type III formula of $\mathbf{Q}_m^{(a)}$ and the type I formula of $\mathcal{Q}^\nu(y)$ for the classical algebras was shown [HKOTY]. In [HKOTT], the type I and type II formulae, and the Q-systems for the twisted algebras $U_q(X_N^{(r)})$ were proposed. The type III formula of $\mathbf{Q}_m^{(a)}$ for $A_{2n}^{(2)}$, $A_{2n-1}^{(2)}$, $D_{n+1}^{(2)}$, $D_4^{(3)}$ was also proposed, and the equivalence to the type I formula was shown in a similar way to the untwisted case.

10 [KN1, KN2]. The second formula in Conjecture 5.6 was proposed and proved for A_1 [KN1] from the formal completeness of the XXZ-type Bethe vectors. The same formula was proposed for X_n, and the equivalence to the type I formula was proved [KN2]. The type I formula is formulated in

the form (5.19), and the characterization of type I formula in [HKOTY] was simplified as the solution of the Q-system with the convergence property (4.15).

11 [C]. Chari proved the type III formula of $\mathbf{Q}^{(a)}_m$ for $U_q(X^{(1)}_n)$ for any a for the classical algebras, and for some a for the exceptional algebras.

12 [OSS]. Okado *et al.* constructed bijections between the rigged configurations and the crystals (resp. virtual crystals) corresponding to $\mathcal{Q}^\nu(y)$, with $\nu^{(a)}_m = 0$ for $m > 1$, for $C^{(1)}_n$ and $A^{(2)}_{2n}$ (resp. $D^{(2)}_{n+1}$). As a corollary, the type II formula of those $\mathcal{Q}^\nu(y)$ was proved for $C^{(1)}_n$ and $A^{(2)}_{2n}$.

Assembling all the above results and the indications to each other, let us summarize the current status of the Kirillov-Reshetikhin conjectures into the following theorem. Here, we mention the results only for the quantum affine algebra $U_q(X^{(r)}_N)$ case. Also, we exclude the isolated results only valid for small m.

Theorem 5.19. *(i) Conjecture 5.5 and the type I formula of $\mathcal{Q}^\nu(y)$ are valid for $A^{(1)}_n$, $B^{(1)}_n$, $C^{(1)}_n$, $D^{(1)}_n$.*

(ii) The type II formula of $\mathcal{Q}^\nu(y)$ is valid for $A^{(1)}_n$ and valid for those ν with $\nu^{(a)}_m = 0$ for $m > 1$ for $C^{(1)}_n$ and $A^{(2)}_{2n}$. The type II formula of $\mathcal{Q}^{(a)}_m(y)$ is valid for the following a in [C]*: any a for $B^{(1)}_n$, $C^{(1)}_n$, $D^{(1)}_n$; $a = 1, 6$ for $E^{(1)}_6$; $a = 7$ for $E^{(1)}_7$.*

(iii) The type III formula of $\mathbf{Q}^{(a)}_m$ is valid for all a for $A^{(1)}_n$, $B^{(1)}_n$, $C^{(1)}_n$, $D^{(1)}_n$, and for those a listed in [C] *for $E^{(1)}_6$, $E^{(1)}_7$, $E^{(1)}_8$, $F^{(1)}_4$, $G^{(1)}_2$. The formula is found in* [C].

Appendix A. The Denominator formulae

We give a proof of Proposition 4.9. The proof is divided into three steps.

A.1. **Step 1. Reduction of the denominator formula.**

In Steps 1 and 2, we consider the unspecialized infinite Q-system (4.8), and we assume that D and G satisfy the condition (KR-II) in Definition 4.5.

For a given positive integer L, let $H_L = \{1, \dots, n\} \times \{1, \dots, L\}$ be the finite subset of H in Section 4.1. With multivariables $v_L = (v^{(a)}_m)_{(a,m)\in H_L}$, $w_L = (w^{(a)}_m)_{(a,m)\in H_L}$, $z_L = (z^{(a)}_m)_{(a,m)\in H_L}$, we define the bijection $v_L \mapsto w_L$ around $v = w = 0$ (cf. (2.1)) by

$$w^{(a)}_m(v_L) = v^{(a)}_m \prod_{(b,k)\in H_L} (1 - v^{(b)}_k)^{-G'_{am,bk}}, \tag{A.1}$$

and the bijection $v_L \mapsto z_L$ around $v = z = 0$ by

$$z_m^{(a)}(v_L) = w_m^{(a)}(v_L) \prod_{(b,k)\in H_L} (1 - v_k^{(b)})^{g_{ab}m} \tag{A.2}$$

$$= v_m^{(a)} \prod_{(b,k)\in H_L} (1 - v_k^{(b)})^{-G'_{am,bk}+g_{ab}m}, \tag{A.3}$$

where g_{ab} is the one in (KR-II). Let us factorize the bijection $w_L \mapsto v_L$ as $w_L \mapsto z_L \mapsto v_L$. The map $w_L \mapsto z_L$ is described as

(A.4)

$$z_m^{(a)}(w_L) = w_m^{(a)} \prod_{b=1}^{n} (Q_b(w_L))^{-g_{ab}m}, \quad Q_b(w_L) := \prod_{k=1}^{L} (1 - v_k^{(b)}(w_L))^{-1}.$$

By the assumption (KR-II) and the expression (A.3), the map $v_L \mapsto z_L$ is lower-triangular in the sense of Example 2.9. Therefore, the following equality holds:

$$\det_{H_L}\Big(\frac{w_k^{(b)}}{v_m^{(a)}} \frac{\partial v_m^{(a)}}{\partial w_k^{(b)}}(w_L)\Big) = \det_{H_L}\Big(\frac{w_k^{(b)}}{z_m^{(a)}} \frac{\partial z_m^{(a)}}{\partial w_k^{(b)}}(w_L)\Big), \tag{A.5}$$

where $\det_{H_L}$ means the abbreviation of $\det_{(a,m),(b,k)\in H_L}$.

We now simultaneously specialize the variables w_L and z_L with the variables $y = (y_a)_{a=1}^n$ and $u = (u_a)_{a=1}^n$ as (cf. (4.5))

$$w_m^{(a)} = w_m^{(a)}(y) = (y_a)^m, \quad z_m^{(a)} = z_m^{(a)}(u) = (u_a)^m. \tag{A.6}$$

This specialization is compatible with (A.4) and the map $y \mapsto u$,

$$u_a(y) = y_a \prod_{b=1}^{n} (q_b(y))^{-g_{ab}}, \quad q_b(y) := Q_b(w_L(y)). \tag{A.7}$$

Proposition A.1. *Let $G'_L = (G'_{am,bk})_{(a,m),(b,k)\in H_L}$ be the H_L-truncation of G', $K^0_{I_L,G'_L}(w_L)$ be the one in (2.34), and $\mathcal{K}^0_{I_L,G'_L}(y) := K^0_{I_L,G'_L}(w_L(y))$ be its specialization by (A.6). Then, the formula (2.34) reduces to*

$$\mathcal{K}^0_{I_L,G'_L}(y) = \det_{1\le a,b\le n}\Big(\frac{y_b}{u_a}\frac{\partial u_a}{\partial y_b}(y)\Big) \prod_{a=1}^{n} q_a(y). \tag{A.8}$$

Proof. Because of (A.5), it is enough to prove the equality

$$\det_{H_L}\Big(\frac{w_k^{(b)}}{z_m^{(a)}} \frac{\partial z_m^{(a)}}{\partial w_k^{(b)}}(w_L(y))\Big) = \det_{1\le a,b\le n}\Big(\frac{y_b}{u_a}\frac{\partial u_a}{\partial y_b}(y)\Big). \tag{A.9}$$

We remark that

$$y_a \frac{\partial}{\partial y_a} = \sum_{m=1}^{L} m w_m^{(a)} \frac{\partial}{\partial w_m^{(a)}}, \tag{A.10}$$

$$\det_{H_L}(\delta_{am,bk} + m\alpha_{abk}) = \det_{1\le a,b\le n}\Big(\delta_{ab} + \sum_{k=1}^{L} k\alpha_{abk}\Big), \tag{A.11}$$

where α_{abk} are arbitrary constants depending on a, b, k. Set $F_a(w_L) = \prod_{b=1}^{n}(Q_b(w_L))^{-g_{ab}}$. Then, (A.9) is obtained as

$$\begin{aligned}(\mathrm{LHS}) &= \det_{H_L}\Big(\delta_{am,bk} + m w_k^{(b)} \frac{\partial}{\partial w_k^{(b)}} \log F_a(w_L(y))\Big)\\ &= \det_{1\le a,b\le n}\Big(\delta_{ab} + \sum_{k=1}^{L} k w_k^{(b)} \frac{\partial}{\partial w_k^{(b)}} \log F_a(w_L(y))\Big)\\ &= \det_{1\le a,b\le n}\Big(\delta_{ab} + y_b \frac{\partial}{\partial y_b} \log F_a(w_L(y))\Big)\\ &= \det_{1\le a,b\le n}\Big(\frac{y_b}{u_a}\frac{\partial u_a}{\partial y_b}(y)\Big),\end{aligned}$$

where we used (A.4), (A.11), (A.10), and (A.7). □

A.2. **Step 2. Change of variables.** We introduce the change of the variables y and u in (A.6) to $x = (x_a)_{a=1}^n$ and $\mathbf{q} = (\mathbf{q}_a)_{a=1}^n$ as

$$y_a(x) = \prod_{b=1}^{n}(x_b)^{-g_{ab}}, \quad u_a(\mathbf{q}) = \prod_{b=1}^{n}(\mathbf{q}_b)^{-g_{ab}}. \tag{A.12}$$

Thus, if $f(y)$ is a power series of y, then $f(y(x))$ is a Laurent series of x because of the assumption in (KR-II) that g_{ab}'s are integers. This specialization is compatible with (A.7) and the map $x \mapsto \mathbf{q}$,

$$\mathbf{q}_a(x) = x_a q_a(y(x)). \tag{A.13}$$

Let us summarize all the maps and variables in a diagram:

$$\begin{array}{ccccc} v & \overset{(A.2)}{\longleftrightarrow} & z & \overset{(A.4)}{\longleftrightarrow} & w \\ & & {\scriptstyle (A.6)}\uparrow & & \uparrow{\scriptstyle (A.6)} \\ & & u & \overset{(A.7)}{\longleftrightarrow} & y \\ & & {\scriptstyle (A.12)}\uparrow & & \uparrow{\scriptstyle (A.12)} \\ & & \mathbf{q} & \overset{(A.13)}{\longleftrightarrow} & x \end{array} \tag{A.14}$$

With these changes of variables, (A.8) becomes the Jacobian of $\mathbf{q}(x)$:

Proposition A.2. *Let $\mathcal{K}^0_{I_L,G'_L}(y)$ be the one in Proposition A.1, and let $\mathbf{K}^0_{I_L,G'_L}(x) := \mathcal{K}^0_{I_L,G'_L}(y(x))$. Then, the formula*

$$\mathbf{K}^0_{I_L,G'_L}(x) = \det_{1\le a,b\le n}\Big(\frac{\partial \mathbf{q}_a}{\partial x_b}(x)\Big) \tag{A.15}$$

holds.

Proof. By (A.12), we have

$$\det_{1\leq a,b\leq n}\Bigl(\frac{\mathbf{q}_b}{u_a}\frac{\partial u_a}{\partial \mathbf{q}_b}\Bigr) = \det_{1\leq a,b\leq n}\Bigl(\frac{x_b}{y_a}\frac{\partial y_a}{\partial x_b}\Bigr) = \det_{1\leq a,b\leq n}(-g_{ab}) \neq 0. \tag{A.16}$$

Using Proposition A.1, (A.13), and (A.16), we obtain

$$\begin{aligned} \mathbf{K}^0_{I_L,G'_L}(x) &= \det_{1\leq a,b\leq n}\Bigl(\frac{y_b}{u_a}\frac{\partial u_a}{\partial y_b}(y(x))\Bigr)\prod_{a=1}^{n} q_a(y(x)) \\ &= \det_{1\leq a,b\leq n}\Bigl(\frac{x_b}{\mathbf{q}_a}\frac{\partial \mathbf{q}_a}{\partial x_b}(x)\Bigr)\prod_{a=1}^{n} q_a(y(x)) = \det_{1\leq a,b\leq n}\Bigl(\frac{\partial \mathbf{q}_a}{\partial x_b}(x)\Bigr). \end{aligned} \tag{A.17}$$

□

A.3. **Step 3. Denominator formula for the Q-systems for KR type.** Now we are ready to prove Proposition 4.9; namely,

Proposition A.3. *Let $\mathbf{K}^0_{D,G}(x) := \mathcal{K}^0_{D,G}(y(x))$, where $\mathcal{K}^0_{D,G}(y)$ is the denominator in (4.11) for the Q-system of KR type (4.14). Then, the formula*

$$\mathbf{K}^0_{D,G}(x) = \det_{1\leq a,b\leq n}\Bigl(\frac{\partial \mathbf{Q}^{(a)}_1}{\partial x_b}(x)\Bigr) \tag{A.18}$$

holds, where we set $\mathbf{Q}^{(a)}_1(x) := x_a \mathcal{Q}^{(a)}_1(y(x))$ for the canonical solution $(\mathcal{Q}^{(a)}_m(y))_{(a,m)\in H}$ of (4.14).

Proof. We recall the following four facts:

Fact 1: By (3.33) and (4.6), we have

$$\mathcal{K}^0_{D,G}(y) \equiv \mathcal{K}^0_{I_L,G'_L}(y) \mod J_L. \tag{A.19}$$

Fact 2: By Theorem 3.7 and the proof therein, the canonical solution $(\mathcal{Q}^{(a)}_m(y))_{(a,m)\in H}$ of (4.14) and the solution $(\mathcal{Q}'^{(a)}_m(y))_{(a,m)\in H}$ of the corresponding standard Q-system are related as

$$\mathcal{Q}'^{(a)}_m(y) = \frac{\mathcal{Q}^{(a)}_{m+1}(y)\mathcal{Q}^{(a)}_{m-1}(y)}{(\mathcal{Q}^{(a)}_m(y))^2}. \tag{A.20}$$

Fact 3: By Propositions 2.1, 3.4, and (4.6), the series $q_b(y)$ in (A.7) satisfies

$$q_a(y) \equiv \prod_{m=1}^{L}(\mathcal{Q}'^{(a)}_m(y))^{-1} \mod J_L, \tag{A.21}$$

where $\mathcal{Q}'^{(a)}_m(y)$ is the one in Fact 2. Note that $q_b(y)$ depends on L.

Fact 4: By the proof of Proposition 4.4, it holds that

$$\mathcal{Q}^{(a)}_L(y) \equiv \mathcal{Q}^{(a)}_{L+1}(y) \mod J_L. \tag{A.22}$$

Combining Facts 2–4, we immediately have $q_a(y) \equiv \mathcal{Q}_1^{(a)}(y) \bmod J_L$. Thus, $\lim_{L\to\infty} q_a(y) = \mathcal{Q}_1^{(a)}(y)$ holds. Therefore, taking the limit $L \to \infty$ of (A.8) with the help of Fact 1, we obtain

$$\mathcal{K}^0_{D,G}(y) = \det_{1\le a,b\le n}\Bigl(\frac{y_b}{U_a}\frac{\partial U_a}{\partial y_b}(y)\Bigr)\prod_{a=1}^{n}\mathcal{Q}_1^{(a)}(y), \tag{A.23}$$

$$U_a(y) := y_a\prod_{b=1}^{n}(\mathcal{Q}_1^{(b)}(y))^{-g_{ab}}. \tag{A.24}$$

The equality (A.18) is obtained from (A.23) in the same way as the proof of Proposition A.2. □

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