

Bethe Equation at $q = 0$, Möbius Inversion Formula, and Weight Multiplicities: I. $\mathfrak{sl}(2)$ case

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ABSTRACT. The $U_q(\widehat{\mathfrak{sl}}(2))$ Bethe equation is studied at $q = 0$. A linear congruence equation is proposed related to the string solutions. The number of its off-diagonal solutions is expressed in terms of an explicit combinatorial formula and coincides with the weight multiplicities of the quantum space.

1. Introduction

1.1. Background. Consider the periodic spin $\frac{1}{2}$ XXX Heisenberg hamiltonian

$$H_{XXX} = J \sum_{n=1}^L (\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \sigma_n^z \sigma_{n+1}^z)$$

acting on the tensor product of the L -copies of the vector representations of $\mathfrak{sl}(2)$:

$$W = \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2.$$

Since H_{XXX} is $\mathfrak{sl}(2)$ -linear, its spectrum is degenerated within the irreducible components in the decomposition:

$$W = \bigoplus_{\lambda \in (\mathbb{Z}_{\geq 0})\Lambda_1} [W : V_\lambda] V_\lambda,$$

where Λ_1 is the fundamental weight and V_λ denotes the irreducible module with highest weight λ . Diagonalization of H_{XXX} was achieved by Bethe [Be] in 1931. Associated to each solution of the simultaneous equations (Bethe equation)

$$\left(\frac{u_j + \sqrt{-1}}{u_j - \sqrt{-1}} \right)^L = - \prod_{j=1}^M \frac{u_j - u_k + 2\sqrt{-1}}{u_j - u_k - 2\sqrt{-1}} \quad j = 1, \dots, M,$$

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he proposed (Bethe ansatz) a vector $\psi \in W$ (Bethe vector) such that

$$\begin{aligned} H_{XXX}\psi &= E\psi & E \in \mathbb{C}, \\ \left(\sum_{n=1}^L \sigma_n^+ \right) \psi &= 0 & \sigma_n^+ = \sigma_n^x + \sqrt{-1}\sigma_n^y, \\ \left(\sum_{n=1}^L \sigma_n^z \right) \psi &= (L - 2M)\psi & 0 \leq M \leq \lfloor \frac{L}{2} \rfloor. \end{aligned}$$

The second and the third properties (cf. [TF]) tell that the Bethe vector is $\mathfrak{sl}(2)$ -highest of weight $(L - 2M)\Lambda_1$. Therefore in order to have the completeness of the Bethe ansatz, there should exist as many solutions to the Bethe equation as the multiplicity of $V_{(L-2M)\Lambda_1}$ in W , $[W : V_{(L-2M)\Lambda_1}] (= \binom{L}{M} - \binom{L}{M-1})$. Actually ψ can be vanishing depending on the solutions $\{u_1, \dots, u_M\}$. In particular, it is so if $u_i = u_j$ for some $i \neq j$.

It was Bethe himself who studied the completeness with the introduction of strings. (He called it ‘‘WellenKomplex’’.) It is a solution in which $\{u_1, \dots, u_M\}$ are arranged as

$$(1.1) \quad \bigcup_{m \in \mathbb{N}} \bigcup_{1 \leq \alpha \leq N_m} \bigcup_{u_{m\alpha} \in \mathbb{R}} \{u_{m\alpha} + \sqrt{-1}(m + 1 - 2i) + \epsilon_{m\alpha i} \mid 1 \leq i \leq m\}$$

for each partition $M = \sum_{m \in \mathbb{N}} mN_m$ ($N_m \in \mathbb{Z}_{\geq 0}$). Here $\mathbb{N} = \mathbb{Z}_{\geq 1}$ denotes the set of positive integers and $\epsilon_{m\alpha i}$ stands for a small deviation. The m -tuple configuration (with negligible $\epsilon_{m\alpha i}$) is called the m -string with string center $u_{m\alpha}$. N_m is the number of m -strings. In general to expect such a behavior for the solutions is called the string hypothesis. Actually in a strict sense, it is known invalid as exemplified already for $M = 2$ and $L > 21$ (cf. [EKS, JD]). Nevertheless Bethe’s counting of the number of string solutions led to the discovery of the identity ($M \leq \lfloor \frac{L}{2} \rfloor$):

$$(1.2) \quad \sum_N \prod_{m \in \mathbb{N}} \binom{L - 2 \sum_{k \geq 1} \min(m, k)N_k + N_m}{N_m} = [W : V_{(L-2M)\Lambda_1}],$$

where \sum_N runs over $N_1, N_2, \dots \in \mathbb{Z}_{\geq 0}$ such that $M = \sum_{m \geq 1} mN_m$. In his counting each summand in the left side represents the number of string solutions corresponding to the prescribed values of N_1, N_2, \dots . The binomial coefficients are originated in the fermionic restriction on the solutions $u_i \neq u_j$ ($i \neq j$). The expression like the left side is called the fermionic formula and the above identity is called the combinatorial completeness of the string hypothesis.

Despite the gap with the completeness in the rigorous sense, the above result opened a fruitful link between quantum integrable systems and representation theory. For a class of Bethe ansatz solvable models with Yangian symmetry $Y(X_n)$, one can set up fermionic formulae following Bethe's counting. If the combinatorial completeness holds, they yield the multiplicities of irreducible X_n -modules in tensor products of a variety of finite dimensional $Y(X_n)$ -modules. The XXX chain corresponds to the $Y(\mathfrak{sl}(2))$ case. The fermionic formula associated with $Y(X_n)$ in such a sense was firstly written down in [KR] for general X_n , where the combinatorial completeness was also announced for the classical types $X_n = A_n, B_n, C_n$ and D_n . The proof of the combinatorial completeness boils down to showing recursion relations (Q -system) among classical characters of certain $Y(X_n)$ -modules (cf. [HKOTY]).

1.2. Present work. The XXX chain admits an integrable q -deformation called the XXZ chain:

$$H_{XXZ} = J \sum_{n=1}^L (\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \frac{q + q^{-1}}{2} \sigma_n^z \sigma_{n+1}^z).$$

In place of the Yangian $Y(\mathfrak{sl}(2))$, the underlying symmetry of the model is the quantum affine algebra $U_q(\hat{\mathfrak{sl}}(2))$ as is well known. Accordingly we regard the space W (called the quantum space) as a $U_q(\hat{\mathfrak{sl}}(2))$ -module. Under the periodic boundary condition the spectrum is determined from the solutions of the Bethe equation ($i = 1, \dots, M$)

$$\left(\frac{\sin \pi(u_i + \sqrt{-1}\hbar)}{\sin \pi(u_i - \sqrt{-1}\hbar)} \right)^L = - \prod_{j=1}^M \frac{\sin \pi(u_i - u_j + 2\sqrt{-1}\hbar)}{\sin \pi(u_i - u_j - 2\sqrt{-1}\hbar)},$$

where \hbar is related to q by $q = e^{-2\pi\hbar}$. When the deformation parameter q tends to 1, the above equation reduces to the one in Section 1.1 by replacing u_j by $\hbar u_j$ and setting $\hbar \rightarrow 0$. A significant difference from the $q = 1$ case is that the hamiltonian is no longer invariant under $\mathfrak{sl}(2)$ nor $U_q(\mathfrak{sl}(2))$ as far as a finite chain ($L < \infty$) is considered under the periodic boundary condition. For the completeness, the number of solutions to the Bethe equation should therefore coincide with the weight multiplicity of $(L - 2M)\Lambda_1$, $\dim W_{(L-2M)\Lambda_1} (= \binom{L}{M})$, rather than the multiplicity of $V_{(L-2M)\Lambda_1}$. Similar facts are valid also for the generalized model in which W is replaced with

$$W(\nu) = \bigotimes_{s \geq 1} (W_s)^{\otimes \nu_s},$$

where $\nu_s \in \mathbb{Z}_{\geq 0}$ and W_s stands for the $(s+1)$ -dimensional irreducible module. See (2.2) for the corresponding Bethe equation.

The purpose of this paper is to study the Bethe equation and to formulate another version of the combinatorial completeness at $q = 0$. This is inspired by the crystal theory, where the simplification at $q = 0$ is known to lead to many fascinating features. In terms of the exponential variables $x_j = e^{2\pi\sqrt{-1}u_j}$, we shall consider a class of meromorphic solutions $x_j = x_j(q)$ around $q = 0$ which correspond to the strings. In a sense we are approaching to the point $q = 0$ within the off-critical regime $\hbar \in \mathbb{R}_{>0}$, avoiding the parity and the arithmetic complexity of strings **[TS]** in the critical regime $\hbar \in \sqrt{-1}\mathbb{R}$.

It is a routine calculation to reduce the Bethe equation to the one for string centers for general $\nu = (\nu_s)$ and $N = (N_m)$. At $q = 0$ the resulting string center equation (SCE) is a linear congruence equation (2.23). As a remnant of the fermionic restriction we seek the off-diagonal solutions (DEFINITION 2.13) of them. They are counted systematically by means of the Möbius inversion formula. When $P_m := \sum_{k \geq 1} \min(m, k)(\nu_k - 2N_k) \geq 0$ for any m such that $N_m > 0$, we find that the result is expressed as (cf. THEOREM 3.5):

$$R(\nu, N) = \sum_{J \subset \mathbb{N}} D_J \prod_{m \in \mathbb{N} \setminus J} \binom{P_m + N_m}{N_m} \prod_{m \in J} \binom{P_m + N_m - 1}{N_m - 1},$$

$$D_J = \begin{cases} 1 & \text{if } J = \emptyset \\ \det_{m, k \in J} (2 \min(m, k) - \delta_{m, k}) & \text{otherwise.} \end{cases}$$

In the XXZ case $\nu_s = L\delta_{s,1}$, the $J = \emptyset$ term here is equal to the summand in the left side of (1.2). With this $R(\nu, N)$ the combinatorial completeness at $q = 0$ is stated as (cf. THEOREM 4.9)

$$\sum_N^{(\lambda)} R(\nu, N) = \dim W(\nu)_\lambda \quad \lambda \in \mathbb{Z}\Lambda_1,$$

where the sum $\sum_N^{(\lambda)}$ runs over $N_1, N_2, \dots \in \mathbb{Z}_{\geq 0}$ such that $\sum_{j \geq 1} j(\nu_j - 2N_j)\Lambda_1 = \lambda$. This is a non-trivial identity even when $\dim W(\nu)_\lambda = 0$ for $\lambda < 0$. Curiously, the left side in general involves the contributions from those N that break the $P_m \geq 0$ condition said above. In the course of the proof we will clarify the relation between the known fermionic formula as in (1.2) and our $R(\nu, N)$. It is most transparently presented in terms of generating functions. See (4.25) and (4.29).

The layout of the paper is as follows. In Section 2, we study the Bethe equation at $q = 0$. We explain the relation between solutions of SCE and string solutions of the Bethe equation. In Section 3 we derive the formula $R(\nu, N)$ by counting the off-diagonal solutions of SCE. In

Section 4 we prove the combinatorial completeness. In Section 5 we give a summary and discussion. Appendix A is a summary of the Möbius inversion trick used in Section 3.

A few remarks are in order. In [KL] combinatorial completeness was investigated when q is a root of unity. The fermionic formula there for weight multiplicities is different from ours. We expect that their result describes the rich singular behavior of the meromorphic solutions (around $q = 0$) of the Bethe equation on the convergence circle $|q| = 1$. In [LS] SCE has been obtained for the XXZ case. There is a statement similar to the combinatorial completeness at $q = 0$ without an explicit formula as $R(\nu, N)$. In [TV] they studied a deformation of the XXZ type Bethe equation and showed the completeness of the Bethe vectors for the admissible off-diagonal solutions at a generic value of the deformation parameter.

Many results in this paper admit generalizations to $U_q(X_n^{(1)})$ case for arbitrary X_n . It will be a subject of our forthcoming paper.

2. Bethe equation at $q = 0$

In this section we start from the Bethe equation and seek string solutions in $q \rightarrow 0$ limit. We introduce an equation for string centers (SCE), which is a linear congruence equation. Later sections will be devoted to studies of the SCE. Our aim here is to explain the precise relation between solutions of SCE and string solutions of the Bethe equation. Our theorems mostly concern what we call generic string solutions.

We let α_1 and Λ_1 denote the simple root and the fundamental weight of $\mathfrak{sl}(2)$. $\alpha_1 = 2\Lambda_1$. In this paper $U_q(\hat{\mathfrak{sl}}(2))$ means the quantum affine algebra without the derivation operator. It is denoted by $U'_q(\hat{\mathfrak{sl}}(2))$ in some literature.

For a meromorphic function $f(q)$ around $q = 0$, we will use the notation

$$\begin{aligned} f(q) &= q^{\text{ord}(f)}(f^0 + f^1q + \cdots) & \text{ord}(f) \in \mathbb{Z}, f^0 \neq 0, \\ \tilde{f}(q) &= q^{-\text{ord}(f)}f(q) = f^0 + f^1q + \cdots. \end{aligned}$$

We call $\text{ord}(f)$ the order, f^0 the leading coefficient of f . \tilde{f} is a holomorphic function around $q = 0$ with nonzero constant term. Note that $f^0 = \tilde{f}^0$.

2.1. Bethe equation. Consider a solvable vertex model associated with $U_q(\widehat{\mathfrak{sl}}(2))$. Let

$$(2.1) \quad W(\nu) = \bigotimes_{s \geq 1} (W_s)^{\otimes \nu_s}$$

be the $U_q(\widehat{\mathfrak{sl}}(2))$ -module (the quantum space) on which the commuting family of row-to-row transfer matrices act. We assume that only finitely many ν_s 's are nonzero throughout this paper. Here W_s stands for an $(s+1)$ -dimensional irreducible module with highest weight $s\Lambda_1$. Each W_s depends on a spectral parameter, which may be interpreted as inhomogeneity of the interaction. The Bethe equation relevant to the spectrum of the transfer matrices also depends on those spectral parameters. In this paper we concentrate on the regime and the situation in which the Bethe equation takes the form ($i = 1, \dots, M$):

$$(2.2) \quad \prod_{s \geq 1} \left(\frac{\sin \pi(u_i + \sqrt{-1}s\hbar)}{\sin \pi(u_i - \sqrt{-1}s\hbar)} \right)^{\nu_s} = - \prod_{j=1}^M \frac{\sin \pi(u_i - u_j + 2\sqrt{-1}\hbar)}{\sin \pi(u_i - u_j - 2\sqrt{-1}\hbar)},$$

Here $\hbar \in \mathbb{R}_{>0}$ and $M \in \mathbb{Z}_{\geq 0}$. This is a regime in which the so-called parity [TS] is irrelevant. Integer shifts of u_j do not lead to a new Bethe vector, hence one should consider $u_j \in \mathbb{R}/\mathbb{Z}$. Setting

$$q = e^{-2\pi\hbar}, \quad x_j = e^{2\pi\sqrt{-1}u_j},$$

(2.2) can be written as polynomial equations on x_j 's:

$$(2.3) \quad F_{i+}G_{i-} = -F_{i-}G_{i+} \quad i = 1, \dots, M,$$

where

$$F_{i+} = \prod_{s \geq 1} (x_i q^s - 1)^{\nu_s}, \quad G_{i+} = \prod_{j=1}^M (x_i q^2 - x_j),$$

$$F_{i-} = \prod_{s \geq 1} (x_i - q^s)^{\nu_s}, \quad G_{i-} = \prod_{j=1}^M (x_i - x_j q^2).$$

The equation is invariant under the permutation of the variables $x_i \leftrightarrow x_j$.

We are interested in meromorphic solutions (x_i) , $x_i = x_i(q)$ of (2.3) around $q = 0$. We set $x_i(q) = q^{d_i} z_i(q)$, where $d_i = \text{ord}(x_i)$, and $z_i(q) = \tilde{x}_i(q)$. Then the Bethe equation for $z_i(q)$ is given by (2.3) with $F_{i\pm}$, $G_{i\pm}$

now specified as

$$(2.4) \quad F_{i+} = \prod_{s \geq 1} (z_i q^{d_i+s} - 1)^{\nu_s}, \quad G_{i+} = \prod_{j=1}^M (z_i q^{d_i+2} - z_j q^{d_j}),$$

$$(2.5) \quad F_{i-} = \prod_{s \geq 1} (z_i q^{d_i} - q^s)^{\nu_s}, \quad G_{i-} = \prod_{j=1}^M (z_i q^{d_i} - z_j q^{d_j+2}).$$

This equation is invariant under the permutation of the variables $z_i \leftrightarrow z_j$, only when $d_i = d_j$.

2.2. String Solution.

DEFINITION 2.1. A meromorphic solution (x_i) of (2.3) is called *inadmissible* (*admissible*) if $F_{i+}G_{i-} = F_{i-}G_{i+} = 0$ for some i as a function of q around $q = 0$ (otherwise).

Let $N = (N_m)$ be an infinite sequence of non-negative integers such that

$$(2.6) \quad M = \sum_{m \geq 1} m N_m.$$

DEFINITION 2.2. A meromorphic solution (x_i) of (2.3) is called a *string solution of pattern* $N = (N_m)$ if

- (i) (x_i) is admissible.
- (ii) (x_i) can be arranged as $(x_{m\alpha i})$ with

$$m = 1, 2, \dots, \quad \alpha = 1, \dots, N_m, \quad i = 1, \dots, m$$

such that

- (a) $d_{m\alpha i} = m + 1 - 2i$ for $d_{m\alpha i} := \text{ord}(x_{m\alpha i})$.
- (b) $z_{m\alpha 1}^0 = z_{m\alpha 2}^0 = \dots = z_{m\alpha m}^0$, where $z_{m\alpha i} = \tilde{x}_{m\alpha i}$.

For each $1 \leq \alpha \leq N_m$, $(z_{m\alpha i})_{i=1}^m$ is called an m -string. N_m is the number of m -strings. When considering string solutions, we denote the quantity in (b) by $z_{m\alpha}^0$, and call it the *string center*. We set

$$q^{\zeta_{m\alpha i}} y_{m\alpha i}(q) = z_{m\alpha i}(q) - z_{m\alpha i-1}(q) \quad 2 \leq i \leq m,$$

where $\zeta_{m\alpha i} = \text{ord}(z_{m\alpha i} - z_{m\alpha i-1}) \in \mathbb{Z}_{\geq 1}$. For a string solution of pattern N , the Bethe equation (2.3) reads

$$(2.7) \quad F_{m\alpha i+} G_{m\alpha i-} = -F_{m\alpha i-} G_{m\alpha i+},$$

where

$$F_{m\alpha i+} = \prod_{s \geq 1} (z_{m\alpha i} q^{d_{m\alpha i} + s} - 1)^{\nu_s}, \quad G_{m\alpha i+} = \prod_{k\beta j} (z_{m\alpha i} q^{d_{m\alpha i} + 2} - z_{k\beta j} q^{d_{k\beta j}}),$$

$$F_{m\alpha i-} = \prod_{s \geq 1} (z_{m\alpha i} q^{d_{m\alpha i}} - q^s)^{\nu_s}, \quad G_{m\alpha i-} = \prod_{k\beta j} (z_{m\alpha i} q^{d_{m\alpha i}} - z_{k\beta j} q^{d_{k\beta j} + 2}).$$

Here $\prod_{k\beta j}$ means $\prod_{k \geq 1} \prod_{\beta=1}^{N_k} \prod_{j=1}^k$. Let us extract the factors $y_{m\alpha i}$ from $G_{m\alpha i \pm}$ by introducing $G'_{m\alpha i \pm}$ as follows:

$$G_{m\alpha i+} = \begin{cases} G'_{m\alpha 1+} & i = 1 \\ G'_{m\alpha i+} q^{d_{m\alpha i} + 2 + \zeta_{m\alpha i}} y_{m\alpha i} & 2 \leq i \leq m, \end{cases}$$

$$G_{m\alpha i-} = \begin{cases} G'_{m\alpha i-} (-q^{d_{m\alpha i} + \zeta_{m\alpha i+1}} y_{m\alpha i+1}) & 1 \leq i \leq m-1 \\ -G'_{m\alpha m-} & i = m. \end{cases}$$

Now the Bethe equation (2.7) takes the form:

$$(2.8) \quad \tilde{F}_{m\alpha 1+} \tilde{G}'_{m\alpha 1-} y_{m\alpha 2} = \tilde{F}_{m\alpha 1-} \tilde{G}'_{m\alpha 1+} \quad i = 1,$$

$$(2.9) \quad \tilde{F}_{m\alpha i+} \tilde{G}'_{m\alpha i-} y_{m\alpha i+1} = \tilde{F}_{m\alpha i-} \tilde{G}'_{m\alpha i+} y_{m\alpha i} \quad 2 \leq i \leq m-1,$$

$$(2.10) \quad \tilde{F}_{m\alpha m+} \tilde{G}'_{m\alpha m-} = \tilde{F}_{m\alpha m-} \tilde{G}'_{m\alpha m+} y_{m\alpha m} \quad i = m.$$

In particular,

$$(2.11) \quad 1 = (-1)^m \prod_{i=1}^m \frac{F_{m\alpha i+} G_{m\alpha i-}}{F_{m\alpha i-} G_{m\alpha i+}} = \prod_{i=1}^m \frac{\tilde{F}_{m\alpha i+} \tilde{G}'_{m\alpha i-}}{\tilde{F}_{m\alpha i-} \tilde{G}'_{m\alpha i+}},$$

where the latter equality is the identity as holomorphic functions around $q = 0$.

2.3. $q \rightarrow 0$ limit of Bethe equation. Suppose that $(x_{m\alpha i})$ is a string solution to the Bethe equation (2.7). Taking the leading coefficients of (2.8)–(2.10) we get the $q \rightarrow 0$ limit:

$$(2.12) \quad F_{m\alpha 1+}^0 G_{m\alpha 1-}^0 y_{m\alpha 2}^0 = F_{m\alpha 1-}^0 G_{m\alpha 1+}^0 \quad i = 1,$$

$$(2.13) \quad F_{m\alpha i+}^0 G_{m\alpha i-}^0 y_{m\alpha i+1}^0 = F_{m\alpha i-}^0 G_{m\alpha i+}^0 y_{m\alpha i}^0 \quad 2 \leq i \leq m-1,$$

$$(2.14) \quad F_{m\alpha m+}^0 G_{m\alpha m-}^0 = F_{m\alpha m-}^0 G_{m\alpha m+}^0 y_{m\alpha m}^0 \quad i = m.$$

In particular,

$$(2.15) \quad 1 = (-1)^m \prod_{i=1}^m \frac{F_{m\alpha i+}^0 G_{m\alpha i-}^0}{F_{m\alpha i-}^0 G_{m\alpha i+}^0} = \prod_{i=1}^m \frac{F_{m\alpha i+}^0 G_{m\alpha i-}^0}{F_{m\alpha i-}^0 G_{m\alpha i+}^0}$$

holds for the leading coefficients.

2.4. Generic string solution. In order to estimate the order of the Bethe equation (2.7) we introduce

$$\begin{aligned}\xi_{m\alpha i+} &= \sum_{s \geq 1} \nu_s \min(m+1-2i+s, 0), \\ \xi_{m\alpha i-} &= \sum_{s \geq 1} \nu_s \min(m+1-2i, s), \\ \eta_{m\alpha i+} &= \sum_{k, \beta j} \min(m+3-2i, k+1-2j), \\ \eta_{m\alpha i-} &= \sum_{k, \beta j} \min(m+1-2i, k+3-2j),\end{aligned}$$

where $\sum_{k, \beta j}$ is the abbreviation of $\sum_{k \geq 1} \sum_{\beta=1}^{N_k} \sum_{j=1}^k$. In general one has $\xi_{m\alpha i \pm} \leq \text{ord}(F_{m\alpha i \pm})$, $\eta_{m\alpha i+} \leq \text{ord}(G_{m\alpha i+}) + (1 - \delta_{i,1})\zeta_{m\alpha i}$ and $\eta_{m\alpha i-} \leq \text{ord}(G_{m\alpha i-}) + (1 - \delta_{i,m})\zeta_{m\alpha i+1}$. Let us consider the simplest situation when these inequalities are saturated.

DEFINITION 2.3. A string solution $(x_{m\alpha i})$ to (2.3) is called *generic* if the following condition is valid:

$$\begin{aligned}\text{ord}(F_{m\alpha i \pm}) &= \xi_{m\alpha i \pm}, \\ \text{ord}(G_{m\alpha i+}) &= \begin{cases} \eta_{m\alpha 1+} & i = 1 \\ \eta_{m\alpha i+} + \zeta_{m\alpha i} & 2 \leq i \leq m, \end{cases} \\ \text{ord}(G_{m\alpha i-}) &= \begin{cases} \eta_{m\alpha i-} + \zeta_{m\alpha i+1} & 1 \leq i \leq m-1 \\ \eta_{m\alpha m-} & i = m. \end{cases}\end{aligned}$$

Our results in the rest of Section 2 mostly concern generic string solutions. Given the quantum space data $\nu = (\nu_s)$ and the string pattern $N = (N_m)$, we put

$$(2.16) \quad P_m = P_m(\nu, N) = \gamma_m - 2 \sum_{k \geq 1} \min(m, k) N_k,$$

$$(2.17) \quad \gamma_m = \gamma_m(\nu) = \sum_{k \geq 1} \min(m, k) \nu_k.$$

LEMMA 2.4. For $1 \leq i \leq m$ we have

$$\begin{aligned} & (\xi_{m\alpha i+} + \eta_{m\alpha i-}) - (\xi_{m\alpha i-} + \eta_{m\alpha i+}) \\ &= \begin{cases} -(P_{m+1-2i} + N_{m+1-2i}) & i < \frac{m+1}{2} \\ 0 & i = \frac{m+1}{2} \\ P_{2i-m-1} + N_{2i-m-1} & i > \frac{m+1}{2}. \end{cases}\end{aligned}$$

PROPOSITION 2.5. *A necessary condition for the existence of a generic string solution of pattern $N = (N_m)$ is as follows: If $N_m \geq 1$, then*

$$\begin{aligned} P_{m-1} + N_{m-1} &\geq 1, \\ (P_{m-1} + N_{m-1}) + (P_{m-3} + N_{m-3}) &\geq 1, \\ &\dots \end{aligned}$$

$$(P_{m-1} + N_{m-1}) + (P_{m-3} + N_{m-3}) + \dots + \begin{cases} (P_1 + N_1) \geq 1 & m : \text{even} \\ (P_2 + N_2) \geq 1 & m : \text{odd}. \end{cases}$$

PROOF. From the condition $\text{ord}(F_{m\alpha i+} G_{m\alpha i-}) = \text{ord}(F_{m\alpha i-} G_{m\alpha i+})$, we have

$$(\xi_{m\alpha i+} + \eta_{m\alpha i-}) - (\xi_{m\alpha i-} + \eta_{m\alpha i+}) = \begin{cases} -\zeta_{m\alpha 2} & i = 1 \\ \zeta_{m\alpha i} - \zeta_{m\alpha i+1} & 2 \leq i \leq m-1 \\ \zeta_{m\alpha m} & i = m. \end{cases}$$

Solving this by using LEMMA 2.4, we get

$$\zeta_{m\alpha i} = \zeta_{m\alpha m+2-i} = \sum_{k=1}^{\min(i-1, m+1-i)} (P_{m+1-2k} + N_{m+1-2k}) \quad 2 \leq i \leq m.$$

In order to have $\zeta_{m\alpha 2}, \zeta_{m\alpha 3}, \dots, \zeta_{m\alpha m} \geq 1$, the condition in the proposition must hold. \square

In the remainder of this section we shall exclusively consider the case $\forall P_m(\nu, N) \geq 0$. Thus the necessary condition in PROPOSITION 2.5 reduces to

$$(2.18) \quad P_m \geq 0 \text{ for any } m, \quad P_{m-1} + N_{m-1} \geq 1 \text{ whenever } N_m \geq 1.$$

2.5. String center equation (SCE).

THEOREM 2.6. *Let $(x_{m\alpha i})$ be a generic string solution of pattern N . Then its string center $(z_{m\alpha}^0)$ satisfies the equation:*

$$(2.19) \quad \prod_{k \geq 1} \prod_{\beta=1}^{N_k} (z_{k\beta}^0)^{A_{m\alpha, k\beta}} = (-1)^{P_m + N_m + 1} \quad m \geq 1, \quad 1 \leq \alpha \leq N_m,$$

$$(2.20) \quad A_{m\alpha, k\beta} := \delta_{mk} \delta_{\alpha\beta} (P_m + N_m) + 2 \min(m, k) - \delta_{mk}.$$

We call (2.19) the string center equation (SCE). It is a linear congruence equation in the sense of (2.23).

PROOF. Let us compute the ratio (2.15) explicitly.

$$\prod_{i=1}^m F_{m\alpha i \epsilon}^0 = \begin{cases} (-1)^{\gamma_m} \prod_{s \geq 1} (f_{m\alpha}^{(s)})^{\nu_s} & \epsilon = + \\ (z_{m\alpha}^0)^{\gamma_m} \prod_{s \geq 1} (f_{m\alpha}^{(s)})^{\nu_s} & \epsilon = -, \end{cases}$$

$$f_{m\alpha}^{(s)} = \begin{cases} 1 & m \leq s \\ (-z_{m\alpha}^0)^{\frac{m-s}{2}} & m > s, s \equiv m \pmod{2} \\ (-z_{m\alpha}^0)^{\frac{m-s-1}{2}} (z_{m\alpha}^0 - 1) & m > s, s \not\equiv m \pmod{2}. \end{cases}$$

In order to calculate $\prod_{i=1}^m (G_{m\alpha i-}^0 / G_{m\alpha i+}^0)$, it is convenient first to evaluate

$$\prod_{i=1}^m \prod_{j=1}^k (z_{m\alpha i} q^{d_{m\alpha i} + 1 + \epsilon} - z_{k\beta j} q^{d_{k\beta j} + 1 - \epsilon})^0$$

$$= \begin{cases} (-z_{k\beta}^0)^{2 \min(m,k) - \delta_{mk}} g_{m\alpha}^{k\beta} & \epsilon = 1 \\ (z_{m\alpha}^0)^{2 \min(m,k) - \delta_{mk}} (-1)^{(m-1)\delta_{mk}\delta_{\alpha\beta}} g_{m\alpha}^{k\beta} & \epsilon = -1, \end{cases}$$

$$g_{m\alpha}^{k\beta} = \begin{cases} (-z_{m\alpha}^0 z_{k\beta}^0)^{\frac{mk}{2} - \min(m,k)} & m \not\equiv k \pmod{2} \\ (-z_{m\alpha}^0 z_{k\beta}^0)^{\frac{mk}{2} - \frac{3}{2} \min(m,k) + \delta_{mk}} \times (z_{m\alpha}^0 - z_{k\beta}^0)^{\min(m,k) - \delta_{mk}} & m\alpha \neq k\beta, m \equiv k \pmod{2} \\ (-z_{m\alpha}^0 z_{k\beta}^0)^{\frac{mk}{2} - \frac{3}{2} \min(m,k) + \delta_{mk}} \times y_{m\alpha 2}^0 \cdots y_{m\alpha m}^0 & m\alpha = k\beta. \end{cases}$$

The factors $\prod_{s \geq 1} (f_{m\alpha}^{(s)})^{\nu_s}$ and $g_{m\alpha}^{k\beta}$ are all nonzero for a generic string solution $(x_{m\alpha i})$. They are canceled in the ratio (2.15) and we find

$$(2.21) \quad 1 = (-1)^m \prod_{i=1}^m \frac{F_{m\alpha i+}^0 G_{m\alpha i-}^0}{F_{m\alpha i-}^0 G_{m\alpha i+}^0} = (-1)^{P_m + N_m + 1} \prod_{k\beta} (z_{k\beta}^0)^{-A_{m\alpha, k\beta}}.$$

□

REMARK 2.7. From the condition $\prod_{s \geq 1} (f_{m\alpha}^{(s)})^{\nu_s}$, $g_{m\alpha}^{k\beta} \neq 0$ in the above proof, we see that a string solution $(x_{m\alpha i})$ is generic if and only if

$$(2.22) \quad \prod_{1 \leq s < m, s \not\equiv m(2)} (z_{m\alpha}^0 - 1)^{\nu_s} \neq 0,$$

$$\prod_{k\beta (\neq m\alpha), k \equiv m(2)} (z_{m\alpha}^0 - z_{k\beta}^0)^{\min(m,k) - \delta_{mk}} \neq 0$$

for any $m \geq 1$, $1 \leq \alpha \leq N_m$. In the latter, the power $\min(m, k) - \delta_{mk}$ implies that the collision $z_{m\alpha}^0 = z_{k\beta}^0$ is allowed only when $m = k = 1$.

DEFINITION 2.8. A solution to SCE (2.19) is called *generic* if it satisfies the condition (2.22).

By the definition the solutions to SCE (2.19) arising from the string centers of generic string solutions as in THEOREM 2.6 are generic.

SCE (2.19) becomes the linear congruence equation in terms of the variables $u_{k\beta} \in \mathbb{R}/\mathbb{Z}$ defined by $z_{k\beta}^0 = \exp(2\pi\sqrt{-1}u_{k\beta})$:

$$(2.23) \quad \sum_{k \geq 1} \sum_{\beta=1}^{N_k} A_{m\alpha, k\beta} u_{k\beta} \equiv \frac{P_m + N_m + 1}{2} \pmod{\mathbb{Z}}.$$

This will also be called SCE. In the limit $\hbar \rightarrow \infty$, the asymptotic behavior of the original variable $u_j = \frac{1}{2\pi\sqrt{-1}} \log(x_{k\beta i})$ in (2.2) is $u_{k\beta} + \sqrt{-1}\hbar(k+1-2i)$.

2.6. Lifting generic solutions of SCE to generic string solutions. In Section 2.5 we have seen that string centers of a generic string solution to the Bethe equation (2.7) yield a generic solution to SCE. Here we show the inverse.

THEOREM 2.9. *Suppose that $N = (N_m)$ obeys the condition (2.18). Let $(z_{m\alpha}^0)$ be a generic solution to SCE (2.19). Then there exists a unique generic string solution $(x_{m\alpha i}(q))$ to the Bethe equation (2.3) such that $z_{m\alpha i}(0) = z_{m\alpha}^0$ ($1 \leq i \leq m$).*

Define the variables (holomorphic functions of q) $w_{m\alpha i}$ by

$$w_{m\alpha i} = \begin{cases} z_{m\alpha i} & i = 1 \\ y_{m\alpha i} & 2 \leq i \leq m. \end{cases}$$

Then

$$(2.24) \quad z_{m\alpha i} = w_{m\alpha 1} + q^{\zeta_{m\alpha 2}} w_{m\alpha 2} + \cdots + q^{\zeta_{m\alpha i}} w_{m\alpha i} \quad 1 \leq i \leq m.$$

Denote the i th equation of (2.8)–(2.10) by $L_{m\alpha i} = R_{m\alpha i}$. Let $A = (A_{m\alpha, k\beta})$ be the matrix of size $N_1 + N_2 + \cdots$ defined by (2.20). Similarly let $J = (J_{m\alpha i, k\beta j})$ be the matrix of size $N_1 + 2N_2 + \cdots (= M)$ defined by $J_{m\alpha i, k\beta j} = \frac{\partial}{\partial w_{k\beta j}} \left(\frac{L_{m\alpha i}}{R_{m\alpha i}} - 1 \right)$.

LEMMA 2.10. *Suppose (2.18) is satisfied. Then $\det J$ is nonzero at $q = 0$ (i.e., $\det(J_{m\alpha i, k\beta j}^0) \neq 0$) if $\det A \neq 0$.*

PROOF. Owing to the assumption (2.18), we have $\forall \zeta_{m\alpha i} \geq 1$. Since $L_{m\alpha i}^0 = R_{m\alpha i}^0 \neq 0$, it suffices to show $\det \mathcal{J} \neq 0$ for $\mathcal{J}_{m\alpha i, k\beta j} = \frac{\partial}{\partial w_{k\beta j}} \log \frac{L_{m\alpha i}}{R_{m\alpha i}}$. From (2.24) both $\frac{\partial \tilde{F}_{m\alpha i \pm}}{\partial w_{k\beta j}}$ and $\frac{\partial \tilde{G}'_{m\alpha i \pm}}{\partial w_{k\beta j}}$ for $j \neq 1$ are zero at $q = 0$. Thus among $\mathcal{J}_{m\alpha i, k\beta j}^0$'s the non-vanishing ones are only $\mathcal{J}_{m\alpha i, k\beta 1}^0$

($1 \leq i \leq m$), $\mathcal{J}_{m\alpha i, m\alpha i}^0 = -1/y_{m\alpha i}^0$ ($2 \leq i \leq m$), and $\mathcal{J}_{m\alpha i, m\alpha i+1}^0 = 1/y_{m\alpha i+1}^0$ ($1 \leq i \leq m-1$). Let $\vec{\mathcal{J}}_{m\alpha i}^0 = (\mathcal{J}_{m\alpha i, k\beta j}^0)_{k\beta j}$ be a row vector of the matrix \mathcal{J} . In view of the above result, the linear dependence $\sum_{m\alpha i} c_{m\alpha i} \vec{\mathcal{J}}_{m\alpha i}^0 = 0$ can possibly holds only when $c_{m\alpha i}$ is independent of i . Consequently we consider the equation $\sum_{m\alpha} c_{m\alpha} \sum_{i=1}^m \vec{\mathcal{J}}_{m\alpha i}^0 = 0$. The $(k\beta 1)$ -th component of the vector $\sum_{i=1}^m \vec{\mathcal{J}}_{m\alpha i}^0$ is given by

$$\lim_{q \rightarrow 0} \frac{\partial}{\partial w_{k\beta 1}} \log \prod_{i=1}^m \frac{\tilde{F}_{m\alpha i+} \tilde{G}'_{m\alpha i-}}{\tilde{F}_{m\alpha i-} \tilde{G}'_{m\alpha i+}} = \frac{\partial}{\partial z_{k\beta}^0} \log \prod_{i=1}^m \frac{-F_{m\alpha i+}^0 G_{m\alpha i-}^0}{F_{m\alpha i-}^0 G_{m\alpha i+}^0},$$

where we have taken (2.11), (2.24) and $\forall \zeta_{m\alpha i} \geq 1$ into account. Due to (2.21) the last expression is equal to $-A_{m\alpha, k\beta} / z_{k\beta}^0$. Therefore the equation $\sum_{m\alpha} c_{m\alpha} \sum_{i=1}^m \vec{\mathcal{J}}_{m\alpha i}^0 = 0$ is equivalent to $\sum_{m\alpha} c_{m\alpha} A_{m\alpha, k\beta} = 0$ for any $k\beta$. This admits only the trivial solution for $c_{m\alpha}$ if $\det A \neq 0$. \square

PROOF OF THEOREM 2.9. The Bethe equations (2.8)–(2.10) are simultaneous equations on the variables $(w_{m\alpha i}, q)$. At $q = 0$, (2.8)–(2.10) reduce to (2.12)–(2.14). The latter fix $(y_{m\alpha i}^0)$ unambiguously once a generic solution $(z_{m\alpha}^0)$ to SCE is given. Denote the resulting value of $w_{m\alpha i}$ by $w_{m\alpha i}^0$. Thus (2.8)–(2.10) are valid at $(w_{m\alpha i}, q) = (w_{m\alpha i}^0, 0)$. From the implicit function theorem, there uniquely exist the functions $w_{m\alpha i}(q)$ satisfying (2.8)–(2.10) and $w_{m\alpha i}(0) = w_{m\alpha i}^0$, if the Jacobian $\det J$ at $(w_{m\alpha i}, q) = (w_{m\alpha i}^0, 0)$ is nonzero. By LEMMA 2.10 and COROLLARY 3.4 this has been guaranteed under (2.18). \square

From THEOREM 2.6 and THEOREM 2.9 we have

COROLLARY 2.11. *Suppose the pattern $N = (N_m)$ satisfies (2.18). Then there is a one-to-one correspondence between generic string solutions to the Bethe equation (2.3) and generic solutions to SCE (2.19).*

2.7. Off-diagonal solution. In COROLLARY 2.11 to restrict oneself to the admissible solutions to the Bethe equation is natural, because otherwise the associated Bethe vectors are vanishing [TV]. On the other hand, the limitation to the generic case has been made by a technical reason. In fact the most essential constraint on the solution of the Bethe equation (2.3) is that x_1, \dots, x_M are all distinct. Otherwise the Bethe vectors are again vanishing. To make it precise in the present context, we introduce

DEFINITION 2.12. A meromorphic solution (x_i) of (2.3) is called *diagonal (off-diagonal)* if $x_i = x_j$ for some $i \neq j$ as a function of q around $q = 0$ (otherwise).

DEFINITION 2.13. A solution $(z_{m\alpha}^0)$ of SCE is called *diagonal (off-diagonal)* if $z_{m\alpha}^0 = z_{m\beta}^0$ for some $\alpha \neq \beta$ (otherwise).

According to the above definition, the string solution $(z_{m\alpha i})$ of the Bethe equation is *diagonal (off-diagonal)* if $z_{m\alpha i}(q) = z_{k\beta j}(q)$ and $d_{m\alpha i} = d_{k\beta j}$ for some $m\alpha i \neq k\beta j$ (otherwise). We conjecture that the number of off-diagonal string solutions of the Bethe equation is equal to the number of off-diagonal solutions of SCE under a certain condition like (2.18). So far we have been unable to solve the discrepancy between “generic” and “off-diagonal”.

REMARK 2.14. In [LS] SCE has been given for the XXZ case ($\nu_s = L\delta_{1s}$). It is claimed that SCE is satisfied irrespective of whether a string solution is generic or not. Moreover, each off-diagonal solution of SCE gives rise to an off-diagonal string solution.

3. Counting of off-diagonal solutions

This section is devoted to an expository proof of THEOREM 3.5. It provides an explicit combinatorial formula counting the number of off-diagonal solutions to SCE (2.19) in the sense of Section 2.7. We assume that the quantum space data $\nu = (\nu_s)$ and the string pattern $N = (N_m)$ satisfy the condition

$$(3.1) \quad P_m(\nu, N) \geq 0 \quad \text{whenever} \quad N_m \geq 1$$

throughout Section 3. This is a milder condition than $\forall P_m \geq 0$.

3.1. Rule of counting. We shall work with the logarithmic form of SCE (2.23) presented as

$$(3.2) \quad A\vec{u} \equiv \vec{c} \pmod{\mathbb{Z}^d},$$

where $d = N_1 + N_2 + \dots$. Here $\vec{u} = (u_{k\beta})$ is the unknown and \vec{c} is some constant vector. The d -dimensional matrix $A = (A_{m\alpha, k\beta})$ is specified by (2.20). It has a block structure according to the string pattern. For example, if only N_1, N_2 and N_3 are non-zero, it looks as ($\mathcal{P}_i = P_i + N_i$)

$$\left(\begin{array}{cccccccc} \mathcal{P}_1 + 1 & \cdots & 1 & 2 & \cdots & 2 & 2 & \cdots & 2 \\ \vdots & \ddots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 1 & \cdots & \mathcal{P}_1 + 1 & 2 & \cdots & 2 & 2 & \cdots & 2 \\ 2 & \cdots & 2 & \mathcal{P}_2 + 3 & \cdots & 3 & 4 & \cdots & 4 \\ \vdots & & \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ 2 & \cdots & 2 & 3 & \cdots & \mathcal{P}_2 + 3 & 4 & \cdots & 4 \\ 2 & \cdots & 2 & 4 & \cdots & 4 & \mathcal{P}_3 + 5 & \cdots & 5 \\ \vdots & & \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ 2 & \cdots & 2 & 4 & \cdots & 4 & 5 & \cdots & \mathcal{P}_3 + 5 \end{array} \right),$$

which consists of 9 sub-matrices of size $N_i \times N_j$ ($1 \leq i, j \leq 3$).

We will seek the number of off-diagonal solutions to SCE (3.2) in the sense of Section 2.7. Let us remember the three essential rules to count it. First, there is no change in (2.2) under any integer shift of u_j . Accordingly we should not distinguish the solutions \vec{u} and \vec{u}' to (3.2) if $\vec{u} - \vec{u}' \in \mathbb{Z}^d$. In other words, we consider $u_{k\beta} \in \mathbb{R}/\mathbb{Z}$ rather than \mathbb{R} . Secondly, the original Bethe equation (2.2) is symmetric with respect to u_1, \dots, u_M , but their permutation does not lead to a new Bethe vector as is well known. Consequently, for a solution $\vec{u} = (u_{k\beta})$ of pattern $N = (N_m)$, we should regard

$$(u_{k1}, u_{k2}, \dots, u_{kN_k}) \in (\mathbb{R}/\mathbb{Z})^{N_k} / \mathfrak{S}_{N_k}$$

for each $k \in \mathbb{N}$. Here \mathfrak{S}_{N_k} stands for the symmetric group permuting the N_k components. Last but not most importantly, DEFINITION 2.13 postulates that $u_{k\beta} \neq u_{k\beta'} \in \mathbb{R}/\mathbb{Z}$ if $1 \leq \beta \neq \beta' \leq N_k$ for each $k \in \mathbb{N}$. To summarize these rules, we start with a fixed string pattern (N_m) and specify A and \vec{c} by (2.23). Then we are to count the number of solutions $\vec{u} = (u_{k\beta})$ to SCE (3.2) such that

$$(u_{k1}, u_{k2}, \dots, u_{kN_k}) \in ((\mathbb{R}/\mathbb{Z})^{N_k} - \Delta_{N_k}) / \mathfrak{S}_{N_k},$$

$$\Delta_n = \{(v_1, \dots, v_n) \in (\mathbb{R}/\mathbb{Z})^n \mid v_\alpha = v_\beta \text{ for some } 1 \leq \alpha \neq \beta \leq n\}$$

for each $k \in \mathbb{N}$. In practice, one just has to find the number of solutions such that $(u_{k1}, u_{k2}, \dots, u_{kN_k}) \in (\mathbb{R}/\mathbb{Z})^{N_k} - \Delta_{N_k}$, and divide afterwards by $N_k!$ for each k .

3.2. Example. Before treating the general case, let us illustrate how to enumerate the off-diagonal solutions with a simplest example $\nu_s = L\delta_{s,1}$. This corresponds to the spin $\frac{1}{2}$ XXZ model with L -sites. Thus the character of the quantum space is expanded as ($x = e^{\Lambda_1}$)

$$(x + x^{-1})^L = x^L + \binom{L}{1} x^{L-2} + \binom{L}{2} x^{L-4} + \binom{L}{3} x^{L-6} + \dots,$$

according to the ‘magnon number’ $M = 0, 1, 2, 3, \dots$. Here and in what follows the symbol $\binom{\cdot}{\cdot}$ will always denote the generalized binomial coefficient:

$$\binom{\xi}{n} = \begin{cases} \xi(\xi - 1) \cdots (\xi - n + 1)/n! & \text{if } n \in \mathbb{Z}_{\geq 1} \\ 1 & \text{if } n = 0 \\ 0 & \text{otherwise,} \end{cases} \quad \xi \in \mathbb{C}.$$

The counting of the off-diagonal solutions sketched below up to $M = 3$ is useful to gain the idea for treating the general case. The result coincides with the weight multiplicity $\binom{L}{M}$ appearing in the above expansion. The proof of the coincidence in the general case will be given in Section 4. According to (3.1) we assume that $P_m = L - 2 \sum_{k \geq 1} \min(m, k) N_k \geq 0$ whenever $N_m > 0$. We use the following fact:

PROPOSITION 3.1. *Let B be an n by n integer matrix with $\det B \neq 0$. Then for any $\vec{b} \in \mathbb{R}^n$, the equation $B\vec{x} \equiv \vec{b} \pmod{\mathbb{Z}^n}$ has exactly $|\det B|$ solutions \vec{x} in $(\mathbb{R}/\mathbb{Z})^n$.*

PROOF. Because of linearity, it is enough to show the statement for $\vec{b} = \vec{0}$. Consider a pair of lattices $L \supset M$, where L consists of the solutions $\vec{x} \in \mathbb{R}^n$ for the homogeneous equation $B\vec{x} \equiv \vec{0} \pmod{\mathbb{Z}^n}$, and $M = \mathbb{Z}^n$. The column vectors $\vec{f}_1, \dots, \vec{f}_n$ of B^{-1} is a basis of L . It is well-known [C, 1.2.2, Lemma 1] that $|L/M|$ equals to $|\det(\vec{f}_1, \dots, \vec{f}_n)|^{-1}$, which is $|\det B|$. \square

As it turns out, the condition (3.1) assures $\det B > 0$ not only for $B = A$ but also for all the relevant matrices B coming into the game. (cf. LEMMA 3.3.)

$M = 0$ Case. The equation (3.2) is void. We just define the number of off-diagonal solutions to be 1 in agreement with $\binom{L}{0}$.

$M = 1$ Case. The only string pattern (2.6) is $N_1 = 1$. Then (3.2) is a scalar equation $Lu_{1,1} \equiv c \pmod{\mathbb{Z}}$ for some $c \in \mathbb{R}$. Thus the number of the off-diagonal solution is L in agreement with $\binom{L}{1}$.

$M = 2$ Case. There are two string patterns, (i) $N_2 = 1$ and (ii) $N_1 = 2$, that satisfy (2.6). (i) The equation (3.2) is $Lu_{2,1} \equiv c$ for some $c \in \mathbb{R}$. Thus there are L off-diagonal solutions. (ii) The equation (3.2) reads

$$A\vec{u} = \begin{pmatrix} L-1 & 1 \\ 1 & L-1 \end{pmatrix} \begin{pmatrix} u_{1,1} \\ u_{1,2} \end{pmatrix} \equiv \vec{c} \pmod{\mathbb{Z}^2}$$

for some \vec{c} . From the assumption $P_1 = L - 4 \geq 0$, there are $\det A = L(L-2)$ solutions to this by PROPOSITION 3.1. However they contain the diagonal ones $u_{1,1} = u_{1,2}$. Under this constraint the above matrix equation reduces to $Lu_{1,1} \equiv c'$ with some c' , telling that the number of the diagonal solutions is L . Therefore the non-diagonal solutions from (ii) is enumerated as $(L(L-2) - L)/2$ by recalling the \mathfrak{S}_2 redundancy. Collecting the contributions from the string patterns (i) and (ii) one finds the number of off-diagonal solutions

$$L + \frac{1}{2}(L(L-2) - L) = \binom{L}{2}.$$

$M = 3$ Case. There are three string patterns, (i) $N_3 = 1$, (ii) $N_1 = N_2 = 1$ and (iii) $N_1 = 3$ that satisfy (2.6). (i) The equation (3.2) is $Lu_{3,1} \equiv c$ for some $c \in \mathbb{R}$. Thus there are L off-diagonal solutions. (ii) The equation (3.2) reads

$$A\vec{u} = \begin{pmatrix} L-2 & 2 \\ 2 & L-2 \end{pmatrix} \begin{pmatrix} u_{1,1} \\ u_{2,1} \end{pmatrix} \equiv \vec{c} \pmod{\mathbb{Z}^2}$$

for some \vec{c} . From the assumption $P_1 = L - 4, P_2 = L - 6 \geq 0$, this has $\det A = L(L - 4)$ solutions by PROPOSITION 3.1. In this case there is no permutation redundancy to remove. (iii) The equation (3.2) reads

$$A^{1/2/3}\vec{u} = \begin{pmatrix} L-2 & 1 & 1 \\ 1 & L-2 & 1 \\ 1 & 1 & L-2 \end{pmatrix} \begin{pmatrix} u_{1,1} \\ u_{1,2} \\ u_{1,3} \end{pmatrix} \equiv \vec{c} \pmod{\mathbb{Z}^3}$$

for some \vec{c} . Here we have written A as $A^{1/2/3}$ to match the notation in Section 3.3. Due to the assumption $P_1 = L - 6 \geq 0$, this has $\det A^{1/2/3} = L(L - 3)^2$ solutions in $(\mathbb{R}/\mathbb{Z})^3$ by PROPOSITION 3.1. However they contain various diagonal solutions. For example under the condition $u_{1,1} = u_{1,2}$, the above equation reduces to

$$A^{12/3}\vec{u} = \begin{pmatrix} L-1 & 1 \\ 2 & L-2 \end{pmatrix} \begin{pmatrix} u_{1,1} \\ u_{1,3} \end{pmatrix} \equiv \vec{c}' \pmod{\mathbb{Z}^2}$$

for some \vec{c}' , which has $\det A^{12/3} = L(L - 3)$ solutions. Similarly there are diagonal solutions counted by $\det A^{13/2}$ and $\det A^{23/1}$, which are both equal to $L(L - 3)$. Finally there is the completely diagonal one $u_{1,1} = u_{1,2} = u_{1,3}$ satisfying $A^{123}u_{1,1} = Lu_{1,1} \equiv c'' \pmod{\mathbb{Z}}$ for some c'' . By the inclusion-exclusion principle, we can now compute the number of off-diagonal solutions in (iii) as

$$(3.3) \quad \frac{1}{3!} (\det A^{1/2/3} - \det A^{12/3} - \det A^{13/2} - \det A^{23/1} + 2 \det A^{123}) \\ = \frac{1}{6} L(L - 4)(L - 5),$$

where we have removed the $3!$ -fold \mathfrak{S}_3 redundancy. Assembling the contributions from (i), (ii) and (iii) we get

$$L + L(L - 4) + \frac{1}{6} L(L - 4)(L - 5) = \binom{L}{3}$$

as desired.

3.3. General case. Let us proceed to the general case where both the quantum space data $\nu = (\nu_s)$ and the magnon number $M \in \mathbb{Z}_{\geq 0}$ are arbitrary. The examples in Section 3.2 already elucidate the essential feature in our counting. We dare do some overcounting by classifying diagonal solutions in terms of their patterns like 12/3, and finally subtract them via a kind of the inclusion-exclusion principle. The most natural framework to systematize such a process is the partition of sets and the Möbius inversion trick summarized in Appendix A. Compare the coefficients 1, -1 , -1 , -1 , 2 appearing in (3.3) with (A.1). Below we will use the terminology and the notation therein.

We start with SCE (3.2). Given $M \in \mathbb{N}$, fix a string pattern (N_m) satisfying (2.6). Set

$$(3.4) \quad \mathcal{J} = \{j \in \mathbb{N} \mid N_j > 0\}, \quad j_0 = \max \mathcal{J}.$$

Thus $d = \sum_{j \in \mathcal{J}} N_j$. Take an element $\pi = (\pi^{(1)}, \pi^{(2)}, \dots, \pi^{(j_0)}) \in L_{N_1} \times \cdots \times L_{N_{j_0}}$ in the sense of Appendix A. Thus for each k , $\pi^{(k)} = (\pi_1^{(k)}, \dots, \pi_l^{(k)})$ is a partition of the set $\{1, \dots, N_k\}$ into the blocks:

$$\{1, \dots, N_k\} = \pi_1^{(k)} \sqcup \cdots \sqcup \pi_l^{(k)}$$

for some l . Here and in what follows, if $N_i = 0$ for $1 \leq i < j_0$, the corresponding i th component should be just dropped. For example $L_{N_1} \times \cdots \times L_{N_{j_0}}$ actually means the product of the set L_{N_j} over $j \in \mathcal{J}$, and $(\pi^{(1)}, \dots, \pi^{(j_0)})$ is in fact $(\pi^{(j)})_{j \in \mathcal{J}}$. To classify the diagonal solutions, we introduce

$$\begin{aligned} \text{Sol}'_{\pi} &= \{\vec{u} = (u_{k\beta}) \mid u_{k\alpha} = u_{k\beta} \text{ if } \alpha \text{ and } \beta \text{ belong} \\ &\quad \text{to the same block of } \pi^{(k)} \text{ for each } k\}, \\ \text{Sol}_{\pi} &= \{\vec{u} = (u_{k\beta}) \mid u_{k\alpha} = u_{k\beta} \text{ if and only if } \alpha \text{ and } \beta \text{ belong} \\ &\quad \text{to the same block of } \pi^{(k)} \text{ for each } k\}. \end{aligned}$$

By the definition it follows that ($|\cdot|$ here denotes the cardinality.)

$$|\text{Sol}'_{\pi}| = \sum_{\pi' \leq \pi} |\text{Sol}_{\pi'}|$$

in terms of the partial order \leq of the poset $L_{N_1} \times \cdots \times L_{N_{j_0}}$ introduced in Appendix A.5. By means of the Möbius inversion formula this is equivalent to

$$|\text{Sol}_{\pi}| = \sum_{\pi' \leq \pi} \mu(\pi', \pi) |\text{Sol}'_{\pi'}| \quad \text{for any } \pi \in L_{N_1} \times \cdots \times L_{N_{j_0}}.$$

The off-diagonal solutions in question are counted by setting $\pi = \pi_{\max}$, which is the maximal element of $L_{N_1} \times \cdots \times L_{N_{j_0}}$ explained in Appendix

A.5. Removing the \mathfrak{S}_{N_m} redundancy for $m \in \mathcal{J}$, they are enumerated as

$$(3.5) \quad \frac{|\text{Sol}_{\pi_{\max}}|}{\prod_{m \in \mathcal{J}} N_m!} = \frac{\sum_{\pi \in L_{N_1} \times \dots \times L_{N_{j_0}}} \mu(\pi, \pi_{\max}) |\text{Sol}'_{\pi}|}{\prod_{m \in \mathcal{J}} N_m!}.$$

Let us evaluate $|\text{Sol}'_{\pi}|$. In the original SCE (3.2), impose the constraint $u_{k\alpha} = u_{k\beta}$ on $\vec{u} = (u_{k\beta})$, if α and β belong to the same block of $\pi^{(k)}$ for each $k \in \mathcal{J}$. As we got $A^{12/3}$ from $A = A^{1/2/3}$ in Section 3.2, the result has the form:

$$(3.6) \quad A^{\pi} \vec{u}_{\pi} \equiv \vec{c}_{\pi} \pmod{\mathbb{Z}^{l(\pi)}} \quad \text{for some } \vec{c}_{\pi} \in \mathbb{R}^{l(\pi)}.$$

Here $l(\pi)$ denotes the length of $\pi = (\pi^{(1)}, \dots, \pi^{(j_0)})$ as specified in Appendix A.5. In the new unknown $\vec{u}_{\pi} = (u_{k\beta})$, β is now labeled by the blocks of $\pi^{(k)}$ for each $k \in \mathcal{J}$. A^{π} is an $l(\pi)$ by $l(\pi)$ integer matrix obtained by a reduction of the matrix A which was d by d originally ($d = \sum_{j \in \mathcal{J}} N_j$). It is formed by summing up the $(k\beta)$ columns of A over those β belonging to the same block of $\pi^{(k)}$, and discarding all but one rows for each block. For example, suppose $\mathcal{J} = \{1, 2\}$. Thus only N_1 and N_2 are non-zero and $\pi = (\pi^{(1)}, \pi^{(2)}) \in L_{N_1} \times L_{N_2}$. Let $\pi^{(1)} = (\pi_1^{(1)}, \pi_2^{(1)}, \pi_3^{(1)})$ and $\pi^{(2)} = (\pi_1^{(2)}, \pi_2^{(2)})$ so that $l(\pi^{(1)}) = 3, l(\pi^{(2)}) = 2$ and $l(\pi) = 5$. Denote the number of elements in the block $\pi_i^{(1)}$ by λ_i (resp. $|\pi_i^{(2)}|$ by μ_i). By the definition $\lambda_1 + \lambda_2 + \lambda_3 = N_1, \mu_1 + \mu_2 = N_2$, and the matrix A^{π} reads ($\mathcal{P}_i = P_i + N_i$)

$$\begin{pmatrix} \mathcal{P}_1 + \lambda_1 & \lambda_2 & \lambda_3 & 2\mu_1 & 2\mu_2 \\ \lambda_1 & \mathcal{P}_1 + \lambda_2 & \lambda_3 & 2\mu_1 & 2\mu_2 \\ \lambda_1 & \lambda_2 & \mathcal{P}_1 + \lambda_3 & 2\mu_1 & 2\mu_2 \\ 2\lambda_1 & 2\lambda_2 & 2\lambda_3 & \mathcal{P}_2 + 3\mu_1 & 3\mu_2 \\ 2\lambda_1 & 2\lambda_2 & 2\lambda_3 & 3\mu_1 & \mathcal{P}_2 + 3\mu_2 \end{pmatrix}.$$

This is easily seen from the example of A in Section 3.1. In general A^{π} is not symmetric. Its matrix element is given by $A_{(m,i),(k,j)}^{\pi} = \delta_{m,k} \delta_{i,j} (P_m + N_m) + (2 \min(m, k) - \delta_{m,k}) |\pi_j^{(k)}|$ for $m, k \in \mathcal{J}, 1 \leq i \leq l(\pi^{(m)})$ and $1 \leq j \leq l(\pi^{(k)})$. Note that $A = A^{\pi_{\max}}$.

By the definition $|\text{Sol}'_{\pi}|$ counts the number of all the solutions to (3.6). Therefore from PROPOSITION 3.1 we have

$$(3.7) \quad |\text{Sol}'_{\pi}| = |\det A^{\pi}|.$$

It is straightforward to show

LEMMA 3.2. *For any $\pi \in L_{N_1} \times \cdots \times L_{N_{j_0}}$ we have*

$$(3.8) \quad \det A^\pi = \det_{m,k \in \mathcal{J}} (F_{m,k}) \prod_{m \in \mathcal{J}} (P_m + N_m)^{l(\pi^{(m)})-1},$$

$$(3.9) \quad F_{m,k} = \delta_{m,k} P_m + 2 \min(m, k) N_k.$$

The lemma holds without assuming (3.1). The dependence on π enters only through $l(\pi^{(m)})$. Like A^π , the matrix F is non-symmetric in general:

$$F = \begin{pmatrix} P_1 + 2N_1 & 2N_2 & 2N_3 & \cdots & 2N_{j_0} \\ 2N_1 & P_2 + 4N_2 & 4N_3 & \cdots & 4N_{j_0} \\ 2N_1 & 4N_2 & P_3 + 6N_3 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 2(j_0 - 1)N_{j_0} \\ 2N_1 & 4N_2 & 6N_3 & \cdots & P_{j_0} + 2j_0 N_{j_0} \end{pmatrix}.$$

We remark that

$$(3.10) \quad \sum_{k \geq 1} F_{m,k} = \gamma_m$$

for any m , where γ_m is defined in (2.17).

LEMMA 3.3. *If $P_m \geq 0$ for any $m \in \mathcal{J}$, then $\det A^\pi > 0$.*

PROOF. Denote $\det_{m,k \in \mathcal{J}} (F_{m,k})$ simply by $\det_{\mathcal{J}} F$. By LEMMA 3.2, it suffices to verify $\det_{\mathcal{J}} F > 0$. We do this by a double induction on $|\mathcal{J}|$ and $\sum_m P_m$ regarding P_m as non-negative variables independent of $\{N_m\}$. First let $\mathcal{J} = \{j_1 < \cdots < j_l\}$ be arbitrary and $\sum_m P_m = 0$. Thus $P_m = 0$ for any $m \in \mathcal{J}$, hence $\det_{\mathcal{J}} F = (\prod_{j \in \mathcal{J}} 2N_j) \det_{m,k \in \mathcal{J}} (\min(m, k)) = (\prod_{j \in \mathcal{J}} 2N_j) j_1(j_2 - j_1) \cdots (j_l - j_{l-1}) > 0$. Next let $\mathcal{J} = \{j\}$. Then $\det_{\mathcal{J}} F = P_j + 2N_j > 0$ because of the assumption $P_j \geq 0$. Finally let \mathcal{J} and $\sum_m P_m > 0$ be arbitrary. Then there exists $i \in \mathcal{J}$ such that $P_i > 0$. Setting $P'_j = P_j - \delta_{j,i}$ and $\mathcal{J}' = \mathcal{J} \setminus \{i\}$, one can expand the determinant as $\det_{\mathcal{J}} F(\{P_m\}) = \det_{\mathcal{J}'} F(\{P_m\}) + \det_{\mathcal{J}} F(\{P'_m\})$. By induction the two terms in the right hand side are both positive. \square

Although it is more direct to expand $\det_{m,k \in \mathcal{J}} (F_{m,k})$ from the beginning, we have presented a proof in the above form because it generalizes to arbitrary simple Lie algebra case that will be treated in our subsequent paper.

The specialization $\pi = \pi_{\max}$ in the above leads to

COROLLARY 3.4. *If $P_m \geq 0$ for any m , then $\det A > 0$.*

Given the quantum space data $\nu = (\nu_s)$ and the string pattern $N = (N_m)$, we define

$$(3.11) \quad R(\nu, N) = \det_{m,k \in \mathcal{J}} (F_{m,k}) \prod_{m \in \mathcal{J}} \frac{1}{N_m} \binom{P_m + N_m - 1}{N_m - 1},$$

when $\mathcal{J} \neq \emptyset$. Here $P_m = P_m(\nu, N)$, $F_{m,k}$ and \mathcal{J} are given by (2.16), (3.9) and (3.4), respectively. When $\mathcal{J} = \emptyset$, namely $\forall N_m = 0$, we set $R(\nu, N) = 1$ irrespective of ν . In the definition itself, we do not need to assume that $P_m \geq 0$ for those $m \in \mathcal{J}$, and (ν_s) can be arbitrary complex parameters.

THEOREM 3.5. *Assume that $P_m \geq 0$ for any $m \in \mathcal{J}$. Then the number of off-diagonal solutions to SCE (3.2) is equal to $R(\nu, N)$.*

PROOF. Under the assumption the number of off-diagonal solutions has already been obtained in (3.5). By virtue of (3.7) and LEMMA 3.3 it is equal to

$$\begin{aligned} & \frac{1}{\prod_{m \in \mathcal{J}} N_m!} \sum_{\pi \in L_{N_1} \times \cdots \times L_{N_{j_0}}} \mu(\pi, \pi_{\max}) \det A^\pi \\ &= \det_{m,k \in \mathcal{J}} (F_{m,k}) \sum_{\pi \in L_{N_1} \times \cdots \times L_{N_{j_0}}} \mu(\pi, \pi_{\max}) \prod_{m \in \mathcal{J}} \frac{(P_m + N_m)^{l(\pi^{(m)})-1}}{N_m!}, \end{aligned}$$

where we have substituted LEMMA 3.2. By means of (A.2) the π -sum can be taken, leading to

$$\det_{m,k \in \mathcal{J}} (F_{m,k}) \prod_{m \in \mathcal{J}} \frac{(P_m + N_m)_{N_m}}{N_m! (P_m + N_m)} = R(\nu, N).$$

□

Under the specialization $\nu_s = L\delta_{1,s}$, the above $R(\nu, N)$ reproduces the number of off-diagonal solutions for each string pattern exemplified in Section 3.2.

Expanding the determinant in (3.11), one can rewrite $R(\nu, N)$ as follows:

$$(3.12) \quad R(\nu, N) = \sum_{J \subset \mathbb{N}} D_J \prod_{m \in \mathbb{N} \setminus J} \binom{P_m + N_m}{N_m} \prod_{m \in J} \binom{P_m + N_m - 1}{N_m - 1},$$

$$(3.13) \quad D_J = \begin{cases} 1 & \text{if } J = \emptyset, \\ \det_{m,k \in J} (2 \min(m, k) - \delta_{m,k}) & \text{otherwise.} \end{cases}$$

In deriving this we have used $\binom{*}{0} = 1$, $\binom{*}{-1} = 0$. From this expression $R(\nu, N) \in \mathbb{Z}$ is manifest if $\forall \nu_s \in \mathbb{Z}$. In Section 4 we will mainly work with the formula (3.12) rather than (3.11).

4. $R(\nu, N)$ as weight multiplicity

Set

$$K(\nu, N) = \prod_{m \geq 1} \binom{P_m + N_m}{N_m}.$$

This is a generalization of Bethe's fermionic formula in (1.2) corresponding to the quantum space (2.1) $[\mathbf{K}]$. The nature of our $R(\nu, N)$ becomes most transparent by a parallel analysis on $K(\nu, N)$. It contains $K(\nu, N)$ as the summand in (3.12) corresponding to $J = \emptyset$.

In this section we fix $l \in \mathbb{N}$. It plays a role of "cut-off" similar to j_0 in (3.4) and has nothing to do with the length function of partitions. We will introduce various functions indexed with l , which tend to the quantities in our problem in the limit $l \rightarrow \infty$. In particular, P_m and γ_m in Sections 4.1–4.3 stand for the truncations of (2.16)–(2.17) by l :

$$(4.1) \quad P_m = P_m(\nu, N) = \gamma_m - 2 \sum_{k=1}^l \min(m, k) N_k,$$

$$(4.2) \quad \gamma_m = \gamma_m(\nu) = \sum_{k=1}^l \min(m, k) \nu_k.$$

We do not prepare new symbols for them as they will be used only in the said sections. We set $\mathbb{N}_l = \{1, 2, \dots, l\}$. The binomial coefficient is the one specified in the beginning of Section 3.2.

4.1. $R_l(\nu, N)$ and $K_l(\nu, N)$. Let $\nu = (\nu_s)$, $\nu_1, \dots, \nu_l \in \mathbb{C}$ and $N = (N_m)$, $N_1, \dots, N_l \in \mathbb{Z}_{\geq 0}$ be arbitrary. Define

$$(4.3) \quad R_l(\nu, N) = \sum_{J \subset \mathbb{N}_l} D_J \prod_{m \in \mathbb{N}_l \setminus J} \binom{P_m + N_m}{N_m} \prod_{m \in J} \binom{P_m + N_m - 1}{N_m - 1},$$

$$(4.4) \quad K_l(\nu, N) = \prod_{m \in \mathbb{N}_l} \binom{P_m + N_m}{N_m},$$

where D_J is specified by (3.13). When $N = 0$ (i.e., $\forall N_m = 0$), we have $R_l(\nu, 0) = K_l(\nu, 0) = 1$ irrespective of ν . Obviously one has $R(\nu, N) = \lim_{l \rightarrow \infty} R_l(\nu, N)$ and $K(\nu, N) = \lim_{l \rightarrow \infty} K_l(\nu, N)$, where

the limits render no subtlety. We will utilize two other expressions of $R_l(\nu, N)$. The first one is the analogue of (3.11):

$$(4.5) \quad R_l(\nu, N) = \left(\det_{m,k \in \{i \in \mathbb{N}_l \mid N_i \neq 0\}} F_{m,k} \right) \prod_{m \in \mathbb{N}_l, N_m \neq 0} \frac{1}{N_m} \binom{P_m + N_m - 1}{N_m - 1}.$$

To match (4.3), the right side of this should be understood as 1 when $N = 0$. To deduce the second expression, for $J \subset \mathbb{N}_l$ we introduce

$$(4.6) \quad \nu[J] = (\nu[J]_s), \quad \nu[J]_s = \nu_s - 2\theta(s \in J),$$

$$(4.7) \quad N[J] = (N[J]_m), \quad N[J]_m = N_m - \theta(m \in J),$$

where $\theta(\text{true}) = 1$ and $\theta(\text{false}) = 0$. With the aid of

$$P_m(\nu[J], N[J]) = P_m(\nu, N),$$

one can rewrite (4.3) as

$$(4.8) \quad R_l(\nu, N) = \sum_{J \subset \mathbb{N}_l} D_J \prod_{m \in \mathbb{N}_l} \binom{P[J]_m + N[J]_m}{N[J]_m}.$$

4.2. Generating functions. Let us introduce generating functions:

$$(4.9) \quad R_l(\nu|w) = \sum_N R_l(\nu, N) w_1^{N_1} \cdots w_l^{N_l},$$

$$(4.10) \quad K_l(\nu|w) = \sum_N K_l(\nu, N) w_1^{N_1} \cdots w_l^{N_l},$$

where $w = (w_1, \dots, w_l)$ and \sum_N extends over $N_1, \dots, N_l \in \mathbb{Z}_{\geq 0}$.

PROPOSITION 4.1. *When $\nu = 0$ (i.e., $\forall \nu_s = 0$) we have*

$$R_l(0|w) = 1.$$

PROOF. We show $R_l(0, N) = 0$ for any $N \neq 0$. Note that $\nu = 0$ implies $\forall \gamma_m = 0$. Therefore the assertion follows from the expression (4.5) and (3.10). \square

This simple observation will eventually lead to a non-trivial consequence (4.24) whose derivation is analogous to the ‘‘denominator formula’’. In contrast, $K_l(0|w)$ is not a simple function. See (4.19).

4.3. Analytic formula for generating functions. Consider the variables $\{z_{j,i-1} \mid 1 \leq i \leq j \leq l\}$ related via

$$(4.11) \quad z_{j,i} = z_{j,i-1} (1 - z_{i,i-1})^{-2(j-i)} \quad 1 \leq i < j \leq l.$$

For $1 \leq i \leq l$ we define the function ψ_i by

$$(4.12) \quad \psi_i = \prod_{j=i}^l (1 - z_{j,j-1})^{-\beta_j-1} \quad 1 \leq i \leq l,$$

where $\beta_1, \dots, \beta_l \in \mathbb{C}$ are parameters.

LEMMA 4.2. ψ_i has a formal power series expansion

$$\psi_i = \sum_{\{N_j\}} \prod_{j=i}^l \binom{\beta_j + 2 \sum_{j < k \leq l} (k-j)N_k + N_j}{N_j} (z_{j,i-1})^{N_j} \quad 1 \leq i \leq l,$$

where the sum $\sum_{\{N_j\}}$ extends over $N_i, \dots, N_l \in \mathbb{Z}_{\geq 0}$.

PROOF. We prove by induction on i . The case $i = l$ is due to the formula

$$(4.13) \quad (1 - z)^{-\beta-1} = \sum_{N=0}^{\infty} \binom{\beta + N}{N} z^N.$$

Assume ψ_{i+1} has the above expansion. Then from (4.12) ψ_i is

$$(1 - z_{i,i-1})^{-\beta_i-1} \sum'_{\{N_j\}} \prod_{j=i+1}^l \binom{\beta_j + 2 \sum_{j < k \leq l} (k-j)N_k + N_j}{N_j} (z_{j,i})^{N_j},$$

where the sum $\sum'_{\{N_j\}}$ is over $N_{i+1}, \dots, N_l \in \mathbb{Z}_{\geq 0}$. Upon substituting (4.11), the right hand side becomes

$$\sum'_{\{N_j\}} \left\{ (1 - z_{i,i-1})^{-\beta_i-2 \sum_{i < k \leq l} (k-i)N_k-1} \right. \\ \left. \times \prod_{j=i+1}^l \binom{\beta_j + 2 \sum_{j < k \leq l} (k-j)N_k + N_j}{N_j} (z_{j,i-1})^{N_j} \right\}.$$

Applying (4.13) again, we obtain the desired expansion. \square

This lemma is originally due to [K]. Here we have quoted the version reproduced in [HKOTY].

In the sequel, we will only work with the variables $z_j = z_{j,0}$, $v_j = z_{j,j-1}$ and w_j ($1 \leq j \leq l$):

$$(4.14) \quad v_j = z_j \prod_{k=1}^{j-1} (1 - v_k)^{-2(j-k)}, \quad w_j = z_j \prod_{k=1}^l (1 - v_k)^{-2j}$$

where the former relation is due to (4.11), while the latter is the definition of w_j . (z_i here should not be confused with the $z_i(q)$ in Section 2.) Note that $v_i = w_i \prod_{k=1}^l (1 - v_k)^{2\min(i,k)}$. Let $z = (z_1, \dots, z_l)$ and $w = (w_1, \dots, w_l)$. We will denote $z_1 = \dots = z_l = 0$ simply by $z = 0$, and $dz_1 \wedge \dots \wedge dz_l / z_1 \cdots z_l$ by dz/z , etc. The variables z are holomorphic functions of w around $w = 0$. This is due to $w = 0$ and $\partial w_i / \partial z_j = \delta_{ij}$ at $z = 0$. Setting $i = 1$ in LEMMA 4.2, we have

$$(4.15) \quad \prod_{j=1}^l \binom{\beta_j + 2 \sum_{j < k \leq l} (k-j) N_k + N_j}{N_j} = \operatorname{Res}_{z=0} \left(\prod_{j=1}^l (1 - v_j)^{-\beta_j - 1} z_j^{-N_j} \right) \frac{dz}{z}.$$

Under a further specialization to $\beta_j = \gamma_j - 2 \sum_{k=1}^l k N_k$, this becomes

$$(4.16) \quad \prod_{m \in \mathbb{N}_l} \binom{P_m + N_m}{N_m} = \operatorname{Res}_{z=0} \left(\prod_{j=1}^l (1 - v_j)^{-\gamma_j - 1} w_j^{-N_j} \right) \frac{dz}{z} \\ = \operatorname{Res}_{w=0} \left(\prod_{j=1}^l (1 - v_j)^{-\gamma_j - l(l+1) - 1} w_j^{-N_j} \right) \frac{\partial z}{\partial w} \frac{dw}{w},$$

where $\partial z / \partial w$ represents the Jacobian $\det_{i,j \in \mathbb{N}_l} (\partial z_i / \partial w_j)$. In (4.16) replace N_m by $N[J]_m$, P_m by $P_m(\nu[J], N[J])$ defined in (4.6)–(4.7), and γ_j by $\gamma_j - 2 \sum_{i \in J} \min(j, i)$. The result reads

$$(4.17) \quad \prod_{m \in \mathbb{N}_l} \binom{P[J]_m + N[J]_m}{N[J]_m} \\ = \operatorname{Res}_{w=0} \left(\prod_{j=1}^l (1 - v_j)^{-\gamma_j - l(l+1) - 1} w_j^{-N_j} \right) \left(\prod_{i \in J} v_i \right) \frac{\partial z}{\partial w} \frac{dw}{w}.$$

Notice that the left side constitutes the summand in (4.8). In terms of the generating functions the results (4.16)–(4.17) are stated as

PROPOSITION 4.3.

$$(4.18) \quad K_l(\nu|w) = K_l(0|w) \prod_{j=1}^l (1 - v_j)^{-\gamma_j},$$

$$(4.19) \quad K_l(0|w) = \frac{\partial z}{\partial w} \prod_{j=1}^l (1 - v_j)^{-l(l+1)-1},$$

$$(4.20) \quad R_l(\nu|w) = R_l(0|w) \prod_{j=1}^l (1 - v_j)^{-\gamma_j},$$

$$(4.21) \quad R_l(0|w) = K_l(0|w) \left(\sum_{J \subset \mathbb{N}_l} D_J \prod_{i \in J} v_i \right).$$

Combining this with PROPOSITION 4.1, we find

THEOREM 4.4.

$$(4.22) \quad R_l(\nu|w) = \frac{K_l(\nu|w)}{K_l(0|w)} = \prod_{j=1}^l (1 - v_j)^{-\gamma_j},$$

$$(4.23) \quad R_l(\nu|w)R_l(\nu'|w) = R_l(\nu + \nu'|w),$$

$$(4.24) \quad \frac{\partial w}{\partial z} \prod_{j=1}^l (1 - v_j)^{l(l+1)+1} = \sum_{J \subset \mathbb{N}_l} D_J \prod_{i \in J} v_i,$$

where $\nu + \nu' = (\nu_s + \nu'_s)_{s=1}^l$ and γ_j is the truncated one (4.2).

The factorization property (4.23) is enjoyed only by R_l and not by K_l . It is due to (4.22) and $\gamma(\nu + \nu')_j = \gamma(\nu)_j + \gamma(\nu')_j$.

4.4. $l \rightarrow \infty$ limit. Let $R(\nu|w) = \lim_{l \rightarrow \infty} R_l(\nu|w)$ and $K(\nu|w) = \lim_{l \rightarrow \infty} K_l(\nu|w)$ be formal power series in infinitely many variables $w = (w_j)_{j \geq 1}$. They can also be viewed as the series in $(v_j)_{j \geq 1}$ upon the substitution $w_i = v_i \prod_{k \geq 1} (1 - v_k)^{-2 \min(i,k)}$. See the remark after (4.14). In the $l \rightarrow \infty$ limit THEOREM 4.4 yields

THEOREM 4.5.

$$(4.25) \quad R(\nu|w) = \frac{K(\nu|w)}{K(0|w)} = \prod_{j \geq 1} (1 - v_j)^{-\gamma_j},$$

$$(4.26) \quad R(\nu|w)R(\nu'|w) = R(\nu + \nu'|w),$$

where $\nu + \nu' = (\nu_s + \nu'_s)_{s \geq 1}$ and γ_j is defined by (2.17).

We specialize $R(\nu|w)$ and $K(\nu|w)$ as follows:

(4.27)

$$R(\nu) := e^{\gamma_\infty(\nu)\Lambda_1} R(\nu|w)|_{w_j=e^{-j\alpha_1}} = \sum_N R(\nu, N) x^{\gamma_\infty(\nu)-2\sum_{j\geq 1} jN_j},$$

(4.28)

$$K(\nu) := e^{\gamma_\infty(\nu)\Lambda_1} K(\nu|w)|_{w_j=e^{-j\alpha_1}} = \sum_N K(\nu, N) x^{\gamma_\infty(\nu)-2\sum_{j\geq 1} jN_j},$$

where the sum \sum_N runs over $N_1, N_2, \dots \in \mathbb{Z}_{\geq 0}$, α_1 and Λ_1 are the simple root and the fundamental weight, respectively. $x = e^{\Lambda_1}$ is a formal variable and $\gamma_\infty(\nu) = \sum_{s\geq 1} s\nu_s$ in accordance with (2.17). From THEOREM 4.5 it follows that

$$(4.29) \quad R(\nu) = \frac{K(\nu)}{K(0)},$$

$$(4.30) \quad R(\nu)R(\nu') = R(\nu + \nu').$$

We remark that PROPOSITION 4.3, THEOREM 4.4 and 4.5 are all valid for $\nu_s \in \mathbb{C}$. The specialization $w_j = e^{-j\alpha_1}$ induces an effect also for v_j (4.14) and $\prod_{j\geq 1} (1 - v_j)^{-\gamma_j}$ in (4.25). It will be worked out in Section 4.5.

4.5. Combinatorial completeness. From now on we assume that $\forall \nu_s \in \mathbb{Z}_{\geq 0}$. For $m \in \mathbb{Z}_{\geq 0}$ we define

$$(4.31) \quad \begin{aligned} \delta_m &= (\nu_s), \quad \nu_s = \delta_{s,m}, \\ Q_m &= R(\delta_m) \in x^m \mathbb{C}[[x^{-2}]]. \end{aligned}$$

From the decomposition $\nu = (\nu_s) = \sum_{s\geq 1} \nu_s \delta_s$ and (4.30) we have

$$(4.32) \quad R(\nu) = \prod_{s\geq 1} Q_s^{\nu_s}$$

for general $\nu = (\nu_s)$.

PROPOSITION 4.6.

$$R(\lambda) = R(\mu) + R(\nu),$$

where $\lambda = (\lambda_s)$, $\mu = (\mu_s)$ and $\nu = (\nu_s)$ are related as ($s \in \mathbb{N}$)

$$\lambda_s = \nu_s + 2\delta_{s,k}, \quad \mu_s = \nu_s + \delta_{s,k+1} + \delta_{s,k-1} \quad \text{for some } k \in \mathbb{N}.$$

PROOF. Put $N' = (N'_m)$, $N'_m = N_m - \delta_{m,k}$. Then it is easy to check

$$(4.33) \quad \gamma_\infty(\lambda) - 2 \sum_{j\geq 1} jN_j = \gamma_\infty(\mu) - 2 \sum_{j\geq 1} jN_j = \gamma_\infty(\nu) - 2 \sum_{j\geq 1} jN'_j,$$

$$(4.34) \quad P_m(\lambda, N) = P_m(\nu, N') = P_m(\mu, N) + \delta_{m,k}.$$

From (4.27) and (4.33) we are to show $R(\lambda, N) = R(\mu, N) + R(\nu, N')$. By expanding a binomial coefficient in (3.12), the $R(\lambda, N)$ is expressed as ($P_m = P_m(\lambda, N)$)

$$\begin{aligned} & \sum_{J \subset \mathbb{N}, k \notin J} D_J(A+B) \prod_{m \in \mathbb{N} \setminus J, m \neq k} \binom{P_m + N_m}{N_m} \prod_{m \in J} \binom{P_m + N_m - 1}{N_m - 1} \\ & + \sum_{J \subset \mathbb{N}, k \in J} D_J \prod_{m \in \mathbb{N} \setminus J} \binom{P_m + N_m}{N_m} (C+D) \prod_{m \in J, m \neq k} \binom{P_m + N_m - 1}{N_m - 1}, \end{aligned}$$

where $A = \binom{P_k + N_k - 1}{N_k}$, $B = \binom{P_k + N_k - 1}{N_k - 1}$, $C = \binom{P_k + N_k - 2}{N_k - 1}$ and $D = \binom{P_k + N_k - 2}{N_k - 2}$. By (4.34) they can also be written as $A = \binom{P_k(\mu, N) + N_k}{N_k}$, $B = \binom{P_k(\nu, N') + N'_k}{N'_k}$, $C = \binom{P_k(\mu, N) + N_k - 1}{N_k - 1}$ and $D = \binom{P_k(\nu, N') + N'_k - 1}{N'_k - 1}$. Thus the contributions containing A and C (resp. B and D) amount to $R(\mu, N)$ (resp. $R(\nu, N')$). \square

Let $\mathbb{Q}((x))$ denote the field of the formal Laurent series in x over \mathbb{Q} with finitely many negative powers. Clearly, $Q_m \in \mathbb{Q}((x^{-1}))$.

PROPOSITION 4.7. (i) Q_m satisfies

(a) (recursion relation)

$$Q_0 = 1, \quad Q_k^2 = Q_{k+1}Q_{k-1} + 1 \quad k \in \mathbb{N},$$

(b) (asymptotic property)

$$\lim_{k \rightarrow \infty} \frac{Q_{k+1}}{Q_k} = x.$$

(ii) Conversely, the properties (a) and (b) above characterize the series $Q_m \in \mathbb{Q}((x^{-1}))$.

PROOF. (i). (a) Put $\nu = 0$ in PROPOSITION 4.6 and apply (4.32). (b) It is enough to show that the limit $\lim_{k \rightarrow \infty} x^{-k}Q_k$ exists in $\mathbb{Q}[[x^{-1}]]$. Note that $P_m(\delta_k, N) = P_m(\delta_{k+1}, N) - \theta(m \geq k+1)$ from (2.16). In the series $x^{-k}Q_k = x^{-k}R(\delta_k)$ in (4.27), those $N = (N_m)$ containing $N_j > 0$ with $j \geq k+1$ make contributions in the order higher than $2k+1$. It follows that $x^{-k}Q_k \equiv x^{-k-1}Q_{k+1} \pmod{x^{-2k-2}\mathbb{Q}[[x^{-1}]]}$. Then, we have

$$x^{-k}Q_k \equiv x^{-k-1}Q_{k+1} \equiv x^{-k-2}Q_{k+2} \equiv \cdots \pmod{x^{-2k-2}\mathbb{Q}[[x^{-1}]]},$$

which means $\lim_{k \rightarrow \infty} x^{-k}Q_k$ exists. (ii). Suppose \tilde{Q}_m satisfies (a) and (b). Setting $v_j = 1 - \frac{\tilde{Q}_{j-1}\tilde{Q}_{j+1}}{\tilde{Q}_j^2}$, we find

$$\prod_{j=1}^l (1 - v_j)^{-\gamma_j} = \left(\frac{\tilde{Q}_l}{\tilde{Q}_{l+1}} \right)^{\gamma_l} \prod_{j=1}^l \tilde{Q}_j^{\nu_j}, \quad w_j = \left(\frac{\tilde{Q}_l}{\tilde{Q}_{l+1}} \right)^{2j}$$

by (4.14). (γ_j here is the truncated one (4.2).) Therefore (4.22) specializes to

$$\sum_N R_l(\nu, N) \prod_{j=1}^l \left(\frac{\tilde{Q}_l}{\tilde{Q}_{l+1}} \right)^{2jN_j} = \left(\frac{\tilde{Q}_l}{\tilde{Q}_{l+1}} \right)^{\gamma_l} \prod_{j=1}^l \tilde{Q}_j^{\nu_j},$$

where \sum_N is over $N_1, \dots, N_l \in \mathbb{Z}_{\geq 0}$. By taking the limit $l \rightarrow \infty$ using (b) for \tilde{Q}_m , this leads to $R(\nu) = \prod_{j \geq 1} \tilde{Q}_j^{\nu_j}$. Since ν_j 's are arbitrary, we obtain $\tilde{Q}_m = R(\delta_m)$. Comparing this with (4.31) we conclude $\tilde{Q}_m = Q_m$. \square

It is immediate to check that the character of the $(m+1)$ -dimensional irreducible $U_q(\hat{\mathfrak{sl}}(2))$ -module W_m (character with respect to the classical Cartan subalgebra)

$$\text{ch } W_m = \frac{x^{m+1} - x^{-m-1}}{x - x^{-1}}, \quad x = e^{\Lambda_1}$$

fulfills the properties (a) and (b) in PROPOSITION 4.7. Thus from (ii) we have

PROPOSITION 4.8.

$$Q_m = \text{ch } W_m \quad m \in \mathbb{Z}_{\geq 0}.$$

Our main result in Section 4 is the following.

THEOREM 4.9 (Combinatorial completeness). *Let $W(\nu)$ be the quantum space in (2.1), $W(\nu) = \bigotimes_{s \geq 1} (W_s)^{\otimes \nu_s}$.*

(i)

$$R(\nu) = \text{ch } W(\nu),$$

$$\sum_N {}^{(\lambda)} R(\nu, N) = \dim W(\nu)_\lambda \quad \lambda \in \mathbb{Z}\Lambda_1.$$

Here the sum $\sum_N {}^{(\lambda)}$ extends over $N_1, N_2, \dots \in \mathbb{Z}_{\geq 0}$ such that $\sum_{j \geq 1} j(\nu_j - 2N_j)\Lambda_1 = \lambda$, and $\dim W(\nu)_\lambda$ denotes the multiplicity of the weight λ . In particular $R(\nu)$ is invariant under the Weyl group.

(ii) (Kirillov [K])

$$K(\nu) = (1 - e^{-\alpha_1}) \text{ch } W(\nu),$$

$$\sum_N {}^{(\lambda)} K(\nu, N) = [W(\nu) : V_\lambda] \quad \lambda \in (\mathbb{Z}_{\geq 0})\Lambda_1.$$

Here $[W(\nu) : V_\lambda]$ denotes the multiplicity of the irreducible $U_q(\hat{\mathfrak{sl}}(2))$ -module V_λ with highest weight λ . The sum $\sum_N {}^{(\lambda)}$ is the same as (i). In particular $e^{\Lambda_1} K(\nu)$ is skew-invariant under the Weyl group.

PROOF. (i) In view of (4.27), the two equalities are equivalent. The first one is due to (4.32) and $\text{ch } W(\nu) = \prod_{s \geq 1} Q_s^{\nu_s}$ by PROPOSITION 4.8. (ii) In view of (4.28), the two equalities are again equivalent. To be self-contained, let us include a quick proof of the first one although this has been done in [K]. Let γ_j be as in (4.2) and $\mu \in \mathbb{Z}_{\geq 0}$. In the expansion of $\prod_{j=1}^l (1 - v_j)^{-\beta_j - 1}$ by means of (4.15), specialize the variables as $v_j = 1 - \frac{Q_{j-1}Q_{j+1}}{Q_j^2}$ (hence $z_j = Q_1^{-2j}$) and $\beta_j = \gamma_j - \mu$. The result reads

$$\begin{aligned} & Q_1^{-\mu+1} \left(\frac{Q_l}{Q_{l+1}} \right)^{\gamma_l - \mu + 1} \text{ch } W(\nu) \\ &= \sum_N Q_1^{-2 \sum_{i=1}^l i N_i} \prod_{j=1}^l \binom{\gamma_j - \mu + 2 \sum_{j < k \leq l} (k - j) N_k + N_j}{N_j}, \end{aligned}$$

where \sum_N is taken over $N_1, N_2, \dots \in \mathbb{Z}_{\geq 0}$. Picking up the coefficient of $Q_1^{-\mu}$, we get

$$\begin{aligned} \sum_{N: 2 \sum_{i=1}^l i N_i = \mu} K_l(\nu, N) &= \text{Res}_{Q_1 = \infty} \left(Q_1 \left(\frac{Q_l}{Q_{l+1}} \right)^{\gamma_l - \mu + 1} \text{ch } W(\nu) \right) \frac{dQ_1}{Q_1} \\ &= \text{Res}_{x = \infty} \left(x(1 - x^{-2}) \left(\frac{Q_l}{Q_{l+1}} \right)^{\gamma_l - \mu + 1} \text{ch } W(\nu) \right) \frac{dx}{x}, \end{aligned}$$

where $Q_1 = x + x^{-1}$ is used. In the limit $l \rightarrow \infty$ this is equivalent to $K(\nu) = (1 - e^{-\alpha_1}) \text{ch } W(\nu)$ due to (4.28) and the property (b) in PROPOSITION 4.7. \square

It is curious that in general the sum $\sum_N^{(\lambda)}$ involves the contributions from those N that do not satisfy the assumption in THEOREM 3.5.

5. Discussion

In this paper we have proposed the string center equation (SCE) relevant to the string solutions of the Bethe equation at $q = 0$. The number of off-diagonal solutions to SCE is identified with the weight multiplicities of the quantum space by constructing an explicit combinatorial formula $R(\nu, N)$.

It is quite common to reduce the Bethe equation to the one for string centers. Indeed such analyses have been done extensively at $q = 1$, and has led to the well known fermionic formula $K(\nu, N)$ [K]. However at $q = 0$, systematic counting of the number of solutions had been left untouched. The result in this paper reveals another aspect of the combinatorial completeness of the string hypothesis. The fermionic form $K(\nu, N)$ is relevant to $q = 1$ and the multiplicity of irreducible

components, while our $R(\nu, N)$ is relevant to $q = 0$ and the weight multiplicities. Their generating functions are simply related as (4.25) and (4.29).

In this paper we have exclusively treated the $U_q(\hat{\mathfrak{sl}}(2))$ case. Many results here admit straightforward generalizations to $U_q(X_n^{(1)})$, which will be the subject of our subsequent paper. In place of (3.12)–(3.13), our main formula is $(\nu = (\nu_s^{(a)}), N = (N_m^{(a)}))$

$$R(\nu, N) = \sum_{J \subset \mathbb{N}^n} D_J \prod_{(a,m) \in \mathbb{N}^n \setminus J} \binom{P_m^{(a)} + N_m^{(a)}}{N_m^{(a)}} \prod_{(a,m) \in J} \binom{P_m^{(a)} + N_m^{(a)} - 1}{N_m^{(a)} - 1},$$

$$D_J = \begin{cases} 1 & \text{if } J = \emptyset, \\ \det_{(a,m),(b,k) \in J} ((\alpha_a | \alpha_b) \min(t_b m, t_a k) - \delta_{a,b} \delta_{m,k}) & \text{otherwise,} \end{cases}$$

where $\mathbb{N}^n = \{(a, m) \mid 1 \leq a \leq n, m \in \mathbb{N}\}$, and the other notations are the same as [HKOTY] under the identification of $(P_j^{(a)}, N_j^{(a)})$ here with $(p_j^{(a)}, m_j^{(a)})$ there. As the $\mathfrak{sl}(2)$ case, the above $R(\nu, N)$ contains the fermionic form in [KR] as the summand corresponding to $J = \emptyset$. With this $R(\nu, N)$, THEOREM 3.5, 4.5, PROPOSITION 4.7 generalize to arbitrary X_n . On the other hand, the identification like PROPOSITION 4.8, hence THEOREM 4.9 are attained for the non-exceptional series $X_n = A_n, B_n, C_n, D_n$ only, due to a technical complexity.

Another direction of the generalization is to seek a q -analogue of $R(\nu, N)$ that expresses the unrestricted one dimensional configuration sums (1dsms) over the quantum space W in the sense of [HKOTY]. So far we have only obtained a conjecture for the XXZ case jointly with G. Hatayama, M. Okado and T. Takagi.

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Appendix A. Möbius function $\mu(\pi, \pi')$

Let us explain a minimum about the partition of sets and the Möbius function on it. For a more extensive treatment see [A, B, S].

A.1. Partition of set. Let $N \in \mathbb{N}$. By definition $\pi = (\pi_1, \dots, \pi_l)$ is called a *partition* of a set $\{1, \dots, N\}$ if

$$\{1, \dots, N\} = \pi_1 \sqcup \dots \sqcup \pi_l$$

is a disjoint union decomposition. Here, the ordering of $\pi_1, \pi_2, \dots, \pi_l$ does not matter, e.g., $(\pi_1, \pi_2, \dots, \pi_l)$ and $(\pi_2, \pi_1, \dots, \pi_l)$ are the same partition. Each π_i is called a *block* of π and l is called a *length* of π . Let L_N denote the set of partitions of $\{1, \dots, N\}$. Here are the first three:

$$\begin{aligned} L_1 &= \{1\}, \\ L_2 &= \{12, 1/2\}, \\ L_3 &= \{123, 12/3, 13/2, 23/1, 1/2/3\}, \end{aligned}$$

where, for example, $23/1$ stands for the partition $\pi = (\pi_1, \pi_2)$ of length $l(\pi) = 2$ consisting of the blocks $\pi_1 = \{2, 3\}$ and $\pi_2 = \{1\}$.

A.2. Poset structure. One can endow a natural partial order “ \leq ” with the set L_N . Given two partitions $\pi, \pi' \in L_N$, we say $\pi \leq \pi'$ if each block of π' is contained in a block of π . For L_2 in the above we have $12 \leq 1/2$, and for L_3

$$\begin{array}{c} 12/3 \\ 123 \leq 13/2 \leq 1/2/3, \\ 23/1 \end{array}$$

where there is no order among the middle three. Sometimes π' is called a *refinement* of π when $\pi \leq \pi'$. The partition $\pi_{\max} = 1/2/\dots/N$ (resp. $\pi_{\min} = 12\dots N$) is the unique maximal (resp. minimal) element in L_N of length $l(\pi_{\max}) = N$ (resp. $l(\pi_{\min}) = 1$). Clearly the following three axioms hold:

1. For any $\pi \in L_N$, $\pi \leq \pi$. (reflexivity)
2. If $\pi \leq \pi'$ and $\pi' \leq \pi$, then $\pi = \pi'$. (antisymmetry)
3. If $\pi \leq \pi'$ and $\pi' \leq \pi''$, then $\pi \leq \pi''$. (transitivity)

Thus L_N equipped with \leq is a partially ordered set (poset) in the sense of [S].

A.3. Möbius function. Consider an $|L_N|$ by $|L_N|$ matrix ζ defined by

$$\zeta = (\zeta(\pi, \pi'))_{\pi, \pi' \in L_N}, \quad \zeta(\pi, \pi') = \begin{cases} 1 & \text{if } \pi \leq \pi' \\ 0 & \text{otherwise.} \end{cases}$$

This matrix is upper triangular with all the diagonal elements being 1. Thus it has the inverse

$$\zeta \mu = 1_{L_N}, \quad \mu = (\mu(\pi, \pi'))_{\pi, \pi' \in L_N}.$$

The matrix elements $\mu(\pi, \pi') \in \mathbb{Z}$ are called the Möbius function of the poset L_N . Note from the definition that $\mu(\pi, \pi) = 1$ for any π and

$\mu(\pi, \pi') = 0$ unless $\pi \leq \pi'$. For example in $N = 2$ and 3 cases in the above they are explicitly given by

$$(A.1) \quad \mu = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad \mu = \begin{pmatrix} 1 & -1 & -1 & -1 & 2 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

For general N , an explicit formula of $\mu(\pi, \pi')$ is available [A, B], but we do not need it in this paper.

A.4. Möbius inversion formula. Given any function $f : L_N \rightarrow \mathbb{C}$, define another function $g : L_N \rightarrow \mathbb{C}$ by

$$g(\pi) = \sum_{\pi' \leq \pi} f(\pi').$$

This is a composition with ζ introduced previously. In the vector-matrix notation it is expressed as $g = f\zeta$, hence is equivalent to $f = g\mu$:

$$f(\pi) = \sum_{\pi' \leq \pi} g(\pi')\mu(\pi', \pi),$$

which is the Möbius inversion formula. Here is a simple example of its application:

PROPOSITION A.1. *Let X be an indeterminate. For any $\pi \in L_N$ we have*

$$\begin{aligned} X^{l(\pi)} &= \sum_{\pi' \leq \pi} (X)_{l(\pi')}, \\ (X)_{l(\pi)} &= \sum_{\pi' \leq \pi} \mu(\pi', \pi) X^{l(\pi')}, \end{aligned}$$

where $(X)_l = X(X-1)\cdots(X-l+1)$.

PROOF. It suffices to show the former assuming that X is any positive integer. Notice that $X^{l(\pi)}$ is the number of maps $\phi : \{1, \dots, N\} \rightarrow \{1, \dots, X\}$ such that $\phi(i) = \phi(j)$ if i and j belong to the same block of π . Similarly, $(X)_{l(\pi)}$ is the number of maps $\phi : \{1, \dots, N\} \rightarrow \{1, \dots, X\}$ such that $\phi(i) = \phi(j)$ if and only if i and j belong to the same block of π . Since “if” case consists of the disjoint union of “if and only if” cases labeled by $\pi' (\leq \pi)$, the former relation holds. \square

A.5. Product poset. Consider the product set $L_{N_1} \times \cdots \times L_{N_m}$ for any positive integers N_1, \dots, N_m . Denote its elements by $\pi = (\pi^{(1)}, \dots, \pi^{(m)})$, where $\pi^{(i)} \in L_{N_i}$. One can equip $L_{N_1} \times \cdots \times L_{N_m}$ with a poset structure by introducing the partial order as

$$(\pi^{(1)}, \dots, \pi^{(m)}) \leq (\pi^{(1)'}, \dots, \pi^{(m)'}) \stackrel{\text{def}}{\iff} \pi^{(i)} \leq \pi^{(i)'} \text{ for any } 1 \leq i \leq m.$$

The length function is also introduced as $l((\pi^{(1)}, \dots, \pi^{(m)})) = l(\pi^{(1)}) + \cdots + l(\pi^{(m)})$. The unique maximal element in $L_{N_1} \times \cdots \times L_{N_m}$ is $\pi_{\max} = (\pi_{\max}^{(1)}, \dots, \pi_{\max}^{(m)})$, where $\pi_{\max}^{(i)} = 1/2/\cdots/N_i$ is the maximal one in L_{N_i} . Obviously the Möbius function of this poset is the direct product of the one for each component:

$$\mu((\pi^{(1)}, \dots, \pi^{(m)}), (\pi^{(1)'}, \dots, \pi^{(m)'})) = \prod_{i=1}^m \mu_{N_i}(\pi^{(i)}, \pi^{(i)'}),$$

where we have written the Möbius function of L_{N_i} as μ_{N_i} . Combining this with PROPOSITION A.1, we get

$$(A.2) \quad \sum_{\pi \in L_{N_1} \times \cdots \times L_{N_m}} \mu(\pi, \pi_{\max}) X_1^{l(\pi^{(1)})} \cdots X_m^{l(\pi^{(m)})} = \prod_{i=1}^m (X_i)_{l(\pi_{\max}^{(i)})} = \prod_{i=1}^m (X_i)_{N_i},$$

where X_1, \dots, X_m are indeterminates. In the main text we use the Möbius inversion formula for the poset $L_{N_1} \times \cdots \times L_{N_m}$ and (A.2).

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