

On the Kodaira Dimension of the Moduli Space of $K3$ Surfaces II

SHIGEYUKI KONDŌ*

Graduate School of Polymathematics, Nagoya University, Nagoya, 464-01 Japan
e-mail: kondo@math.nagoya-u.ac.jp

(Received: 12 May 1997; accepted in final form: 21 October 1997)

Abstract. We show that the Kodaira dimension of the moduli space of polarized $K3$ surfaces of degree $2n$ is nonnegative if $n = 42, 43, 51, 53, 55, 57, 59, 61, 66, 67, 69, 74, 83, 85, 105, 119$ or 133 . We use an automorphic form associated with the fake monster Lie algebra constructed by Borcherds.

Mathematics Subject Classifications (1991): 14J28, 11F55.

Key words: $K3$ surface, automorphic forms, moduli.

1. Introduction

Let \mathcal{K}_{2n} be the coarse moduli space of $K3$ surfaces with a primitive polarization of degree $2n$. It follows from the Torelli theorem for $K3$ surfaces that \mathcal{K}_{2n} is described as an arithmetic quotient $\mathcal{D}_{2n}/\Gamma_{2n}$ where \mathcal{D}_{2n} is a 19-dimensional bounded symmetric domain of type IV and Γ_{2n} is an arithmetic subgroup of $O(2, 19)_{\mathbb{Q}}$. Recently, Borcherds [2] constructed a remarkable automorphic form Φ of weight 12 on a 26-dimensional bounded symmetric domain of type IV. By dividing Φ by a product of linear functions vanishing on \mathcal{D}_{2n} and by restricting it to \mathcal{D}_{2n} , we have an automorphic form Φ_{2n} on \mathcal{D}_{2n} with respect to Γ_{2n} (see Borcherds *et al.* [3]). The purpose of this note is to show the following theorem:

THEOREM. *Assume that $n = 42, 43, 51, 53, 55, 57, 59, 61, 66, 67, 69, 74, 83, 85, 105, 119$ or 133 . Then Φ_{2n} is a cusp form of weight 19 on \mathcal{D}_{2n} with respect to Γ_{2n} .*

An automorphic form of weight 19 on \mathcal{D}_{2n} gives a section of the canonical line bundle of the smooth locus of $\mathcal{D}_{2n}/\Gamma_{2n}$. Moreover, it is known that a cusp form of weight 19 gives a global section of the canonical line bundle of a smooth model of a compactification of $\mathcal{D}_{2n}/\Gamma_{2n}$ (Bauermann [1]). Thus we have the corollary:

COROLLARY 1.1. *Assume that n is the same as above. Then the Kodaira dimension of $\mathcal{K}_{2n} = \mathcal{D}_{2n}/\Gamma_{2n}$ is nonnegative.*

* Supported by Max-Planck-Institut in Bonn.

If $m = n \cdot l^2$ for some natural number l , then $\Gamma_{2n} \supset \Gamma_{2m}$ and we have a dominant map from $\mathcal{D}_{2m}/\Gamma_{2m}$ to $\mathcal{D}_{2n}/\Gamma_{2n}$ (O'Grady [12], Kondō [6], Lemma 3.2). Hence, we have

COROLLARY 1.2. *Assume that n is the same as above and $m = n \cdot l^2$ for some l . Then the Kodaira dimension of \mathcal{K}_{2m} is nonnegative.*

We remark that if $1 \leq n \leq 11, 17$ or 19 , then \mathcal{K}_{2n} is unirational and, in particular, its Kodaira dimension is $-\infty$. For $n \leq 4$, this is classical and for other n 's, this was shown by Mukai [7], [8], [9]. On the other hand, in the paper [6], the author proved that if $n = p^2$ and p is a sufficiently large prime number, then \mathcal{K}_{2n} is of a general type, i.e. the Kodaira dimension of \mathcal{K}_{2n} is equal to $\dim \mathcal{K}_{2n}$ ($= 19$). However, the author did not have an effective estimate of p . Also, Gritsenko [5] proved that the Kodaira dimension of the double cover $\mathcal{D}_{2n}/\bar{\Gamma}_{2n}$ of $\mathcal{D}_{2n}/\Gamma_{2n}$ is nonnegative if n is not perfect square and positive if n is square free and $n > 3$ by using the lifting of Jacobi forms, where $\bar{\Gamma}_{2n} = \Gamma_{2n} \cap \mathrm{SO}(2, 19)_{\mathbf{Q}}$.

2. Preliminaries

A lattice L is a free \mathbf{Z} -module of finite rank endowed with an integral symmetric bilinear form $\langle \cdot, \cdot \rangle$. If L_1 and L_2 are lattices, then $L_1 \oplus L_2$ denotes the orthogonal direct sum of L_1 and L_2 . An isomorphism of lattices preserving the bilinear forms is called an *isometry*. A sublattice S of L is called *primitive* if L/S is torsion free.

A lattice L is *even* if $\langle x, x \rangle$ is even for each $x \in L$. A lattice L is *non-degenerate* if the discriminant $d(L)$ of its bilinear form is nonzero, and *unimodular* if $d(L) = \pm 1$. If L is a nondegenerate lattice, the *signature* of L is a pair (t_+, t_-) where t_{\pm} denotes the multiplicity of the eigenvalues ± 1 for the quadratic form on $L \otimes \mathbf{R}$.

Let L be a nondegenerate even lattice. The bilinear form of L determines a canonical embedding $L \subset L^* = \mathrm{Hom}(L, \mathbf{Z})$. The factor group L^*/L , which is denoted by A_L , is an Abelian group of order $|d(L)|$. We extend the bilinear form on L to the one on L^* , taking value in \mathbf{Q} , and define

$$q_L: A_L \rightarrow \mathbf{Q}/2\mathbf{Z}, \quad q_L(x + L) = \langle x, x \rangle + 2\mathbf{Z} \ (x \in L^*).$$

We call q_L the *discriminant quadratic form* of L .

We denote by H the hyperbolic lattice defined by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ which is an even unimodular lattice of signature $(1, 1)$, and by A_m , D_n or E_l an even negative definite lattice associated with the Dynkin diagram of type A_m , D_n or E_l ($m \geq 1$, $n \geq 4$, $l = 6, 7, 8$). A *root* of a lattice L is a vector r in L with $\langle r, r \rangle = -2$. For an even negative definite lattice L , we denote by $R(L)$ the sublattice of L generated by all roots in L which is called the *root sublattice* of L and isometric to a direct sum of some A_m, D_n, E_l .

Let X be a $K3$ surface. Put $L = H^2(X, \mathbf{Z})$. Then L admits a canonical structure of a lattice induced from the cup product. It is an even unimodular lattice with signature $(3, 19)$ and, hence, isometric to $H \oplus H \oplus H \oplus E_8 \oplus E_8$ (e.g., Nikulin [11], Thm. 1.1.1). Let h be a primitive vector of L with $\langle h, h \rangle = 2n$. Then the orthogonal complement of h in L is isometric to $L_{2n} = H \oplus H \oplus E_8 \oplus E_8 \oplus \langle -2n \rangle$. Put $\Omega_{2n} = \{[\omega] \in \mathbf{P}(L_{2n} \otimes \mathbf{C}) : \langle \omega, \omega \rangle = 0, \langle \omega, \bar{\omega} \rangle > 0\}$. Then Ω_{2n} consists of two connected components. We denote by \mathcal{D}_{2n} either one connected component, which is a bounded symmetric domain of type IV and of dimension 19. Let Γ_{2n} be the group of isometries of L which fix h and preserve the component \mathcal{D}_{2n} . Then Γ_{2n} acts on \mathcal{D}_{2n} properly discontinuously and, hence, by Cartan's theorem $\mathcal{D}_{2n}/\Gamma_{2n}$ has a canonical structure of normal analytic space. We call \mathcal{D}_{2n} the *period space* of $K3$ surfaces with a primitive polarization of degree $2n$. It follows from Torelli theorem for $K3$ surfaces that $\mathcal{D}_{2n}/\Gamma_{2n}$ is a coarse moduli space of $K3$ surface with a primitive polarization of degree $2n$.

A *boundary component* of \mathcal{D}_{2n} is a maximal connected complex analytic subset in $\mathcal{D}_{2n} \setminus \mathcal{D}_{2n}$ where $\bar{\mathcal{D}}_{2n}$ is the topological closure of \mathcal{D}_{2n} in $\{[\omega] \in \mathbf{P}(L_{2n} \otimes \mathbf{C}) : \langle \omega, \omega \rangle = 0\}$. A boundary component is called *rational* if its stabilizer subgroup of $O(L_{2n} \otimes \mathbf{R})$ is defined over \mathbf{Q} . It is known that the set of rational boundary components of \mathcal{D}_{2n} bijectively corresponds to the set of all primitive totally isotropic sublattices of L_{2n} (Scattone [13], (2.1.7)). If E is a totally isotropic sublattice, then the corresponding rational boundary component is defined by $\mathbf{P}(E \otimes \mathbf{C}) \cap \mathcal{D}_{2n}$. Since the signature of L_{2n} is $(2, 19)$, the dimension of a rational boundary component is either 0 or 1. For simplicity, we now assume that n is square-free. Let F be a rational boundary component and E the corresponding primitive totally isotropic sublattice. If $\dim F = 0$, then F is unique modulo Γ_{2n} and if $\dim F = 1$, then we have an orthogonal decomposition

$$L_{2n} = H \oplus H \oplus K,$$

where $H \oplus H$ contains E and $K \simeq E^\perp/E$ (Scattone [13], Thm. 4.0.1, Lemma 5.2.1). Let $\{e_1, \dots, e_{21}\}$ be a base of L_{2n} such that

$$(\langle e_i, e_j \rangle)_{1 \leq i, j \leq 21} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & Q & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where $\{e_1, e_2\}$ is a base of E and Q is the intersection matrix of K . In case $\dim F = 0$, F is unique modulo Γ_{2n} and, hence, we may assume that E is generated by e_1 . Denote by q_0 (resp. q_1) the symmetric bilinear form associated with the matrix $(\langle e_i, e_j \rangle)_{2 \leq i, j \leq 20}$ (resp. Q). Also denote by L_0 the sublattice e_1^\perp/e_1

of L_{2n} . Let $z = \sum z_i e_i$ be a homogeneous coordinate of $\mathbf{P}(L_{2n} \otimes \mathbf{C})$. Then we have the following unbounded realization of \mathcal{D}_{2n} .

(1.1) In case $\dim F = 0$:

$$\mathcal{D}_{2n} \simeq \{z = (z_2, \dots, z_{20}) \in \mathbf{C}^{19}: (\operatorname{Im}(z_i)) \in C(F)\},$$

where $C(F) = \{y = (y_2, \dots, y_{20}) \in \mathbf{R}^{19}: q_0(y, y) > 0, y_2 > 0\}$.

(1.2) In case $\dim F = 1$:

$$\begin{aligned} \mathcal{D}_{2n} \simeq \{(z, w, \tau) = (z_2, z_3, \dots, z_{19}, z_{20}) \\ \in H^+ \times \mathbf{C}^{17} \times H^+: \operatorname{Im}(z) - \operatorname{Re}(h_\tau(w, w)) > 0\}, \end{aligned}$$

where $h_\tau(w, w') = \{-q_1(w, \bar{w}') + q_1(w, w')\}/4 \operatorname{Im}(\tau)$ (see Kondo [6], Sect. 2).

Let f be an automorphic form on \mathcal{D}_{2n} with respect to Γ_{2n} . Then f has an expansion with respect to (1.1) which is called the *Fourier expansion* of f at F :

$$f = \sum_{\rho \in \bar{C}(F) \cap L_0^*} c_\rho \exp(2\pi\sqrt{-1} \cdot q_0(\rho, z)),$$

where $\bar{C}(F)$ is the closure of $C(F)$. Also with respect to (1.2), f has an expansion which is called the *Fourier–Jacobi expansion* of f at F :

$$f = \sum_{m \geq 0} \theta_m(\tau, w) \exp(2\pi\sqrt{-1}mz).$$

We call f a *cuspidal form* if for any rational boundary components the initial term of the above expansion vanishes, i.e. $c_0 \equiv 0$ if $\dim F = 0$ and $\theta_0(\tau, w) \equiv 0$ if $\dim F = 1$. Note that $\theta_0(\tau, w)$ does not depend on w . For more details, we refer the reader to Borcherds [2], Gritsenko [5], Kondo [6].

3. Automorphic Forms on the Period Domain of $K3$ Surfaces

In the following, as in Borcherds *et al.* [3], we identify $L_{2n} = H \oplus H \oplus E_8 \oplus E_8 \oplus \langle -2n \rangle$ with the sublattice $H \oplus H \oplus E_8 \oplus E_8 \oplus \mathbf{Z}v$ of $II_{2,26} = H \oplus H \oplus E_8 \oplus E_8 \oplus E_8$, where v is a primitive vector in E_8 with $\langle v, v \rangle = -2n$. Let U be the orthogonal complement of v in E_8 . Let \mathcal{D} be the 26-dimensional bounded symmetric domain of type IV associated with $II_{2,26}$. Under the above identification, $\mathcal{D}_{2n} \subset \mathcal{D}$. In the paper [2], Example 2 of section 10, Borcherds constructed an automorphic form Φ of weight 12 on \mathcal{D} with respect to the group of isometries of $II_{2,26}$ which preserve \mathcal{D} . The restriction of Φ on \mathcal{D}_{2n} is identically 0 whenever U contains a root. So first divide Φ by a product of linear functions vanishing on the divisors of each of these roots in U , and then restrict it on \mathcal{D}_{2n} . Then we have an automorphic form Φ_{2n} on \mathcal{D}_{2n} with respect to Γ_{2n} which has the following properties (Borcherds *et al.* [3], also see Borcherds [2], Thm. 13.1 and its proof):

(2.1) Φ_{2n} vanishes on the hyperplanes of \mathcal{D}_{2n} each of which is orthogonal to a vector $r' \in L_{2n}^*$ with $-2 \leq \langle r', r' \rangle < 0$. Here r' is the projection of a root r of $II_{2,26}$.

(2.2) The weight of Φ_{2n} is equal to the weight (= 12) of Φ plus half the number of roots of U .

Let $\{e_i\}_{1 \leq i \leq 8}$ be a set of simple roots of E_8 which satisfies the following:

$$\begin{aligned} \langle e_i, e_i \rangle &= -2, \langle e_1, e_2 \rangle = \langle e_2, e_3 \rangle = \langle e_3, e_4 \rangle \\ &= \langle e_3, e_5 \rangle = \langle e_5, e_6 \rangle = \langle e_6, e_7 \rangle = \langle e_7, e_8 \rangle = 1 \quad \text{and} \\ \langle e_i, e_j \rangle &= 0 \quad \text{for other } i, j. \end{aligned}$$

Let R be a root lattice isometric to $A_2 \oplus A_2 \oplus A_1$ or $A_3 \oplus A_1$. Note that the number of roots in R is 14. In the case where $R = A_2 \oplus A_2 \oplus A_1$, we consider R as one of the following sublattices of E_8 :

$$\begin{aligned} \text{(a)} \langle e_1, e_2, e_4, e_6, e_7 \rangle, \quad \text{(b)} \langle e_1, e_2, e_5, e_6, e_8 \rangle, \quad \text{(c)} \langle e_1, e_2, e_5, e_7, e_8 \rangle, \\ \text{(d)} \langle e_1, e_3, e_4, e_6, e_7 \rangle, \quad \text{(e)} \langle e_1, e_3, e_4, e_7, e_8 \rangle. \end{aligned}$$

In the case where $R = A_3 \oplus A_1$, we consider R as one of the following sublattices of E_8 :

$$\begin{aligned} \text{(f)} \langle e_1, e_2, e_3, e_6 \rangle, \quad \text{(g)} \langle e_1, e_2, e_3, e_7 \rangle, \quad \text{(h)} \langle e_1, e_2, e_3, e_8 \rangle, \\ \text{(i)} \langle e_2, e_3, e_4, e_7 \rangle, \quad \text{(j)} \langle e_2, e_3, e_5, e_7 \rangle, \quad \text{(k)} \langle e_2, e_3, e_5, e_8 \rangle, \\ \text{(l)} \langle e_3, e_4, e_5, e_7 \rangle, \quad \text{(m)} \langle e_1, e_5, e_6, e_7 \rangle, \quad \text{(n)} \langle e_4, e_5, e_6, e_7 \rangle, \\ \text{(o)} \langle e_1, e_6, e_7, e_8 \rangle, \quad \text{(p)} \langle e_2, e_6, e_7, e_8 \rangle, \quad \text{(q)} \langle e_4, e_6, e_7, e_8 \rangle. \end{aligned}$$

If $R = \langle e_\alpha, e_\beta, e_\gamma, e_\delta, e_\kappa \rangle$ (resp. $R = \langle e_\alpha, e_\beta, e_\gamma, e_\delta \rangle$), then we take a vector v as a sum:

$$v = e_\lambda^* + e_\mu^* + e_\nu^* \quad (\text{resp. } v = e_\kappa^* + e_\lambda^* + e_\mu^* + e_\nu^*),$$

where $\{\alpha, \beta, \dots, \nu\} = \{1, 2, \dots, 8\}$ and e_α^* is the dual of e_α (see Bourbaki [4], Planche VII). Since E_8 is unimodular, $v \in E_8$. A direct calculation shows that the number $-\langle v, v \rangle / 2$ is equal to

$$\begin{aligned} \text{(a)} 61, \text{(b)} 55, \text{(c)} 67, \text{(d)} 43, \text{(e)} 66, \text{(f)} 57, \text{(g)} 69, \text{(h)} 83, \text{(i)} 59, \text{(j)} 42, \\ \text{(k)} 53, \text{(l)} 51, \text{(m)} 85, \text{(n)} 74, \text{(o)} 133, \text{(p)} 105, \text{(q)} 119. \end{aligned}$$

Since $\langle v, v \rangle / 2$ is square free, v is primitive in E_8 . Obviously any root in $U (= v^\perp$ in $E_8)$ is contained in R . Hence, the number of roots of U is 14. Thus, we have an automorphic form Φ_{2n} of weight 19 (= 12 + 14/2) by (2.2).

THEOREM. Φ_{2n} is a cusp form of weight 19.

Proof. It suffices to see that Φ_{2n} vanishes on the top dimensional (= 1-dim.) rational boundary components of \mathcal{D}_{2n} . Since n is square free, the set of primitive totally isotropic sublattices of rank two corresponds to the set of isomorphism classes of even negative definite lattices K such that $L_{2n} \simeq H \oplus H \oplus K$ (Scattone [13], Thm. 5.0.2). Here totally isotropic sublattice is contained in $H \oplus H$. Since $q_K \simeq q_{L_{2n}} \simeq -q_U$, we have an even negative definite unimodular lattice N of rank 24 such that K and U are primitive sublattices of N , $K^\perp \simeq U$ and $U^\perp \simeq K$ (Nikulin [11], Cor. 1.6.2). Thus, each K is obtained as the orthogonal complement of a primitive sublattice U of some N . Let $U \subset N$ be a primitive embedding. Since U contains R , N also contains a root. Such lattices N are classified into 23 isomorphism classes which are characterized by its root sublattice $R(N)$ (Niemeier [10]). Since $\text{rank } R(N) = 24$, there exists a root r of N whose projection r' into K^* has a negative norm: $-2 \leq \langle r', r' \rangle < 0$. Note that the totally isotropic primitive sublattice of rank 2 corresponding to K is contained in the orthogonal complement of r' . It follows from (2.1) that Φ_{2n} vanishes the hyperplane orthogonal to r' . Let

$$\Phi_{2n} = \Sigma \theta_m(\tau, w) \cdot \exp(2\pi\sqrt{-1}mz)$$

be the Fourier–Jacobi expansion of Φ_{2n} at this boundary component. Since

$$\theta_0(\tau, w) = \lim_{\text{Im}(z) \rightarrow \infty} \Phi_{2n},$$

and $\theta_0(\tau, w)$ does not depend on w , the above implies that $\theta_0(\tau, w) \equiv 0$. \square

Remark. There are another embeddings of R into E_8 . In the above, we take embeddings such that the number $\langle v, v \rangle / 2$ is square free. Also we can take v as a linear combination of $e_\lambda^*, e_\mu^*, e_\nu^*$ (resp. $e_\kappa^*, e_\lambda^*, e_\mu^*, e_\nu^*$) with positive coefficients. Then Φ_{2n} is a cusp form of weight 19 whenever $\langle v, v \rangle / 2$ is square free and Corollaries 1, 2 hold for these n .

Acknowledgements

This paper was written during the author’s stay at the Max-Planck-Institut für Mathematik in Bonn 1996–1997. He expresses his thanks to the Institute for giving him the opportunity to work in an stimulating working atmosphere.

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