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Asymptotic series for double zeta, double gamma, and Hecke L-functions

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Abstract

Asymptotic expansions of the Barnes double zeta-function

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(\alpha+m+nw)^{-v}$$

and the double gamma-function $\Gamma_2(\alpha,(1,w))$, with respect to the parameter w, are proved. An application to Hecke L-functions of real quadratic fields is also discussed.

1. *Asymptotic expansions of double zeta and double gamma-functions*

Let α, w be positive numbers, v a complex number, and

$$\zeta_2(v;\alpha,w)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(\alpha+m+nw)^{-v}. \tag{1·1}$$

This double series is convergent absolutely for $\operatorname{Re} v>2$, and can be continued meromorphically to the whole complex plane. The 'critical strip' is $0\leqslant \operatorname{Re} v\leqslant 2$. This function was studied extensively by Barnes [**5**] and is therefore called the double zeta-function of Barnes. The first purpose of the present paper is to prove the asymptotic expansion of this function with respect to w. Let

$$\zeta(v)=\sum_{n=1}^{\infty}n^{-v},\quad \zeta(v,\alpha)=\sum_{n=0}^{\infty}(n+\alpha)^{-v}$$

be the Riemann zeta and the Hurwitz zeta-function (with parameter $\alpha > 0$), respectively, and put

$$\binom{v}{n} = \begin{cases} v(v-1)\ldots(v-n+1)/n! & \text{if } n \text{ is a positive integer,} \\ 1 & \text{if } n = 0. \end{cases}$$

Throughout this paper the empty sum is to be considered as zero.

THEOREM 1. *For any positive integer N and* $\mathrm{Re}\, v > -N+1$ (*except for the points* $v \in \{2, 1, 0, -1, -2, -3, \ldots\}$), *we have*

$$\zeta_2(v;\alpha,w) = \zeta(v,\alpha) + \frac{\zeta(v-1)}{v-1} w^{1-v} + \sum_{n=0}^{N-1} \binom{-v}{n} \zeta(-n,\alpha)\, \zeta(v+n)\, w^{-v-n} + R_N(v;\alpha,w) \tag{1·2}$$

with the estimate

$$R_N(v;\alpha,w) = O(w^{-\mathrm{Re}\, v - N}), \tag{1·3}$$

where the O-constant depends only on v and N.

It is possible to generalize this result to the case of any complex w with positive real part.

Later we shall prove another type of asymptotic series for $\zeta_2(v;\alpha,w)$, in which the error estimate will be given by a strict inequality (Theorem 6, stated in the last section). Note that if $v \in \{2, 1, 0, -1, -2, -3, \ldots\}$, then some factor in the right-hand side of (1·2) has a singularity. At $v = 1$ and $v = 2$, the function $\zeta_2(v;\alpha,w)$ has poles. If v is a non-positive integer, we can calculate the value of $\zeta_2(v;\alpha,w)$ exactly (see Theorem 5 in Section 3).

Our method is a variant of Atkinson's device. Inspired by Atkinson [**3**], Motohashi [**16**] invented a new method of studying certain mean square of Dirichlet L-functions, and this method was further developed by Katsurada–Matsumoto [**10**, **13**] and Katsurada [**8**, **9**]. The same idea can be applied to the mean values of Hurwitz zeta-functions of the type

$$\int_0^1 |\zeta(v,\alpha) - \alpha^{-v}|^2 \, d\alpha, \tag{1·4}$$

as was shown in Katsurada–Matsumoto [**11**, **12**, **14**]. Our argument here is a modification of the method developed in these papers of Katsurada–Matsumoto and Katsurada.

One advantage of this method is that it is also useful for the study of derivatives of zeta-functions, as was first observed by Katsurada [**9**]. In fact, we can prove the asymptotic expansion of $\zeta_2'(v;\alpha,w)$ (throughout this paper 'prime' denotes the differentiation with respect to v), from which we can deduce:

THEOREM 2. *For any $N \geqslant 2$ we have*

$$\begin{aligned} \zeta_2'(0;\alpha,w) = {} & -\tfrac{1}{12} w \log w + \left(\tfrac{1}{12} - \zeta'(-1)\right) w + \tfrac{1}{2}\left(\tfrac{1}{2} - \alpha\right) \log w + \log \Gamma(\alpha) \\ & + \left(\tfrac{1}{2}\alpha - \tfrac{3}{4}\right) \log 2\pi + \zeta(-1,\alpha)\, w^{-1} \log w - \zeta(-1,\alpha)\, \gamma w^{-1} \\ & + \sum_{n=2}^{N-1} \frac{(-1)^n}{n} \zeta(-n,\alpha)\, \zeta(n)\, w^{-n} + R_N'(0;\alpha,w) \end{aligned} \tag{1·5}$$

with

$$R'_N(0;\alpha,w) = O_N(w^{-N}(|\log w|+1)), \tag{1·6}$$

where $\Gamma(\alpha)$ *is the gamma-function,* γ *is Euler's constant, and* O_N *means (throughout this paper) that the implied constant depends only on* N.

Another asymptotic series for $\zeta_2'(0;\alpha,w)$, in which the error term will be estimated by a strict inequality, will be given (Theorem 7) in the last section. It is to be stressed that in Theorem 7, the error estimate has no log-factor.

Shintani[**19**, proposition 4] proved a different type of approximation formula for $\zeta_2'(0;\alpha,w)$, which should be compared with our Theorems 2 and 7. (See also Shintani[**20**].)

The double gamma-function $\Gamma_2(\alpha,(1,w))$ is defined by

$$\log\left(\frac{\Gamma_2(\alpha,(1,w))}{\rho_2(1,w)}\right) = \zeta_2'(0;\alpha,w), \tag{1·7}$$

where

$$-\log\rho_2(1,w) = \lim_{\alpha\to 0}\{\zeta_2'(0;\alpha,w)+\log\alpha\}. \tag{1·8}$$

Double gamma-functions have a long history associated with the names of Méray, Pincherle, Alexeiewsky, and, especially, Barnes[**4**, **5**]. Today double gamma-functions are quite important in number theory (cf. Shintani[**18**, **19**, **20**]); it is therefore highly desirable to study the behaviour of these functions. From Theorem 2 we can deduce

Theorem 3. *For any* $N \geqslant 2$ *we have*

$$\begin{aligned}\log\Gamma_2(\alpha,(1,w)) = {} & -\tfrac{1}{2}\alpha\log w+\log\Gamma(\alpha)+\tfrac{1}{2}\alpha\log 2\pi \\ & +(\zeta(-1,\alpha)-\zeta(-1))\,w^{-1}\log w-(\zeta(-1,\alpha)-\zeta(-1))\,\gamma w^{-1} \\ & +\sum_{n=2}^{N-1}\frac{(-1)^n}{n}(\zeta(-n,\alpha)-\zeta(-n))\,\zeta(n)\,w^{-n}+S_N(\alpha,(1,w))\end{aligned} \tag{1·9}$$

with

$$S_N(\alpha,(1,w)) = O_N(w^{-N}(|\log w|+1)). \tag{1·10}$$

In the next section we combine this theorem with Shintani's 'Kronecker limit formula'[**19**] to obtain an asymptotic series for the special values at $s=1$ of Hecke L-functions of real quadratic fields. In this application it is important that the implied constant in (1·10) is independent of α. Another asymptotic series for double gamma-functions, in which the error estimate will be given by a strict inequality without log-factor, will be proved in the last section (Theorem 8).

Double zeta-functions and double gamma-functions are now very frequently used in mathematical physics. Our results may be useful in zeta-function regularization methods; for example, compare our Theorem 1 with various formulas, in particular (4·8), in Elizalde–Odintsov–Romeo–Bytsenk–Zerbini[**6**]. Double gamma-functions have also appeared in Sarnak[**17**], Vardi[**22**] and Voros[**23**] in connection with the evaluation of the determinants of Laplacians which are important in superstring theory.

2. *Asymptotic series for certain special values of Hecke L-functions of real quadratic fields*

Let F be a real quadratic field, $\mathfrak{f}$ an integral ideal of F, $I(\mathfrak{f})$ the group of fractional ideals of F which are prime to $\mathfrak{f}$, and $P(\mathfrak{f})$ the subgroup of $I(\mathfrak{f})$ consisting of all the principal ideals generated by totally positive μ with the congruence condition $\mu \equiv 1 \pmod{\mathfrak{f}}$. Let χ be a character of $I(\mathfrak{f})/P(\mathfrak{f})$ such that

$$\chi((\mu)) = \operatorname{sgn}(\mu)\,\chi_0(\mu)$$

for any principal integral ideal (μ) of F, where χ_0 is a character of the group of invertible residue class mod $\mathfrak{f}$.

We can choose integral ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_h$ (where h is the number of narrow ideal classes of F), which form a complete set of representatives of narrow ideal classes of F. Let E_+ be the group of totally positive units of F, and $\epsilon > 1$ the generator of E_+. We denote by R_k $(1 \leqslant k \leqslant h)$ the set of all numbers $z = x + \epsilon y \in (\mathfrak{a}_k \mathfrak{f})^{-1}$, where x and y are rational numbers satisfying $0 < x \leqslant 1$, $0 \leqslant y < 1$, and $(\mathfrak{a}_k \mathfrak{f}(z), \mathfrak{f}) = 1$. It can be shown that R_k is a finite set. For any $u \in E_+$ and any $z \in R_k$, there exists a uniquely determined element $\psi(u,z) \in R_k$ such that

$$\psi(u,z) - uz \in \mathbb{Z} + \mathbb{Z}\epsilon$$

(where $\mathbb{Z}$ is the ring of rational integers). Then E_+ acts on R_k via the mapping ψ, and each E_+-orbit in R_k is called a cycle. We can write

$$\chi_k(C) = \chi(\mathfrak{a}_k \mathfrak{f}(z)) \quad (z \in C)$$

for each cycle C, because this value depends only on C.

Define the permutation $z \to \overline{-z}$ on R_k by

$$\overline{-z} = \begin{cases} 1+\epsilon-z & \text{if} \quad 0<x<1, 0<y<1, \\ 1-z & \text{if} \quad 0<x<1, y=0, \\ 1+\epsilon(1-y) & \text{if} \quad x=1, 0<y<1, \end{cases}$$

and put $-C = \{\overline{-z} \mid z \in C\}$. Then we have the decomposition of the form

$$R_k = \bigcup_{j=1}^{l(k)} \{C_j^{(k)} \cup (-C_j^{(k)})\} \quad \text{(disjoint union)}.$$

Furthermore, define the conjugate $\tilde{C} = \{\tilde{z} \mid z \in C\}$ of C by

$$\tilde{z} = \tilde{x} + \tilde{y}\epsilon = \begin{cases} y + x\epsilon & \text{if} \quad 0<x<1, 0<y<1, \\ 1 + x\epsilon & \text{if} \quad 0<x<1, y=0, \\ y & \text{if} \quad x=1, 0<y<1. \end{cases}$$

Shintani[**19**, p. 187, corollary 2 of theorem 1] proved

$$\frac{1}{2\pi}\overline{w(\chi)}\,(dN(\mathfrak{f}))^{1/2} L_F(1,\chi) = \sum_{k=1}^{h} \sum_{C} \chi_k(C)^{-1} \times \log\left\{\prod_{z \in C} \frac{\Gamma_2(z,(1,\epsilon))}{\Gamma_2(1+\epsilon-z,(1,\epsilon))} \prod_{\tilde{z} \in \tilde{C}} \frac{\Gamma_2(\tilde{z},(1,\epsilon))}{\Gamma_2(1+\epsilon-\tilde{z},(1,\epsilon))}\right\}, \quad (2\cdot 1)$$

where d is the discriminant of F, $N(\mathfrak{f})$ is the norm of $\mathfrak{f}$, $L_F(s,\chi)$ is the Hecke L-function attached to χ, $w(\chi)$ is the complex number of modulus 1 determined by

$$\xi(s,\chi) = w(\chi)\,\xi(1-s,\chi^{-1})$$

(where $\xi(s,\chi) = \pi^{-s}(dN(\mathfrak{f}))^{s/2}\,\Gamma(s/2)\,\Gamma((s+1)/2)\,L_F(s,\chi)$), and the summation $\sum_C$ runs over all C belonging to the set $\{C_1^{(k)},\ldots,C_{l(k)}^{(k)}\}$ and satisfying $\tilde{C} \neq -C$.

Applying Theorem 3 to (2.1), we obtain

THEOREM 4. *For any $N \geqslant 2$, we have*

$$\begin{aligned}
&\frac{1}{2\pi}\overline{w(\chi)}\,(dN(f))^{1/2}\,L_F(1,\chi) = \sum_{k=1}^{h}\sum_{C}\chi_k^{-1}(C)\\
&\quad\times\sum_{z\in C}\left(\log\frac{\Gamma(x+y\epsilon)}{\Gamma((1-x)+(1-y)\,\epsilon)}+\log\frac{\Gamma(\tilde{x}+\tilde{y}\epsilon)}{\Gamma((1-\tilde{x})+(1-\tilde{y})\,\epsilon)}\right.\\
&\quad+(1-y-\tilde{y})\,\epsilon\log\epsilon+((y+\tilde{y}-1)\log 2\pi)\,\epsilon\\
&\quad+(1-x-\tilde{x})\log\epsilon+(x+\tilde{x}-1)\log 2\pi\\
&\quad+\{\zeta(-1,x+y\epsilon)-\zeta(-1,1-x+(1-y)\,\epsilon)\}\,\epsilon^{-1}\log\epsilon\\
&\quad+\{\zeta(-1,\tilde{x}+\tilde{y}\epsilon)-\zeta(-1,1-\tilde{x}+(1-\tilde{y})\,\epsilon)\}\,\epsilon^{-1}\log\epsilon\\
&\quad-\{\zeta(-1,x+y\epsilon)-\zeta(-1,1-x+(1-y)\,\epsilon)\}\,\gamma\epsilon^{-1}\\
&\quad-\{\zeta(-1,\tilde{x}+\tilde{y}\epsilon)-\zeta(-1,1-\tilde{x}+(1-\tilde{y})\,\epsilon)\}\,\gamma\epsilon^{-1}\\
&\quad+\sum_{n=2}^{N-1}\frac{(-1)^n}{n}\{\zeta(-n,x+y\epsilon)-\zeta(-n,1-x+(1-y)\,\epsilon)\}\,\zeta(n)\,\epsilon^{-n}\\
&\quad\left.+\sum_{n=2}^{N-1}\frac{(-1)^n}{n}\{\zeta(-n,\tilde{x}+\tilde{y}\epsilon)-\zeta(-n,1-\tilde{x}+(1-\tilde{y})\,\epsilon)\}\,\zeta(n)\,\epsilon^{-n}+O_N(\epsilon^{-N}\log\epsilon)\right).
\end{aligned}\tag{2·2}$$

We remark that (2·2) is not the asymptotic expansion with respect to ϵ in the strict sense, because 'coefficients' such as $x+\tilde{x}-1$ and $\zeta(-n,x+y\epsilon)$ also depend on ϵ. However, $x+\tilde{x}-1$ and $y+\tilde{y}-1$ are bounded because $0<x\leqslant 1$, $0<\tilde{x}\leqslant 1$, $0\leqslant y<1$ and $0\leqslant\tilde{y}<1$. The term $\log\{\Gamma(x+y\epsilon)/\Gamma(1-x+(1-y)\,\epsilon)\}$ can be written in the asymptotic expansion form by using Stirling's formula. Moreover, it is known that

$$\begin{aligned}
\zeta(u-n,\beta) = \frac{\Gamma(-u+1+n)}{(2\pi)^{-u+1+n}}\{&e^{\pi i(u-1-n)/2}\,\phi(-u+1+n,\beta)\\
&+e^{-\pi i(u-1-n)/2}\,\phi(-u+1+n,-\beta)\}
\end{aligned}\tag{2·3}$$

(see (2·17·3) in Titchmarsh [**21**]), where $\phi(s,\beta)$ is Lerch's zeta-function defined by

$$\phi(s,\beta) = \sum_{m=1}^{\infty} e^{2\pi i m\beta}\,m^{-s}. \tag{2·4}$$

Putting $u=0$, $\beta=x+y\epsilon$ or $1-x+(1-y)\,\epsilon$ in (2·3) we see that the factors $\zeta(-n,x+y\epsilon)$ and $\zeta(-n,1-x+(1-y)\,\epsilon)$ $(n\geqslant 1)$ are uniformly bounded with respect to ϵ. Lastly, since the implied constant in (1·10) is independent of α, we see that the implied constant in the error term in (2·2) is independent of ϵ. In view of these remarks, we can regard (2·2) as a kind of asymptotic series with respect to the totally positive fundamental unit ϵ.

Arakawa[**1**, proposition 1·3] proved that certain special values of Hecke L-functions with Grössencharacters of real quadratic fields can be expressed as sums of double zeta-functions. (See also Arakawa's related article [**2**].) Therefore, combining with our Theorem 1, we obtain another asymptotic result on certain special values of Hecke L-functions, though we do not give the statement here.

3. *The generalized double zeta-functions*

To study the mean square (1·4), Katsurada–Matsumoto[**14**] considered the product

$$\zeta(u,\alpha)\,\zeta(v,\alpha) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(m+\alpha)^{-u}(n+\alpha)^{-v},$$

where u and v are independent complex variables. Dividing the right-hand side into three parts corresponding to the conditions $m = n$, $m < n$ and $m > n$, we obtain

$$\zeta(u,\alpha)\,\zeta(v,\alpha) = \zeta(u+v,\alpha)+f(u,v;\alpha)+f(v,u;\alpha),$$

where

$$f(u,v;\alpha) = \sum_{m=0}^{\infty}(m+\alpha)^{-u}\sum_{n=1}^{\infty}(m+n+\alpha)^{-v}. \tag{3·1}$$

This is Atkinson's 'dissection device', and in [**14**] the properties of the function $f(u,v;\alpha)$ were studied in detail.

Inspired by the similarity of (1·1) and (3·1), we now introduce

$$\tilde{\zeta}_2(u,v;\alpha,w) = \sum_{m=0}^{\infty}(\alpha+m)^{-u}\sum_{n=1}^{\infty}(\alpha+m+nw)^{-v}. \tag{3·2}$$

This function is a generalization of both the double zeta-function $\zeta_2(v;\alpha,w)$ and the function $f(u,v;\alpha)$.

Remark. Comparing with the definition of Shintani zeta-functions (Shintani[**18**, **19**]), the special case $u = v$ in (3·2) can be regarded as the degenerate case of Shintani zeta-functions. Therefore the function (3·2) itself may be of some interest.

If $\mathrm{Re}\,u > 1$ and $\mathrm{Re}\,v > 1$, then (3·2) is absolutely convergent. But (3·2) is also absolutely convergent in the region $\mathrm{Re}\,u > -\delta$, $\mathrm{Re}\,v > 2+\delta$ for any positive δ, because

$$\begin{aligned}\sum_{m=0}^{\infty}&(\alpha+m)^{-\mathrm{Re}\,u}\sum_{n=1}^{\infty}(\alpha+m+nw)^{-\mathrm{Re}\,v}\\ &\leqslant \sum_{m=0}^{\infty}(\alpha+m)^{\delta}\sum_{n=1}^{\infty}(\alpha+m+nw)^{-\mathrm{Re}\,v+\delta}(\alpha+m+nw)^{-\delta}\\ &\leqslant \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}(\alpha+m+nw)^{-\mathrm{Re}\,v+\delta}.\end{aligned}$$

In particular, we can put $u = 0$ in (3·2) if $\mathrm{Re}\,v > 2$. Hence we obtain:

PROPOSITION 1. *For* $\mathrm{Re}\,v > 2$, *we have*

$$\zeta_2(v;\alpha,w) = \zeta(v,\alpha)+\tilde{\zeta}_2(0,v;\alpha,w). \tag{3·3}$$

We study $\tilde{\zeta}_2(u,v;\alpha,w)$ by a simple generalization of the method developed in [**14**] for the purpose of analysing $f(u,v;\alpha)$. First we shall prove that the function $\tilde{\zeta}_2(u,v;\alpha,w)$ can be continued to the region $\operatorname{Re} u > 0$, $\operatorname{Re} v > 1$ and $\operatorname{Re}(u+v) > 2$, and in this region

$$\Gamma(u)\,\Gamma(v)\,\tilde{\zeta}_2(u,v;\alpha,w) = \int_0^\infty \frac{y^{v-1}}{e^{wy}-1}\int_0^\infty \frac{e^{(1-\alpha)(x+y)}\,x^{u-1}}{e^{x+y}-1}\,dx\,dy. \tag{3·4}$$

Remark. If we restrict ourselves to the case $u=0$, we cannot consider this situation. This is one advantage of introducing the new variable u. Later in the proof we sometimes encounter similar situations.

Now we prove this claim. If $y>0$ and $\operatorname{Re} u > 0$, then

$$\int_0^\infty \frac{e^{(1-\alpha)(x+y)}\,x^{u-1}}{e^{x+y}-1}\,dx = \sum_{m=0}^\infty \int_0^\infty e^{-(\alpha+m)(x+y)}\,x^{u-1}\,dx = \Gamma(u)\sum_{m=0}^\infty e^{-(\alpha+m)y}\,(\alpha+m)^{-u}.$$

Therefore, if $\delta>0$, $\operatorname{Re} u>0$, $\operatorname{Re} v>1$ and $\operatorname{Re}(u+v)>2$, we have

$$\int_\delta^\infty \frac{y^{v-1}}{e^{wy}-1}\int_0^\infty \frac{e^{(1-\alpha)(x+y)}\,x^{u-1}}{e^{x+y}-1}\,dx\,dy = \Gamma(u)\sum_{m=0}^\infty (\alpha+m)^{-u}\int_\delta^\infty \frac{e^{-(\alpha+m)y}\,y^{v-1}}{e^{wy}-1}\,dy, \tag{3·5}$$

where the change of the integration and the summation is possible because, for any fixed w,

$$\begin{aligned}
&\sum_{m=0}^\infty (\alpha+m)^{-\operatorname{Re} u}\int_\delta^\infty \frac{e^{-(\alpha+m)y}\,y^{\operatorname{Re} v-1}}{|e^{wy}-1|}\,dy\\
&\quad\ll \sum_{m=0}^\infty (\alpha+m)^{-\operatorname{Re} u}\int_\delta^1 e^{-(\alpha+m+w)y}\,y^{\operatorname{Re} v-2}\,dy\\
&\qquad+\sum_{m=0}^\infty (\alpha+m)^{-\operatorname{Re} u}\int_1^\infty e^{-(\alpha+m+w)y}\,y^{\operatorname{Re} v-1}\,dy\\
&\quad\ll \sum_{m=0}^\infty (\alpha+m)^{-\operatorname{Re} u-\operatorname{Re} v+1}\,\Gamma(\operatorname{Re} v-1)+\sum_{m=0}^\infty (\alpha+m)^{-\operatorname{Re} u-\operatorname{Re} v}\,\Gamma(\operatorname{Re} v).
\end{aligned}$$

Moreover, this convergence is uniform with respect to δ, hence we can take $\delta\to 0$ termwise in (3·5) to get

$$\int_0^\infty \frac{y^{v-1}}{e^{wy}-1}\int_0^\infty \frac{e^{(1-\alpha)(x+y)}\,x^{u-1}}{e^{x+y}-1}\,dx\,dy = \Gamma(u)\sum_{m=0}^\infty (\alpha+m)^{-u}\int_0^\infty \frac{e^{-(\alpha+m)y}\,y^{v-1}}{e^{wy}-1}\,dy. \tag{3·6}$$

Also we have

$$\int_0^\infty \frac{e^{-(\alpha+m)y}\,y^{v-1}}{e^{wy}-1}\,dy = \lim_{\delta\to 0}\sum_{n=1}^\infty \int_\delta^\infty e^{-(\alpha+m+nw)y}\,y^{v-1}\,dy = \Gamma(v)\sum_{n=1}^\infty (\alpha+m+nw)^{-v},$$

which can be justified as above. Substituting this into (3·6), we obtain (3·4).

For any complex z, put

$$h(z;\alpha) = \frac{e^{(1-\alpha)z}}{e^z-1}-\frac{1}{z},$$

and let $h^{(N)}(z;\alpha)$ mean the Nth derivative of $h(z;\alpha)$ with respect to z. Let $\mathscr{C}$ be the contour which consists of the half-line on the positive real axis from infinity to a small positive constant δ, a circle of radius δ counterclockwise round the origin, and the other half-line on the positive real axis from δ to infinity. Note that $h(z;\alpha)$ is holomorphic at $z=0$. Lemma 1 of Katsurada–Matsumoto[**14**] asserts that for any $N\geqslant 0$, the estimate

$$h^{(N)}(x+\tau y;\alpha)=O(\alpha^N e^{-K\alpha|x|}+(|x|+1)^{-N-1}) \tag{3·7}$$

holds with a positive absolute constant K, uniformly for any $x, y\in\mathscr{C}\cup[0,+\infty)$, $\tau\in[0,1]$ and $\alpha\geqslant 0$.

It is known that

$$\int_0^\infty \frac{x^{u-1}}{x+y}dx = y^{u-1}\,\Gamma(u)\,\Gamma(1-u)$$

for $y>0$, $0<\operatorname{Re}u<1$. Therefore, noting (3·7) with $N=0$, for any $\delta>0$ we can write

$$\begin{aligned}\int_\delta^\infty \frac{y^{v-1}}{e^{wy}-1}\int_0^\infty \frac{e^{(1-\alpha)(x+y)}\,x^{u-1}}{e^{x+y}-1}dx\,dy \\ = \int_\delta^\infty \frac{y^{v-1}}{e^{wy}-1}\int_0^\infty h(x+y;\alpha)\,x^{u-1}\,dx\,dy+\Gamma(u)\,\Gamma(1-u)\int_\delta^\infty \frac{y^{u+v-2}}{e^{wy}-1}dy,\end{aligned} \tag{3·8}$$

and the integrals in the right-hand side are convergent absolutely, uniformly with respect to δ, in the region $0<\operatorname{Re}u<1$, $\operatorname{Re}(u+v)>2$. Therefore, letting $\delta\to 0$ in (3·8) and combining with (3·4), we obtain

$$\begin{aligned}\tilde{\zeta}_2(u,v;\alpha,w) &= \frac{\Gamma(1-u)}{\Gamma(v)}\Gamma(u+v-1)\,\zeta(u+v-1)\,w^{1-u-v} \\ &\quad + \frac{1}{\Gamma(u)\,\Gamma(v)}\int_0^\infty \frac{y^{v-1}}{e^{wy}-1}\int_0^\infty h(x+y;\alpha)\,x^{u-1}\,dx\,dy\end{aligned} \tag{3·9}$$

for $0<\operatorname{Re}u<1$, $\operatorname{Re}(u+v)>2$. It is easy to see that the second term in the right-hand side of (3·9) is equal to

$$g(u,v;\alpha,w)=\frac{1}{\Gamma(u)\,\Gamma(v)\,(e^{2\pi iu}-1)\,(e^{2\pi iv}-1)}\int_{\mathscr{C}} \frac{y^{v-1}}{e^{wy}-1}\int_{\mathscr{C}} h(x+y;\alpha)\,x^{u-1}\,dx\,dy. \tag{3·10}$$

Since (3·10) is absolutely convergent for $\operatorname{Re}u<1$ and any v, we now obtain:

PROPOSITION 2. *For* $\operatorname{Re}u<1$ *and any* v, *we have*

$$\tilde{\zeta}_2(u,v;\alpha,w)=\frac{\Gamma(1-u)}{\Gamma(v)}\Gamma(u+v-1)\,\zeta(u+v-1)\,w^{1-u-v}+g(u,v;\alpha,w). \tag{3·11}$$

The exact formula of the values of double zeta-functions at non-positive integers can be easily deduced from Proposition 2. The kth Bernoulli polynomial $B_k(t)$ is defined by

$$\frac{ze^{tz}}{e^z-1}=\sum_{k=0}^\infty \frac{B_k(t)}{k!}z^k \quad (|z|<2\pi); \tag{3·12}$$

hence it follows that

$$h(z;\alpha) = \sum_{k=0}^{\infty} \frac{B_{k+1}(1-\alpha)}{(k+1)!} z^k. \tag{3·13}$$

Let m be a non-negative integer. From Propositions 1 and 2 we have

$$\zeta_2(-m;\alpha,w) = \zeta(-m,\alpha) - \frac{\zeta(-m-1)}{m+1} w^{m+1} + g(0,-m;\alpha,w) \tag{3·14}$$

and

$$g(0,-m;\alpha,w) = \frac{(-1)^m m!}{(2\pi i)^2} \int_{\mathscr{C}} \frac{y^{-m-1}}{e^{wy}-1} \int_{\mathscr{C}} h(x+y;\alpha)\, x^{-1}\, dx\, dy.$$

We can replace the contour $\mathscr{C}$ by a small circle $\mathscr{C}(\delta)$ of radius δ round the origin. Then, by residue calculus, we get

$$g(0,-m;\alpha,w) = (-1)^m m! \sum_{j=0}^{m+1} \frac{B_j\, h^{(m+1-j)}(0;\alpha)}{j!(m+1-j)!} w^{j-1} \quad (B_j = B_j(0)).$$

Here we have used (3·12) with $t=0$, $z = wy$. Substituting this into (3·14), and noting (3·13), the facts $\zeta(-m-1) = -B_{m+2}/(m+2)$ and $\zeta(-m,\alpha) = -B_{m+1}(\alpha)/(m+1)$, we obtain

THEOREM 5. *For any non-negative integer m, we have*

$$\zeta_2(-m;\alpha,w) = -\frac{B_{m+1}(\alpha)}{m+1} + (-1)^m m! \sum_{j=0}^{m+2} \frac{B_j B_{m+2-j}(1-\alpha)}{j!(m+2-j)!} w^{j-1}.$$

When m is even, this formula agrees with Arakawa's result [**1**, (2·6)].

4. *Proof of asymptotic expansions*

First we write

$$\int_{\mathscr{C}} h(x+y;\alpha)\, x^{u-1}\, dx = \int_{\mathscr{C}} \{h(x+y;\alpha) - h(x;\alpha)\}\, x^{u-1}\, dx$$
$$+ \int_{\mathscr{C}} \frac{e^{(1-\alpha)x}\, x^{u-1}}{e^x - 1} dx - \int_{\mathscr{C}} x^{u-2}\, dx = g_1(u;\alpha) + g_2(u;\alpha) - g_3(u),$$

say. If $\operatorname{Re} u < 1$, then $g_3(u) = 0$. Also it is well known that

$$g_2(u;\alpha) = (e^{2\pi i u} - 1)\, \Gamma(u)\, \zeta(u,\alpha) \tag{4·1}$$

(see Whittaker–Watson [**24**, chapter 13]). Therefore we obtain

$$g(u,v;\alpha,w) = \frac{1}{\Gamma(u)\,\Gamma(v)(e^{2\pi i u}-1)(e^{2\pi i v}-1)} \int_{\mathscr{C}} \frac{y^{v-1} g_1(u;\alpha)}{e^{wy}-1} dy$$
$$+ \frac{\zeta(u,\alpha)}{\Gamma(v)(e^{2\pi i v}-1)} \int_{\mathscr{C}} \frac{y^{v-1}}{e^{wy}-1} dy. \tag{4·2}$$

Let N be a positive integer. Since

$$h(x+y;\alpha) - h(x;\alpha) = \sum_{n=1}^{N-1} h^{(n)}(x;\alpha) \frac{y^n}{n!} + y^N \int_0^1 \frac{(1-\tau)^{N-1}}{(N-1)!} h^{(N)}(x+\tau y;\alpha)\, d\tau,$$

we get

$$g_1(u;\alpha) = \sum_{n=1}^{N-1} \frac{(-1)^n}{n!} (e^{2\pi i u}-1)\,\Gamma(u)\,\zeta(u-n,\alpha)\,y^n$$
$$+y^N \int_{\mathscr{C}} \int_0^1 \frac{(1-\tau)^{N-1}}{(N-1)!} h^{(N)}(x+\tau y;\alpha)\,x^{u-1}\,d\tau\,dx. \qquad (4{\cdot}3)$$

Here we have used

$$\int_{\mathscr{C}} h^{(n)}(x;\alpha)\,x^{u-1}\,dx = (-1)^n(u-1)\dots(u-n)\int_{\mathscr{C}} h(x;\alpha)\,x^{u-n-1}\,dx$$
$$= (-1)^n(u-1)\dots(u-n)\int_{\mathscr{C}} \frac{e^{(1-\alpha)x}}{e^x-1}x^{u-n-1}\,dx$$
$$= (-1)^n(e^{2\pi i u}-1)\,\Gamma(u)\,\zeta(u-n,\alpha),$$

which can be shown by repeated integration by parts, using (3·7) and (4·1). Substituting (4·3) into (4·2), and noting

$$\int_{\mathscr{C}} \frac{y^{v+n-1}}{e^{wy}-1}\,dy = (e^{2\pi i v}-1)\,\Gamma(v+n)\,\zeta(v+n)\,w^{-v-n} \quad (n=0,1,2,\dots),$$

we obtain

$$g(u,v;\alpha,w) = \sum_{n=1}^{N-1} \frac{(-1)^n}{n!} \frac{\Gamma(v+n)}{\Gamma(v)} \zeta(u-n,\alpha)\,\zeta(v+n)\,w^{-v-n}$$
$$+\zeta(u,\alpha)\,\zeta(v)\,w^{-v} + R_N(u,v;\alpha,w), \qquad (4{\cdot}4)$$

where

$$R_N(u,v;\alpha,w) = \frac{1}{\Gamma(u)\,\Gamma(v)(e^{2\pi i u}-1)(e^{2\pi i v}-1)} \int_{\mathscr{C}} \frac{y^{v+N-1}}{e^{wy}-1}$$
$$\times \int_{\mathscr{C}} \int_0^1 \frac{(1-\tau)^{N-1}}{(N-1)!} h^{(N)}(x+\tau y;\alpha)\,x^{u-1}\,d\tau\,dx\,dy. \qquad (4{\cdot}5)$$

In view of (3·7), we see that (4·5) is convergent absolutely for $\operatorname{Re} u < N+1$ and any v. Therefore, (4·4) implies

PROPOSITION 3. *Let N be a positive integer. For* $\operatorname{Re} u < N+1$ *and any v, we have*

$$g(u,v;\alpha,w) = \sum_{n=0}^{N-1} \binom{-v}{n} \zeta(u-n,\alpha)\,\zeta(v+n)\,w^{-v-n} + R_N(u,v;\alpha,w). \qquad (4{\cdot}6)$$

Let

$$I_N(y;\alpha) = \int_{\mathscr{C}} \int_0^1 \frac{(1-\tau)^{N-1}}{(N-1)!} h^{(N)}(x+\tau y;\alpha)\,x^{-1}\,d\tau\,dx.$$

We claim

$$I_N(y;\alpha) = O_N(1). \qquad (4{\cdot}7)$$

(Recall that O_N means that the implied constant depends only on N.) In fact, by using (3·7) we have

$$I_N(y;\alpha) \ll \int_{\mathscr{C}} \alpha^N e^{-K\alpha|x|}\,|x|^{-1}|\,dx| + \int_{\mathscr{C}} (|x|+1)^{-N-1}\,|x|^{-1}|\,dx| = I'_N + I''_N,$$

say. We have $I''_N = O_N(1)$, because δ can be considered as an absolute constant. Also we have

$$I'_N = \int_{\mathscr{C}(\delta)} \alpha^N e^{-K\alpha\delta} \delta^{-1} |dx| + 2\int_\delta^\infty \alpha^N e^{-K\alpha x} x^{-1} dx$$

$$= 2\pi\alpha^N e^{-K\alpha\delta} + 2\alpha^N\left(\frac{e^{-K\alpha\delta}}{K\alpha}\delta^{-1} - \frac{1}{K\alpha}\int_\delta^\infty e^{-K\alpha x} x^{-2} dx\right).$$

Since

$$\int_\delta^\infty e^{-K\alpha x} x^{-2} dx \leqslant e^{-K\alpha\delta}\int_\delta^\infty x^{-2} dx \ll e^{-K\alpha\delta},$$

we get

$$I'_N = O(\alpha^N e^{-K\alpha\delta} + \alpha^{N-1} e^{-K\alpha\delta}) = O_N(1),$$

hence (4·7) follows. The uniformity in α in (4·7) is important, because this is the reason why the error estimates in Theorems 1, 2 and 3 are uniform in α.

Using (4·7), we see that

$$\int_{\mathscr{C}(\delta)} \frac{y^{v+N-1}}{e^{wy}-1} I_N(y;\alpha)\, dy \to 0 \quad (\text{as } \delta \to 0)$$

if $\operatorname{Re} v > -N+1$. Hence, in this case,

$$R_N(0,v;\alpha,w) = \frac{1}{2\pi i \Gamma(v)}\int_0^\infty \frac{y^{v+N-1}}{e^{wy}-1} I_N(y;\alpha)\, dy,$$

and applying (4·7) again we obtain

$$R_N(0,v;\alpha,w) = O(w^{-\operatorname{Re} v - N}) \tag{4·8}$$

for $\operatorname{Re} v > -N+1$, where the implied constant depends only on v and N. Combining Propositions 1, 2 and 3 with (4·8), and denoting $R_N(v;\alpha,w) = R_N(0,v;\alpha,w)$, we obtain Theorem 1.

Next we proceed to the proof of Theorem 2. From Propositions 1 and 2 we have

$$\zeta'_2(v;\alpha,w) = \zeta'(v,\alpha) - \frac{\zeta(v-1)}{v-1} w^{1-v}\log w - \frac{\zeta(v-1)}{(v-1)^2} w^{1-v}$$
$$+ \frac{\zeta'(v-1)}{v-1} w^{1-v} + g'(0,v;\alpha,w), \tag{4·9}$$

and from Proposition 3 we have

$$g'(0,v;\alpha,w) = \sum_{n=0}^{N-1}\binom{-v}{n}\zeta(-n,\alpha)$$
$$\times\left\{\left(\sum_{j=0}^{n-1}\frac{1}{v+j}\right)\zeta(v+n) + \zeta'(v+n) - \zeta(v+n)\log w\right\} w^{-v-n} + R'_N(0,v;\alpha,w). \tag{4·10}$$

Letting $v \to 0$ in these formulas, and noticing

$$\zeta'(0,\alpha) = \log\Gamma(\alpha) - \tfrac{1}{2}\log 2\pi, \quad \zeta(-1) = -\tfrac{1}{12}, \quad \zeta(0,\alpha) = \tfrac{1}{2} - \alpha,$$

$$\lim_{v\to 0} v\left\{\frac{1}{v}\zeta(v+1) + \zeta'(v+1) - \zeta(v+1)\log w\right\} = \gamma - \log w,$$

and

$$\lim_{v\to 0}\binom{-v}{n}\left(\frac{1}{v}+\ldots+\frac{1}{v+n-1}\right)=\frac{(-1)^n}{n}\quad (n\geqslant 2),$$

we get, for $N\geqslant 2$,

$$\zeta_2'(0;\alpha,w)=-\tfrac{1}{12}w\log w+(\tfrac{1}{12}-\zeta'(-1))\,w+\log\Gamma(\alpha)$$
$$-\tfrac{1}{2}\log 2\pi+g'(0,0;\alpha,w), \tag{4·11}$$

$$g'(0,0;\alpha,w)=\tfrac{1}{2}(\tfrac{1}{2}-\alpha)\,(\log w-\log 2\pi)-\zeta(-1,\alpha)\,(\gamma-\log w)\,w^{-1}$$
$$+\sum_{n=2}^{N-1}\frac{(-1)^n}{n}\zeta(-n,\alpha)\,\zeta(n)\,w^{-n}+R_N'(0,0;\alpha,w), \tag{4·12}$$

and therefore (1·5). From (4·5) we have

$$R_N'(0,v;\alpha,w)=-\frac{\Gamma'(v)(e^{2\pi iv}-1)+2\pi i\Gamma(v)\,e^{2\pi iv}}{2\pi i\Gamma(v)^2(e^{2\pi iv}-1)^2}\int_{\mathscr{C}}\frac{y^{v+N-1}}{e^{wy}-1}I_N(y;\alpha)\,dy$$
$$+\frac{1}{2\pi i\Gamma(v)(e^{2\pi iv}-1)}\int_{\mathscr{C}}\frac{y^{v+N-1}\log y}{e^{wy}-1}I_N(y;\alpha)\,dy;$$

hence

$$R_N'(0,0;\alpha,w)=-\frac{\pi i-\gamma}{(2\pi i)^2}\int_{\mathscr{C}}\frac{y^{N-1}}{e^{wy}-1}I_N(y;\alpha)\,dy$$
$$+\frac{1}{(2\pi i)^2}\int_{\mathscr{C}}\frac{y^{N-1}\log y}{e^{wy}-1}I_N(y;\alpha)\,dy. \tag{4·13}$$

Therefore, applying (4·7) to the above, we obtain (1·6), so the proof of Theorem 2 is completed. We note that the asymptotic expansions for derivatives of higher order may be deduced in a similar manner.

Now we can prove the asymptotic expansion of $\log\Gamma_2(\alpha,(1,\ w))$. Since $\zeta(-n,\alpha)\to\zeta(-n)$ (as $\alpha\to 0$) for any positive integer n, from (1·5) and (1·8) we get

$$-\log\rho_2(1,w)=-\tfrac{1}{12}w\log w+(\tfrac{1}{12}-\zeta'(-1))\,w+\tfrac{1}{4}\log w-\tfrac{3}{4}\log 2\pi$$
$$+\zeta(-1)\,w^{-1}\log w-\zeta(-1)\,\gamma w^{-1}+\sum_{n=2}^{N-1}\frac{(-1)^n}{n}\zeta(-n)\,\zeta(n)\,w^{-n}$$
$$+\lim_{\alpha\to 0}R_N'(0,0;\alpha,w). \tag{4·14}$$

Letting $\alpha\to 0$ in (4·13), with noticing (4·7), we see that

$$\lim_{\alpha\to 0}R_N'(0,0;\alpha,w)=O_N(w^{-N}(|\log w|+1)).$$

Combining this estimate with (1·5), (1·7) and (4·14), we obtain Theorem 3.

5. *The appearance of confluent hypergeometric functions*

In the error estimates in Theorems 1, 2 and 3 proved above, the implied constants depend implicitly on some parameters such as N and v. It would be better if we could obtain certain asymptotic series with more explicit error estimates. For this purpose, however, the expression (4·5) is not suitable. In the rest of this paper, via another expression of the error term, we will prove another type of asymptotic series for double zeta and double gamma-functions, in which the error estimate will be given as a strict inequality.

We write $\alpha = A+\beta$, where A is a non-negative integer and $0<\beta\leqslant 1$. Then

$$h(z;\alpha) = -\sum_{j=1}^{A} e^{(1-\beta-j)z} + h(z;\beta);$$

hence

$$\begin{aligned}\int_{\mathscr{C}} h(x+y;\alpha)\,x^{u-1}\,dx &= -\sum_{j=1}^{A}\int_{\mathscr{C}} e^{(1-\beta-j)(x+y)}\,x^{u-1}\,dx + \int_{\mathscr{C}} h(x+y;\beta)\,x^{u-1}\,dx\\ &= -\sum_{j=1}^{A}(e^{2\pi i u}-1)\,e^{-(j-1+\beta)y}(j-1+\beta)^{-u}\,\Gamma(u) + \int_{\mathscr{C}} h(x+y;\beta)\,x^{u-1}\,dx.\end{aligned}$$

Substituting this into (3·10), we obtain

$$g(u,v;\alpha,w) = -\sum_{j=1}^{A}\frac{(j-1+\beta)^{-u}}{\Gamma(v)(e^{2\pi i v}-1)}\int_{\mathscr{C}}\frac{y^{v-1}}{e^{wy}-1}\,e^{-(j-1+\beta)y}\,dy + g(u,v;\beta,w).$$

If $\mathrm{Re}\, v>1$, we have

$$\begin{aligned}\int_{\mathscr{C}}\frac{y^{v-1}}{e^{wy}-1}\,e^{-(j-1+\beta)y}\,dy &= (e^{2\pi i v}-1)\sum_{m=1}^{\infty}\int_0^{\infty} y^{v-1}\,e^{-(j-1+\beta+mw)y}\,dy\\ &= (e^{2\pi i v}-1)\,w^{-v}\,\Gamma(v)\,\zeta\left(v,\frac{j-1+\beta}{w}+1\right),\end{aligned}$$

but this is valid for any complex v by analytic continuation. Therefore we have

Proposition 4.

$$g(u,v;\alpha,w) = -w^{-v}\sum_{j=1}^{A}(j-1+\beta)^{-u}\,\zeta\left(v,\frac{j-1+\beta}{w}+1\right) + g(u,v;\beta,w). \tag{5·1}$$

Next, let R be a large positive integer, and

$$\mathscr{C}_R = \{x = -y+2\pi(R+\tfrac{1}{2})\,e^{i\phi}\,|\,0\leqslant\phi<2\pi\}.$$

We claim that

$$h(x+y;\beta) = O(1) \tag{5·2}$$

for any $x\in\mathscr{C}_R$, where the implied constant is absolute. In fact, if $\cos\phi\leqslant 0$, then $|e^{(1-\beta)(x+y)}|\leqslant 1$ because $1-\beta\geqslant 0$. Since $|e^{x+y}-1|$ is bounded from below, (5·2) follows. If $\cos\phi>0$, we consider the expression

$$h(x+y;\beta) = \frac{e^{-\beta(x+y)}}{1-e^{-(x+y)}} - \frac{1}{x+y}.$$

Since $|e^{-\beta(x+y)}|\leqslant 1$ and $|e^{-(x+y)}-1|$ is bounded from below, (5·2) follows in this case too.

It should be emphasized that in this proof of (5·2), the condition $0<\beta\leqslant 1$ is essential. This is the reason why we have introduced the new parameter β.

From (5·2) we have that if $\operatorname{Re} u < 0$ then

$$\int_{\mathscr{C}_R} h(x+y;\beta)\, x^{u-1}\, dx \to 0$$

as R tends to infinity. (Here we again meet the situation we could not consider if we would restrict ourselves to the case $u = 0$.) Hence, by residue calculus we get

$$g(u,v;\beta,w) = \frac{-2\pi i}{\Gamma(u)\,\Gamma(v)(e^{2\pi iu}-1)(e^{2\pi iv}-1)} \int_{\mathscr{C}} \frac{y^{v-1}}{e^{wy}-1} \sum_{n\neq 0} e^{-2\pi in\beta}(-y+2\pi in)^{u-1}\, dy, \tag{5·3}$$

and therefore

$$\begin{aligned} g(u,v;\beta,w) = {} & \frac{\Gamma(1-u)}{\Gamma(v)}(2\pi)^{u+v-1} \\ & \times \left\{ e^{\pi i(u+v-1)/2} \sum_{n=1}^{\infty} \sigma_{u+v-1}(n,\beta) \int_0^{-i\infty} \eta^{v-1}(1+\eta)^{u-1}\, e^{-2\pi inw\eta}\, d\eta \right. \\ & \left. + e^{-\pi i(u+v-1)/2} \sum_{n=1}^{\infty} \sigma_{u+v-1}(n,-\beta) \int_0^{i\infty} \eta^{v-1}(1+\eta)^{u-1}\, e^{2\pi inw\eta}\, d\eta \right\} \end{aligned} \tag{5·4}$$

for $\operatorname{Re} u < 0$, $\operatorname{Re} v > 1$, where

$$\sigma_a(n,\beta) = \sum_{d|n} d^a\, e^{2\pi id\beta}.$$

The argument for deducing (5·4) from (5·3) is quite similar to that in section 5 of Katsurada–Matsumoto [**10**], so we omit the details. We should mention that, in the special case $0 < \alpha \leqslant 1$ and $w = 1$, the expression (5·4) was shown by M. Katsurada in his unpublished 1992 manuscript, which is a part of the joint research by him and the author on the mean square (1·4).

Now, as in Katsurada [**8**], we write the right-hand side of (5·4) in terms of confluent hypergeometric functions, and apply the transformation formula. The confluent hypergeometric function $\Psi(a,c;x)$ is defined by

$$\Psi(a,c;x) = \frac{1}{\Gamma(a)} \int_0^{\infty e^{i\phi}} e^{-xy}\, y^{a-1}(1+y)^{c-a-1}\, dy$$

for $\operatorname{Re} a > 0$, $-\pi < \phi < \pi$, $|\phi + \arg x| < \pi/2$ (see Erdélyi *et al.* [**7**], formula 6·5(3)). Therefore, putting $a = v$, $c = u+v$, $\phi = \mp\pi/2$, $x = \pm 2\pi inw$, (5·4) can be written as

$$\begin{aligned} g(u,v;\beta,w) = {} & \Gamma(1-u)\,(2\pi)^{u+v-1} \\ & \times \{ e^{\pi i(u+v-1)/2} \sum_{n=1}^{\infty} \sigma_{u+v-1}(n,\beta)\, \Psi(v,u+v;2\pi inw) \\ & + e^{-\pi i(u+v-1)/2} \sum_{n=1}^{\infty} \sigma_{u+v-1}(n,-\beta)\, \Psi(v,u+v;-2\pi inw) \}. \end{aligned} \tag{5·5}$$

We quote the transformation formula

$$\Psi(a,c;x) = x^{1-c}\,\Psi(a-c+1, 2-c; x) \tag{5·6}$$

(Erdélyi *et al.* [**7**], formula 6·5(6)) and the asymptotic expansion

$$\Psi(a,c;x) = \sum_{k=0}^{N-1} \frac{(-1)^k (a-c+1)_k (a)_k}{k!} x^{-a-k} + \rho_N(a,c;x), \tag{5·7}$$

where $(a)_k = \Gamma(a+k)/\Gamma(a)$ (the Pochhammer symbol) and

$$\rho_N(a,c;x) = \frac{(-1)^N (a-c+1)_N}{\Gamma(a)} \int_0^{\infty e^{i\phi}} e^{-xy}\, y^{a+N-1} \int_0^1 \frac{(1-\tau)^{N-1}}{(N-1)!} (1+\tau y)^{c-a-N-1}\, d\tau\, dy \tag{5·8}$$

(Katsurada [**8**, (2·12) and (2·13)]; or implicitly included in Erdélyi *et al.* [**7**], formula 6·13·1(1)). Applying (5·6), and then (5·7), to (5·5), we obtain

$$\begin{aligned} g(u,v;\beta,w) &= \Gamma(1-u)\, w^{1-u-v} \\ &\quad \times \Bigg[\sum_{n=1}^{\infty} \sigma_{u+v-1}(n,\beta)\, n^{1-u-v} \Bigg\{ \sum_{k=0}^{N-1} \frac{(-1)^k (1-u)_k (v)_k}{k!} (2\pi i n w)^{u-1-k} \\ &\quad + \rho_N(1-u, 2-u-v; 2\pi i n w) \Bigg\} \\ &\quad + \sum_{n=1}^{\infty} \sigma_{u+v-1}(n,-\beta)\, n^{1-u-v} \Bigg\{ \sum_{k=0}^{N-1} \frac{(-1)^k (1-u)_k (v)_k}{k!} (-2\pi i n w)^{u-1-k} \\ &\quad + \rho_N(1-u, 2-u-v; -2\pi i n w) \Bigg\} \Bigg] \\ &= \Gamma(1-u)\, w^{1-u-v} \times \Bigg[\sum_{k=0}^{N-1} \frac{(-1)^k (1-u)_k (v)_k}{k!} (2\pi w)^{u-1-k} \zeta(v+k) \\ &\quad \times \{ e^{\pi i (u-1-k)/2} \phi(-u+1+k,\beta) + e^{-\pi i (u-1-k)/2} \phi(-u+1+k,-\beta) \} \\ &\quad + \sum_{n=1}^{\infty} \sigma_{u+v-1}(n,\beta)\, n^{1-u-v} \rho_N(1-u, 2-u-v; 2\pi i n w) \\ &\quad + \sum_{n=1}^{\infty} \sigma_{u+v-1}(n,-\beta)\, n^{1-u-v} \rho_N(1-u, 2-u-v; -2\pi i n w) \Bigg] \end{aligned} \tag{5·9}$$

for $\mathrm{Re}\, u < 0$ and $\mathrm{Re}\, v > 1$, where $\phi(s,\beta)$ is Lerch's zeta-function defined by (2·4). Applying (2·3) we see that the first sum in the right-hand side of (5·9) is equal to

$$\sum_{k=0}^{N-1} \frac{(-1)^k (v)_k}{k!\,\Gamma(1-u)} \zeta(v+k)\, \zeta(u-k,\beta)\, w^{u-1-k}.$$

Therefore, comparing with Proposition 3, we find that

$$\begin{aligned} R_N(u,v;\beta,w) &= \Gamma(1-u)\, w^{1-u-v} \\ &\quad \times \Bigg\{ \sum_{n=1}^{\infty} \sigma_{u+v-1}(n,\beta)\, n^{1-u-v} \rho_N(1-u, 2-u-v; 2\pi i n w) \\ &\quad + \sum_{n=1}^{\infty} \sigma_{u+v-1}(n,-\beta)\, n^{1-u-v} \rho_N(1-u, 2-u-v; -2\pi i n w) \Bigg\}. \end{aligned} \tag{5·10}$$

In the next section we will estimate ρ_N and deduce a new estimate of R_N.

6. *Another type of asymptotic series for double zeta and double gamma-functions*

Now we estimate ρ_N. Putting $\pm 2\pi i n w y = \eta$, we have

$$\rho_N(1-u, 2-u-v; \pm 2\pi i n w) = \frac{(-1)^N (v)_N}{\Gamma(1-u)} (\pm 2\pi i n w)^{u-N-1}$$
$$\times \int_0^\infty e^{-\eta}\, \eta^{-u+N} \int_0^1 \frac{(1-\tau)^{N-1}}{(N-1)\,!} \left(1 \pm \frac{\tau\eta}{2\pi i n w}\right)^{-v-N} d\tau\, d\eta. \quad (6{\cdot}1)$$

If $\operatorname{Re} v \geqslant -N$, then

$$\left|1 \pm \frac{\tau\eta}{2\pi i n w}\right|^{-\operatorname{Re} v-N} \leqslant 1,$$

so

$$\left|\left(1 \pm \frac{\tau\eta}{2\pi i n w}\right)^{-v-N}\right| \leqslant e^{\pi|\operatorname{Im} v|/2}.$$

We also note that

$$\int_0^1 (1-\tau)^{N-1}\, d\tau = \frac{1}{N}.$$

Using these facts, and assuming $\operatorname{Re} u < N+1$, we obtain

$$|\rho_N(1-u, 2-u-v; \pm 2\pi i n w)|$$
$$\leqslant \frac{|(v)_N|\, \Gamma(-\operatorname{Re} u+N+1)}{N\,!|\Gamma(1-u)|} e^{\pi(|\operatorname{Im} u|+|\operatorname{Im} v|)/2} (2\pi n w)^{\operatorname{Re} u-N-1}; \quad (6{\cdot}2)$$

hence

$$\sum_{n=1}^\infty |\sigma_{u+v-1}(n, \pm\beta)\, n^{1-u-v} \rho_N(1-u, 2-u-v; \pm 2\pi i n w)|$$
$$\leqslant \frac{|(v)_N|\, \Gamma(-\operatorname{Re} u+N+1)}{N\,!|\, \Gamma(1-u)|} e^{\pi(|\operatorname{Im} u|+|\operatorname{Im} v|)/2} (2\pi w)^{\operatorname{Re} u-N-1} \sum_{n=1}^\infty \sigma_{\operatorname{Re}(u+v)-1}(n)\, n^{-\operatorname{Re} v-N}, \quad (6{\cdot}3)$$

and the last sum is equal to $\zeta(\operatorname{Re} v+N)\,\zeta(-\operatorname{Re} u+N+1)$ if $\operatorname{Re} u < N$ and $\operatorname{Re} v > -N+1$. Therefore, the last two infinite series in the right-hand side of (5·10) are convergent absolutely in the region $\operatorname{Re} u < N$, $\operatorname{Re} v > -N+1$. Putting $u = 0$ in (5·10) and combining with Propositions 1, 2, 3 and 4, we now obtain

THEOREM 6. *For any positive integer N and $\operatorname{Re} v > -N+1$ (except for the points $v \in \{2, 1, 0, -1, -2, -3, \ldots\}$), we have*

$$\zeta_2(v; \alpha, w) = \zeta(v, \alpha) + \frac{\zeta(v-1)}{v-1} w^{1-v} - w^{-v} \sum_{j=1}^{A} \zeta\left(v, \frac{j-1+\beta}{w} + 1\right)$$
$$+ \sum_{n=0}^{N-1} \binom{-v}{n} \zeta(-n, \beta)\, \zeta(v+n)\, w^{-v-n} + R_N(v; \beta, w) \quad (6{\cdot}4)$$

with

$$|R_N(v; \beta, w)| \leqslant 2(2\pi)^{-N-1} e^{\pi|\operatorname{Im} v|/2} |(v)_N|\, \zeta(\operatorname{Re} v+N)\, \zeta(N+1)\, w^{-\operatorname{Re} v-N}. \quad (6{\cdot}5)$$

Remark 1. It is easy to express the terms

$$\zeta\left(v, \frac{j-1+\beta}{w} + 1\right)$$

as expansions with respect to w, because the formula

$$\zeta(v,\alpha)=\alpha^{-v}+\zeta(v)+\sum_{k=1}^{\infty}\binom{-v}{k}\zeta(v+k)\,\alpha^{k}$$

is known (see Mikolás[**15**, (4·6)]).

Remark 2. The existence of the factor $e^{\pi|\mathrm{Im}\,v|/2}$ is not satisfactory. In the appendix we will prove another upper-bound of $|R_N(v;\beta,w)|$, which is of polynomial order of growth with respect to v.

Next, putting $u=0$ in (5·10) and differentiating with respect to v, we get

$$\begin{aligned}R'_N(0,v;\beta,w)=&-w^{1-v}\log w\left\{\sum_{n=1}^{\infty}\sigma_{v-1}(n,\beta)\,n^{1-v}\rho_+(v)+\sum_{n=1}^{\infty}\sigma_{v-1}(n,-\beta)\,n^{1-v}\rho_-(v)\right\}\\&+w^{1-v}\left\{\sum_{n=1}^{\infty}\sigma'_{v-1}(n,\beta)\,n^{1-v}\rho_+(v)+\sum_{n=1}^{\infty}\sigma'_{v-1}(n,-\beta)\,n^{1-v}\rho_-(v)\right.\\&-\sum_{n=1}^{\infty}\sigma_{v-1}(n,\beta)\,n^{1-v}\rho_+(v)\log n-\sum_{n=1}^{\infty}\sigma_{v-1}(n,-\beta)\,n^{1-v}\rho_-(v)\log n\\&\left.+\sum_{n=1}^{\infty}\sigma_{v-1}(n,\beta)\,n^{1-v}\rho'_+(v)+\sum_{n=1}^{\infty}\sigma_{v-1}(n,-\beta)\,n^{1-v}\rho'_-(v)\right\}\end{aligned}\tag{6·6}$$

for $\mathrm{Re}\,v>-N+1$, where we write

$$\rho_{\pm}(v)=\rho_N(1,2-v;\pm 2\pi i n w)$$

as an abbreviation. We see from (6·2) that $\rho_{\pm}(0)=0$. Hence if $N\geqslant 2$, it follows from (6·6) that

$$R'_N(0,0;\beta,w)=w\left\{\sum_{n=1}^{\infty}\sigma_{-1}(n,\beta)\,n\rho'_+(0)+\sum_{n=1}^{\infty}\sigma_{-1}(n,-\beta)\,n\rho'_-(0)\right\}.\tag{6·7}$$

Putting $u=0$ in (6·1) and differentiating with respect to v, we get

$$\begin{aligned}\rho'_{\pm}(v)=&(-1)^N\frac{d(v)_N}{dv}(\pm 2\pi i n w)^{-N-1}\int_0^{\infty}e^{-\eta}\,\eta^N\\&\times\int_0^1\frac{(1-\tau)^{N-1}}{(N-1)!}\left(1\pm\frac{\tau\eta}{2\pi i n w}\right)^{-v-N}d\tau\,d\eta\\&-(-1)^N(v)_N(\pm 2\pi i n w)^{-N-1}\int_0^{\infty}e^{-\eta}\,\eta^N\\&\times\int_0^1\frac{(1-\tau)^{N-1}}{(N-1)!}\left(1\pm\frac{\tau\eta}{2\pi i n w}\right)^{-v-N}\log\left(1\pm\frac{\tau\eta}{2\pi i n w}\right)d\tau\,d\eta\\=&\,Q_1(v)+Q_2(v),\end{aligned}$$

say. It is easy to see that

$$|Q_1(v)|\leqslant\left|\sum_{j=0}^{N-1}\frac{(v)_N}{v+j}\right|(2\pi n w)^{-N-1}\,e^{\pi|\mathrm{Im}\,v|/2}$$

and $Q_2(0)=0$, hence

$$|\rho'_{\pm}(0)|\leqslant(N-1)!\,(2\pi n w)^{-N-1}.\tag{6·8}$$

From this estimate and (6·7) we obtain

$$|R'_N(0,0;\beta,w)| \leqslant 2w(N-1)!\,(2\pi w)^{-N-1}\sum_{n=1}^{\infty}\sigma_{-1}(n)\,n^{-N}$$
$$= 2(N-1)!\,(2\pi)^{-N-1}\zeta(N)\,\zeta(N+1)\,w^{-N}. \qquad (6\cdot 9)$$

Also, from Proposition 4 it follows that

$$g'(0,v;\alpha,w) = w^{-v}\log w\sum_{j=1}^{A}\zeta\left(v,\frac{j-1+\beta}{w}+1\right)$$
$$-w^{-v}\sum_{j=1}^{A}\zeta'\left(v,\frac{j-1+\beta}{w}+1\right)+g'(0,v;\beta,w).$$

Putting $v=0$ in this formula, and combining with (4·11), (4·12) (with $\alpha=\beta$) and (6·9), we obtain

THEOREM 7. *For any $N \geqslant 2$ we have*

$$\zeta_2'(0;\alpha,w) = -\tfrac{1}{12}w\log w+(\tfrac{1}{12}-\zeta'(-1))\,w$$
$$+\left\{\tfrac{1}{2}(\tfrac{1}{2}-\beta)+\sum_{j=1}^{A}\zeta\left(0,\frac{j-1+\beta}{w}+1\right)\right\}\log w+\log\Gamma(\alpha)$$
$$+(\tfrac{1}{2}\beta-\tfrac{3}{4})\log 2\pi-\sum_{j=1}^{A}\zeta'\left(0,\frac{j-1+\beta}{w}+1\right)+\zeta(-1,\beta)\,w^{-1}\log w$$
$$-\zeta(-1,\beta)\,\gamma w^{-1}+\sum_{n=2}^{N-1}\frac{(-1)^n}{n}\zeta(-n,\beta)\,\zeta(n)\,w^{-n}+R'_N(0;\beta,w) \qquad (6\cdot 10)$$

with

$$|R'_N(0;\beta,w)| \leqslant 2(N-1)!\,(2\pi)^{-N-1}\zeta(N)\,\zeta(N+1)\,w^{-N}. \qquad (6\cdot 11)$$

We mention here that, for any v, it is possible to prove the asymptotic series of the same type for $\zeta_2'(v;\alpha,w)$. In fact, let us assume $w \geqslant 1$. Using

$$\left|\log\left(1\pm\frac{\tau\eta}{2\pi inw}\right)\right| \leqslant \left\{\left(\log\left|1\pm\frac{\tau\eta}{2\pi inw}\right|\right)^2+\frac{\pi^2}{4}\right\}^{1/2}$$
$$\leqslant \begin{cases} c_1 & \text{if} \quad \eta \leqslant 1 \\ \log\eta+c_1 & \text{if} \quad \eta > 1 \end{cases}$$

(where $c_1 = ((\log 2)^2+\pi^2/4)^{1/2}$) and

$$\int_1^{\infty} e^{-\eta}\,\eta^N\log\eta\,d\eta \leqslant \Gamma'(N+1) = N!\left(\sum_{j=1}^{N}\frac{1}{j}-\gamma\right) \leqslant N!(\log N+1-\gamma),$$

we can show

$$|Q_2(v)| \leqslant |(v)_N|\,e^{\pi|\mathrm{Im}\,v|/2}\,(\log N+3)\,(2\pi nw)^{-N-1}.$$

From this inequality and (6·6) we can easily deduce an upper bound of $|R'_N(0,v;\beta,w)|$.

Finally, from (6·10) and (1·8) we get

$$-\log\rho_2(1,w) = -\tfrac{1}{12}w\log w+(\tfrac{1}{12}-\zeta'(-1))\,w+\tfrac{1}{4}\log w-\tfrac{3}{4}\log 2\pi$$
$$+\zeta(-1)\,w^{-1}\log w-\zeta(-1)\,\gamma w^{-1}+\sum_{n=2}^{N-1}\frac{(-1)^n}{n}\zeta(-n)\,\zeta(n)\,w^{-n}+\lim_{\beta\to 0}R'_N(0;\beta,w).$$

Therefore we arrive at the following asymptotic series for double gamma-functions.

THEOREM 8. *For any $N \geqslant 2$ we have*

$$\log \Gamma_2(\alpha,(1,w)) = \left\{-\tfrac{1}{2}\beta + \sum_{j=1}^{A} \zeta\left(0, \frac{j-1+\beta}{w}+1\right)\right\} \log w + \log \Gamma(\alpha)$$
$$+\tfrac{1}{2}\beta \log 2\pi - \sum_{j=1}^{A} \zeta'\left(0, \frac{j-1+\beta}{w}+1\right)$$
$$+(\zeta(-1,\beta)-\zeta(-1))\, w^{-1}\log w - (\zeta(-1,\beta)-\zeta(-1))\,\gamma w^{-1}$$
$$+\sum_{n=2}^{N-1} \frac{(-1)^n}{n}(\zeta(-n,\beta)-\zeta(-n))\,\zeta(n) w^{-n} + S_N(\beta,(1,w)) \tag{6·12}$$

with

$$|S_N(\beta,(1,w))| \leqslant 4(N-1)!\,(2\pi)^{-N-1}\zeta(N)\,\zeta(N+1)\,w^{-N}. \tag{6·13}$$

We note that in Theorems 7 and 8, it is easy to expand the terms

$$\zeta\left(0, \frac{j-1+\beta}{w}+1\right) \quad \text{and} \quad \zeta'\left(0, \frac{j-1+\beta}{w}+1\right)$$

with respect to w, by using the remark just after Theorem 6. It is also to be noted that, combining Theorem 8 with Shintani's result (2·1), we can deduce an asymptotic series for $L_F(1,\chi)$, whose error estimate is given by a strict inequality.

Appendix

Here we show an upper-bound estimate of $|R_N(v;\beta,w)|$ which is of polynomial order with respect to v.

Consider the special case $u = 0$ in (6·1). If $\operatorname{Re} v \geqslant -N$, then the double integral in (6·1) is absolutely convergent and hence we may change the order of integration and put $\eta = \pm 2\pi i n w y$ to get

$$\rho_N(1, 2-v; \pm 2\pi i n w) = (-1)^N (v)_N \int_0^1 \frac{(1-\tau)^{N-1}}{(N-1)!} K_{\pm}(\tau)\, d\tau, \tag{A 1}$$

where

$$K_{\pm}(\tau) = \int_0^{\mp i\infty} y^N (1+\tau y)^{-v-N} e^{\mp 2\pi i n w y}\, dy. \tag{A 2}$$

The following evaluation of $K_{\pm}(\tau)$ is similar to the argument of Katsurada–Matsumoto [**10**, section 6] or Katsurada [**8**, section 3]. First, it is easy to see that we may rotate the path of the integral (A 2) to the positive real axis if $\operatorname{Re} v > 0$ and $\tau > 0$. Then we integrate by parts $(N+1)$-times to obtain

$$\begin{aligned} K_{\pm}(\tau) &= \frac{N!}{(\pm 2\pi i n w)^{N+1}} + (-1)^{N+1} \int_0^\infty \frac{\partial^{N+1}}{\partial y^{N+1}} \{y^N (1+\tau y)^{-v-N}\} \frac{e^{\mp 2\pi i n w y}}{(\mp 2\pi i n w)^{N+1}}\, dy \\ &= \frac{N!}{(\pm 2\pi i n w)^{N+1}} + \frac{1}{(\pm 2\pi i n w)^{N+1}} \sum_{h=1}^{N+1} \binom{N+1}{h} A(N,h) \\ &\quad \times (-v-N)(-v-N-1)\ldots(-v-N-h+1)\,\tau^h \\ &\quad \times \int_0^\infty y^{h-1}(1+\tau y)^{-v-N-h} e^{\mp 2\pi i n w y}\, dy, \end{aligned}$$

where

$$A(N,h) = \begin{cases} N(N-1)\dots(h+1)h & \text{if} \quad 1 \leqslant h \leqslant N, \\ 1 & \text{if} \quad h = N+1. \end{cases}$$

Noting

$$\begin{aligned} &\left| \int_0^\infty y^{h-1}(1+\tau y)^{-v-N-h}\, e^{\mp 2\pi i n w y}\, dy \right| \\ &\quad \leqslant \int_0^{1/\tau} y^{h-1}\, dy + \int_{1/\tau}^\infty y^{h-1}(\tau y)^{-\operatorname{Re} v - N - h}\, dy \\ &\quad = \left(\frac{1}{h} + \frac{1}{\operatorname{Re} v + N} \right) \tau^{-h}, \end{aligned}$$

we get

$$|K_{\pm}(\tau)| \leqslant \frac{1}{(2\pi n w)^{N+1}} \{N! + S(N)\}, \tag{A 3}$$

where

$$S(N) = \sum_{h=1}^{N+1} \binom{N+1}{h} A(N,h) \left| \prod_{j=0}^{h-1} (v+N+j) \right| \left(\frac{1}{h} + \frac{1}{\operatorname{Re} v + N} \right).$$

Also we have

$$\begin{aligned} S(N) &\leqslant 2 \sum_{h=1}^{N} \binom{N+1}{h} \frac{A(N,h)}{h} \left| \prod_{j=0}^{h-1} (v+N+j) \right| + \frac{2}{N} \left| \prod_{j=0}^{N} (v+N+j) \right| \\ &\leqslant \left\{ 2(N!)^2(N+1) \sum_{h=1}^{N} (h!)^{-2} + \frac{2}{N} \right\} |(v+N)_{N+1}| \\ &\leqslant 3(N!)^2(N+1)|\,(v+N)_{N+1}|. \end{aligned}$$

From (A 1), (A 3) and the above inequality, it follows that

$$|\rho_N(1, 2-v; \pm 2\pi i n w)| \leqslant 4(2\pi n w)^{-N-1} N!\,(N+1)|\,(v)_N (v+N)_{N+1}|. \tag{A 4}$$

Substituting (A 4) into (5·10) (with $u = 0$), we obtain the desired estimate

$$|R_N(v; \beta, w)| \leqslant 8(2\pi)^{-N-1}(N+1)!\,|\,(v)_{2N+1}|\, \zeta(\operatorname{Re} v + N)\, \zeta(N+1)\, w^{-\operatorname{Re} v - N} \tag{A 5}$$

which is valid for $\operatorname{Re} v > 0$.

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