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On Riesz means of the coefficients of the Rankin–Selberg series

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Abstract

We study $\Delta(x;\varphi)$, the error term in the asymptotic formula for $\sum_{n\leqslant x} c_n$, where the c_ns are generated by the Rankin–Selberg series. Our main tools are Voronoï-type formulae. First we reduce the evaluation of $\Delta(x;\varphi)$ to that of $\Delta_1(x;\varphi)$, the error term of the weighted sum $\sum_{n\leqslant x}(x-n)c_n$. Then we prove an upper bound and a sharp mean square formula for $\Delta_1(x;\varphi)$, by applying the Voronoï formula of Meurman's type. We also prove that an improvement of the error term in the mean square formula would imply an improvement of the upper bound of $\Delta(x;\varphi)$. Some other related topics are also discussed.

1. *Introduction and statement of results*

Let $\varphi(z)$ be a holomorphic cusp form of weight κ with respect to the full modular group $SL(2,\mathbb{Z})$, and denote by $a(n)$ the nth Fourier coefficient of $\varphi(z)$. We suppose that $\varphi(z)$ is a normalized eigenfunction for the Hecke operators $T(n)$, i.e.

$$a(1) = 1$$

and

$$T(n)\varphi = a(n)\varphi$$

for every $n \in \mathbb{N}$. Rankin [**15**] and Selberg [**18**] independently introduced the function

$$Z(s) = \zeta(2s)\sum_{n=1}^{\infty} |a(n)|^2 n^{1-\kappa-s},$$

where $\zeta(s)$ is the Riemann zeta-function. In the half plane $\sigma = \mathrm{Re}(s) > 1$ the function $Z(s)$ has the absolutely convergent Dirichlet series expansion

$$Z(s) = \sum_{n=1}^{\infty} c_n n^{-s} \tag{1·1}$$

with

$$c_n = n^{1-\kappa} \sum_{m^2|n} m^{2(\kappa-1)} |a(n/m^2)|^2.$$

Deligne's estimate $|a(n)| \leqslant n^{(\kappa-1)/2} d(n)$ (cf. [**4**]), where $d(n)$ is the number of positive divisors of n, implies

$$c_n = O(n^{\varepsilon}). \tag{1·2}$$

Here and in what follows ε denotes an arbitrarily small positive number which is not necessarily the same at each occurrence.

Rankin [**15**] considered the analytic behaviour of $Z(s)$, and consequently he obtained

$$\sum_{n \leqslant x} c_n = \tfrac{1}{6}\pi^2 \kappa R_0 x + \Delta(x;\varphi)$$

with

$$\Delta(x;\varphi) = O(x^{\frac{3}{5}}) \tag{1·3}$$

for any $x > 1$, where

$$R_0 = \frac{12(4\pi)^{\kappa-1}}{\Gamma(\kappa+1)} \iint_{\mathfrak{F}} y^{\kappa-2} |\varphi(z)|^2 dxdy,$$

the integral being taken over a fundamental domain $\mathfrak{F}$ of $SL(2,\mathbb{Z})$. Selberg [**18**] also sketched briefly how to get these results. We will give an alternative proof of (1·3), in the frame of our method, in Section 4.

The first author [**9**, theorem 5] proved $\Delta(x;\varphi) = \Omega_{\pm}(x^{\frac{3}{8}})$ and conjectured that

$$\Delta(x;\varphi) = O(x^{\frac{3}{8}+\varepsilon}). \tag{1·4}$$

However, no improvement of (1·3) has been known after Rankin and Selberg. The reasons for this are, broadly speaking, of 'global' and 'local' nature. The 'global' reason is that the generating function $Z(s)$ of c_n, given by (1·1), does not seem representable as a square of a 'nice' Dirichlet series. This was discussed on p. 156 of [**9**]. The 'local' reason is that, via Theorem 1 and Lemma 2 (with $\rho = 1, N = x$) below, the problem of the estimation of $\Delta(x;\varphi)$ is reduced to the problem of the estimation of exponential sums of the form

$$\Sigma(M) := \sum_{M < m \leqslant M' \leqslant 2M} c_m \exp\left(8\pi i (xm)^{\frac{1}{4}}\right) \quad (1 \ll M \ll x).$$

It seems that at present one can only trivially estimate $\Sigma(M)$ as $\Sigma(M) \ll M$, which vitiates the efforts to improve (1·3).

The purpose of the present paper is to study $\Delta(x;\varphi)$ by the method of Voronoï-type formulae. We consider the Riesz mean of the type

$$D_\rho(x;\varphi) = \frac{1}{\Gamma(\rho+1)}\sum_{n\leqslant x}(x-n)^\rho c_n$$

for any fixed $\rho \geqslant 0$ and define the error term $\Delta_\rho(x;\varphi)$ by

$$D_\rho(x;\varphi) = \frac{\pi^2\kappa R_0}{6\Gamma(\rho+2)}x^{\rho+1} + \frac{Z(0)}{\Gamma(\rho+1)}x^\rho + \Delta_\rho(x;\varphi). \tag{1·5}$$

Clearly we have

$$\Delta_{\rho+1}(x;\varphi) = \int_0^x \Delta_\rho(t;\varphi)dt. \tag{1·6}$$

Since

$$\Delta(x;\varphi) = \Delta_0(x;\varphi) + Z(0),$$

hereafter we will discuss $\Delta_0(x;\varphi)$ instead of $\Delta(x;\varphi)$.

The Voronoï-type infinite series expression for $\Delta_\rho(x;\varphi)$ can be obtained in a standard way, but we find that the Voronoï series for $\Delta_0(x;\varphi)$ diverges. This phenomenon makes the study of $\Delta_0(x;\varphi)$ quite difficult.

However we can reduce, to some extent, the problem of evaluating $\Delta_0(x;\varphi)$ to that of evaluating $\Delta_1(x;\varphi)$, for which the Voronoï series converges.

THEOREM 1. *If*

$$\Delta_1(x;\varphi) = O(x^\alpha) \tag{1·7}$$

for some $\alpha \geqslant 0$, *then we have*

$$\Delta_0(x;\varphi) = O(x^{\frac{1}{2}\alpha}). \tag{1·8}$$

Conversely, if

$$\Delta_0(x;\varphi) = O(x^\beta) \tag{1·9}$$

for some $\beta \geqslant \frac{3}{8}$, *then we have*

$$\Delta_1(x;\varphi) = O(x^{\frac{1}{3}\beta+1}). \tag{1·10}$$

This result shows the importance of the study of the function $\Delta_1(x;\varphi)$ (see the Remark at the end of Section 3). We will see in Lemma 1 below that the Voronoï series for $\Delta_\rho(x;\varphi)$ converges only for $\rho > \frac{1}{2}$. Therefore the situation of the convergence of the Voronoï series for $\Delta_1(x;\varphi)$ is similar to the classical error term

$$\Delta(x) = \sum_{n\leqslant x} d(n) - x\log x - (2\gamma - 1)x$$

in the Dirichlet divisor problem (see chapter 3 of [**8**]), where γ is Euler's constant. Much deep information on $\Delta(x)$ was obtained by the use of the Voronoï formula for $\Delta(x)$, so we can expect that the Voronoï formula for $\Delta_1(x;\varphi)$ will also provide information on $\Delta_1(x;\varphi)$. Combining (1·3) with the second half of Theorem 1, we obtain the following

COROLLARY. *For any* $x > 1$ *we have*

$$\Delta_1(x;\varphi) = O(x^{\frac{6}{5}}). \tag{1·11}$$

(See the Remark after Lemma 2.)

We shall also prove a result on the mean square of $\Delta_1(x;\varphi)$, contained in

Theorem 2. *For any $X > 1$ we have*

$$\int_1^X \Delta_1(x;\varphi)^2 dx = \tfrac{2}{13}(2\pi)^{-4}\left(\sum_{n=1}^{\infty} c_n^2 n^{-\frac{7}{4}}\right) X^{\frac{13}{4}} + F_1(X;\varphi) \tag{1·12}$$

with

$$F_1(X;\varphi) = O(X^{3+\varepsilon}). \tag{1·13}$$

The estimate (1·13) suggests that perhaps the true form of $F_1(X;\varphi)$ is

$$F_1(X;\varphi) = X^3 P(\log X) + O(X^{\theta}), \tag{1·14}$$

where $P(y)$ is a certain polynomial of y which may be identically equal to zero and $\theta \leqslant 3$. In this paper we cannot prove any improvement on (1·11), but we shall prove

Theorem 3. *If we assume* (1·14)*, then we have*

$$\Delta_1(x;\varphi) = O(x^{\max(\frac{9}{8},\ \frac{1}{3}\theta + \frac{1}{5})}). \tag{1·15}$$

Therefore, if we could reduce the exponent of X in (1·13), then, via Theorem 3 and Theorem 1, we could improve the classical estimate (1·3).

In Section 2 we shall prove the Voronoï-type formulae for $\Delta_1(x;\varphi)$, both of the infinite series type (Lemma 1) and of the truncated type (Lemma 2). To prove Theorem 2, we shall use both of these two lemmas. If we use Lemma 2 only and apply the method of proof of theorem 13·5 of [**8**], we obtain (1·12) with the weaker error estimate

$$F_1(X;\varphi) = O(X^{\frac{25}{8}+\varepsilon}).$$

To obtain the sharper estimate (1·13), we should combine Lemmas 1 and 2 by appealing to Meurman's method [**13**]. Theorem 2 is the analogue of the mean square formula

$$\int_1^X \Delta(x)^2 dx = \frac{1}{6\pi^2}\left(\sum_{n=1}^{\infty} d(n)^2 n^{-\frac{3}{2}}\right) X^{\frac{3}{2}} + O(X\log^5 X)$$

originally due to Tong [**21**], and Meurman [**13**] gave a simpler proof of this result.

The asymptotic formula (1·12) implies $\Delta_1(x;\varphi) = \Omega(x^{\frac{9}{8}})$, but the stronger result $\Delta_1(x;\varphi) = \Omega_{\pm}(x^{\frac{9}{8}})$ follows by applying the general result of Ivić (theorem 1 of [**9**]). In fact, this result implies that there exist two positive constants B_1, B_2 such that every interval of the form $[x, x+B_1x^{\frac{3}{4}}]$ for large x contains two points x_1, x_2 for which $\Delta_1(x_1;\varphi) > B_2 x_1^{\frac{9}{8}}$ and $\Delta_1(x_2;\varphi) < -B_2 x_2^{\frac{9}{8}}$ hold. Hence it is a plausible conjecture that

$$\Delta_1(x;\varphi) = O(x^{\frac{9}{8}+\varepsilon}).$$

In view of Theorem 1, this conjecture would imply

$$\Delta_0(x;\varphi) = O(x^{\frac{9}{16}+\varepsilon}),$$

which is still far from the conjectural bound (1·4).

In the last section we shall consider another type of Riesz means, namely the logarithmic means

$$\tilde{D}_\xi(x;\varphi) = \frac{1}{\Gamma(\xi+1)} \sum_{n\leqslant x} c_n \log^\xi \left(\frac{x}{n}\right) \qquad (\xi \geqslant 0), \tag{1·16}$$

and we shall discuss some consequences of a recent result of Vorhauer [**22**].

2. *Voronoï-type formulae*

There are several papers which study Voronoï-type formulae for fairly general classes of zeta-functions. Here we shall apply a result of Hafner [**6**] to our $\Delta_\rho(x;\varphi)$. The function $Z(s)$ is holomorphic on the whole complex plane except for a simple pole at $s = 1$ with the residue $\frac{1}{6}\pi^2\kappa R_0$. Moreover, it satisfies the functional equation

$$\Gamma(s+\kappa-1)\Gamma(s)Z(s) = (2\pi)^{4s-2}\Gamma(\kappa-s)\Gamma(1-s)Z(1-s) \tag{2·1}$$

(see theorem 3 of [**15**]). Therefore we can apply theorem A, theorem B and lemma 2·1 of [**6**] with (in Hafner's notation)

$$a(n) = b(n) = c_n, \qquad \lambda_n = \mu_n = 4\pi^2 n,$$
$$\phi(s) = \psi(s) = (2\pi)^{-2s}Z(s), \qquad \Delta(s) = \Gamma(s+\kappa-1)\Gamma(s),$$
$$\sigma_a = \sigma_a^* = 1, \quad N = 2, \quad \alpha_1 = \alpha_2 = 1, \quad \alpha = 2,$$
$$\beta_1 = \kappa - 1, \quad \beta_2 = 0, \quad r = 1, \quad S = \{0,1\},$$
$$D = \mathbb{C}\backslash S, \quad a = 0, \quad s_0 = 1, \quad t_0 = \kappa - 1,$$
$$\mu = \kappa - \tfrac{3}{2}, \quad h = 4, \quad \theta_\rho = \tfrac{3}{4}\rho + \tfrac{3}{8}, \quad \beta_\rho = -\kappa - \tfrac{1}{2}\rho + \tfrac{1}{4}.$$

(Note that in Hafner's paper, the meaning of β_ρ is different from that of β_ν.) The explicit values of μ and h can be found in [**2**], which are

$$\mu = \tfrac{1}{2} + \sum_{\nu=1}^{N} (\beta_\nu - \tfrac{1}{2})$$

and

$$h = 2\exp\left\{-\frac{1}{\alpha}\left(\sum_{\nu=1}^{N} \alpha_\nu \log \alpha_\nu - \alpha \log \alpha\right)\right\}$$

in Hafner's notation. The correct value of $e_0(\rho)$ in Hafner's lemma 2·1 should be

$$e_0(\rho) = \left(2\alpha/h\right)^\rho \left(2/h\pi\right)^{\frac{1}{2}}.$$

Let L be the oriented polygonal path with vertices $-i\infty$, $-iT$, $b-iT$, $b+iT$, iT, and $i\infty$ in that order, where b and T are real numbers satisfying $b > \kappa-1$, $T > \kappa-1$, and b is not an integer. Define the integral

$$f_\rho(y) = \frac{1}{2\pi i}\int_L \frac{\Gamma(s+\kappa-1)\Gamma(s)}{\Gamma(2+\rho-s)\Gamma(\kappa-s)} y^{1+\rho-s} ds$$

for $y > 0$. Hafner's results then imply

Lemma 1. *We have*

$$\Delta_\rho(x;\varphi) = (4\pi^2)^{-(2\rho+1)} \sum_{n=1}^{\infty} c_n n^{-\rho-1} f_\rho(16\pi^4 nx) \tag{2·2}$$

for $x > 0$, where the infinite series on the right-hand side converges for $\rho > \frac{1}{2}$ uniformly on any finite closed x-interval. Moreover we have

$$\frac{d}{dx} f_{\rho+1}(x) = f_\rho(x) \tag{2·3}$$

and

$$f_\rho(y) = (2\pi)^{-\frac{1}{2}} y^{\frac{3}{4}\rho+\frac{3}{8}} \sin\left(4y^{\frac{1}{4}} + (\tfrac{3}{4} - \tfrac{1}{2}\rho)\pi\right) + O(y^{\frac{3}{4}\rho+\frac{1}{8}}) + O(y^{1+\rho-b}) \tag{2·4}$$

for $\rho \geqslant 0$.

For our purpose it is also necessary to prove the truncated Voronoï-type formula for $\Delta_\rho(x;\varphi)$. The result is the following

Lemma 2. *Let $N \gg 1$, $x > 1$ and $0 \leqslant \rho \leqslant 1$. Then we have*

$$\Delta_\rho(x;\varphi) = (2\pi)^{-\rho-1} x^{\frac{3}{4}\rho+\frac{3}{8}} \sum_{n\leqslant N} c_n n^{-\frac{1}{4}\rho-\frac{5}{8}} \sin\left(8\pi(nx)^{\frac{1}{4}} + (\tfrac{3}{4} - \tfrac{1}{2}\rho)\pi\right)$$
$$+O(x^{\frac{3}{4}\rho+\frac{1}{4}} N^{-\frac{1}{4}\rho+\frac{1}{4}+\varepsilon}) + O(x^{\frac{3}{4}\rho+\frac{3}{4}+\varepsilon} N^{-\frac{1}{4}\rho-\frac{1}{4}}). \tag{2·5}$$

Remark. If we choose $\rho = 1$ and $N = x^{\frac{3}{5}}$ in (2·5), we can get immediately

$$\Delta_1(x;\varphi) \ll x^{\frac{6}{5}+\varepsilon}.$$

It should be stressed that this estimate, slightly weaker than (1·11), is obtained without the use of (1·3).

The formula (2·5) can be shown in a standard way (cf. chapter XII of [**20**] or chapter I of [**11**]). By the same method it is possible to prove the truncated Voronoï-type formula for $\rho > 1$, but with somewhat different error estimates.

Now we sketch the proof. The starting point is the formula

$$D_\rho(x;\varphi) = I_\rho(x; 1+\delta) + O(x^{\rho+\varepsilon}) + O(T^{-\rho-1}x^{\rho+1+\varepsilon}), \tag{2·6}$$

where $T > \kappa - 1$, δ is a small positive number with $\delta > \varepsilon$ and

$$I_\rho(x;c) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{\Gamma(s)}{\Gamma(s+\rho+1)} Z(s) x^{s+\rho} ds. \tag{2·7}$$

We can show (2·6) by substituting (1·1) into the right-hand side of (2·7), changing the order of integration and summation and evaluating each integral by residue calculus. Here (and later in the proof) we always use the estimate (1·2).

Next, we replace the path of $I_\rho(x; 1+\delta)$ by the rectangle joining the points $1+\delta-iT$, $-\delta-iT$, $-\delta+iT$ and $1+\delta+iT$. The integrals over the horizontal sides of the rectangle can be estimated by using the convexity estimate

$$Z(\sigma+it) = O(|t|^{2(1+\delta-\sigma)}) \quad (-\delta \leqslant \sigma \leqslant 1+\delta,\ |t| \geqslant 1),$$

hence we obtain

$$\Delta_\rho(x;\varphi) = I_\rho(x;-\delta) + O(x^{\rho+\varepsilon} + T^{-\rho-1}x^{\rho+1+\delta} + T^{1-\rho+4\delta}x^{\rho-\delta}). \tag{2·8}$$

We then substitute the functional equation (2·1) into the integrand of $I_\rho(x;-\delta)$ and use the expression (1·1) to obtain

$$I_\rho(x;-\delta) = -2\pi i x^{\rho+1} \sum_{n=1}^{\infty} c_n J_\rho(n)$$

with

$$J_\rho(n) = \int_{-\delta-iT}^{-\delta+iT} \frac{\Gamma(\kappa-s)\Gamma(1-s)}{\Gamma(s+\kappa-1)\Gamma(s+\rho+1)}(16\pi^4 nx)^{s-1}ds.$$

Without loss of generality we assume that $N\in\mathbb{N}$. We choose $T = 2\pi(x(N+\frac{1}{2}))^{\frac{1}{4}}$ and use lemma 4·3 of [**20**] to find that

$$J_\rho(n) = O\biggl((nx)^{-1-\delta}T^{1-\rho+4\delta}\biggl(\Bigl(\log\frac{n}{N+\frac{1}{2}}\Bigr)^{-1}+1\biggr)\biggr)$$

for $n \geqslant N+1$. In case $n\leqslant N$ we replace the path of $J_\rho(n)$, with the error

$$O\biggl(n^{-1}x^{-\frac{1}{4}\rho-\frac{3}{4}}N^{-\frac{1}{4}\rho+\frac{1}{4}}\Bigl\{\Bigl(\frac{n}{N}\Bigr)^{-\delta}+\Bigl(\frac{n}{N}\Bigr)^{\mu}\Bigl(\Bigl(\log\frac{N+\frac{1}{2}}{n}\Bigr)^{-1}+1\Bigr)\Bigr\}\biggr),$$

by the polygonal path with vertices $\mu-i\infty$, $\mu-iT$, $-\delta-iT$, $-\delta+iT$, $\mu+iT$ and $\mu+i\infty$ in that order, where μ is a real number satisfying $\mu>\frac{1}{4}$. It is easily seen that the resulting integral is equal to $2\pi i(16\pi^4nx)^{-\rho-1}f_\rho(16\pi^4nx)$. Hence we get

$$I_\rho(x;-\delta) = (2\pi)^{-4\rho-2}\sum_{n\leqslant N} c_n n^{-\rho-1}f_\rho(16\pi^4nx) + O(x^{\frac{3}{4}\rho+\frac{1}{4}}N^{-\frac{1}{4}\rho+\frac{1}{4}+\delta}) + O(x^{\frac{3}{4}\rho+\frac{3}{4}+\delta}N^{-\frac{1}{4}\rho-\frac{1}{4}}).$$

Now we apply (2·4) to the right-hand side and then substitute the resulting expression into (2·8). Lastly, putting $\delta=2\varepsilon$, we obtain the assertion of Lemma 2.

Remark. From the case $\rho=0$ of (2·5), we can easily deduce

$$\int_1^X \Delta_0(x;\varphi)^2dx = O(X^{2+\varepsilon}). \tag{2·9}$$

The first author [**9**, theorem 5] proved this estimate by a different method based on the reflection principle. It is also possible to show (2·9) by applying a general theorem of [**3**]. By an argument analogous to that given in pp. 60–61 of [**10**], we see that the estimate $\Delta_0(x;\varphi)=O(x^{\theta/3})$ would follow if one could prove

$$\int_1^X \Delta_0(x;\varphi)^2dx = O(X^{\theta}). \tag{2·10}$$

In particular we could improve (1·3) if (2·10) with $\theta<\frac{9}{5}$ would hold.

3. *Proof of Theorem* 1

First we consider the sum of c_n in short intervals. For this purpose we quote the following result of Perelli [**14**].

LEMMA 3. *Let a and q be co-prime positive integers, $N>0$ and $\Lambda(n)$ the von Mangoldt function. Then*

$$\sum_{\substack{n\leqslant x\\ n\equiv a (\mathrm{mod}\ q)}} a(n)^2\Lambda(n) = \frac{x^\kappa}{\kappa\varphi(q)} + O\bigl(x^\kappa\exp(-c\sqrt{\log x})\bigr) \tag{3·1}$$

uniformly for $q \ll (\log x)^N$ where c is a positive constant depending only on N.

This result refines and generalizes theorem 2 of [**16**]. In Perelli's paper, the factor κ^{-1} in the first term on the right-hand side of (3·1) is missing. We use Lemma 3 with $q = 1$. We note that when $q > 1$, there is a gap in Perelli's argument concerning Siegel's zero (see p. 177 of [**1**]), but it was filled in a recent work of Ichihara [**7**].

Lemma 4. *Let ε be an arbitrary positive number. Then, as $x \to \infty$,*

$$\sum_{x-y<n\leqslant x} c_n \ll y \tag{3·2}$$

uniformly in y for $x^\varepsilon < y \leqslant x$.

Proof. Since $c_{p^l} \ll l^3$ and $c_n = O(n^\varepsilon)$ by Deligne's estimate, we can apply Shiu's result [**19**, theorem 1] and get

$$\sum_{x-y<n\leqslant x} c_n \ll \frac{y}{\log x} \exp\Big(\sum_{p\leqslant x} \frac{c_p}{p}\Big).$$

From (3·1) with $q = 1$, we obtain

$$\sum_{p\leqslant x} c_p p^{\kappa-1} \log p = \frac{1}{\kappa} x^\kappa + O\big(x^\kappa \exp(-c\sqrt{\log x})\big),$$

because

$$\sum_{n\leqslant x} a(n)^2 \Lambda(n) = \sum_{p\leqslant x} a(p)^2 \log p + \sum_{\substack{p^\nu\leqslant x\\ \nu\geqslant 2}} a(p^\nu)^2 \log p$$
$$= \sum_{p\leqslant x} c_p p^{\kappa-1} \log p + O(x^{\kappa-\frac{1}{2}} \log^3 x).$$

Hence by partial summation, we have

$$\sum_{p\leqslant x} \frac{c_p}{p} = \log\log x + O(1),$$

and therefore the assertion of the Lemma.

Proof of Theorem 1. We may assume $\alpha \leqslant 2$. From the definition of $\Delta_0(x;\varphi)$ and (3·2) we have, for $x^\varepsilon \ll H_1 \leqslant x$,

$$\int_{x-H_1}^{x} \Delta_0(t;\varphi)dt - H_1\Delta_0(x;\varphi) = \int_{x-H_1}^{x} \Big(\tfrac{1}{6}\pi^2\kappa R_0(x-t) - \sum_{t<n\leqslant x} c_n\Big)dt$$
$$\ll H_1^2 + H_1 \sum_{x-H_1<n\leqslant x} c_n \ll H_1^2.$$

Hence we have

$$H_1\Delta_0(x;\varphi) = \int_{x-H_1}^{x} \Delta_0(t;\varphi)dt + O(H_1^2), \tag{3·3}$$

and similarly

$$H_2\Delta_0(x;\varphi) = \int_{x}^{x+H_2} \Delta_0(t;\varphi)dt + O(H_2^2) \tag{3·4}$$

for $x^\varepsilon \ll H_2 \leqslant x$. From (1·6), (3·3) and (3·4) we obtain

$$\begin{aligned}\Delta_0(x;\varphi) &= \frac{1}{H_1+H_2}\int_{x-H_1}^{x+H_2}\Delta_0(t;\varphi)dt + O(H_1+H_2)\\ &= \frac{1}{H_1+H_2}\Big(\Delta_1(x+H_2;\varphi)-\Delta_1(x-H_1;\varphi)\Big)+O(H_1+H_2). \qquad (3{\cdot}5)\end{aligned}$$

Putting $H_1+H_2 = x^{\frac{1}{2}\alpha}$ in (3·5) and using (1·7) we obtain the first half of Theorem 1.

Next we assume that (1·9) holds. Then, for $0 < H \leqslant x$, we have

$$\begin{aligned}\Delta_1(x;\varphi) &= \frac{1}{H}\int_x^{x+H}\Delta_1(t;\varphi)dt - \frac{1}{H}\int_x^{x+H}\int_x^t \Delta_0(u;\varphi)dudt\\ &= \frac{1}{H}\int_x^{x+H}\Delta_1(t;\varphi)dt + O(x^\beta H). \qquad (3{\cdot}6)\end{aligned}$$

To estimate the integral on the right-hand side of (3·6), we split $\Delta_1(t;\varphi)$ as

$$\Delta_1(t;\varphi) = \delta_1(t;N) + E_N(t), \qquad (3{\cdot}7)$$

where, in view of Lemma 1,

$$\delta_1(t;N) = (2\pi)^{-2}t^{\frac{9}{8}}\sum_{n\leqslant N}\frac{c_n}{n^{\frac{7}{8}}}\sin\big(8\pi(nt)^{\frac{1}{4}}+\pi/4\big) \qquad (3{\cdot}8)$$

and

$$E_N(t) = (2\pi)^{-2}t^{\frac{9}{8}}\sum_{n>N}\frac{c_n}{n^{\frac{7}{8}}}\sin\big(8\pi(nt)^{\frac{1}{4}}+\pi/4\big) + O(t^{\frac{7}{8}}). \qquad (3{\cdot}9)$$

The contribution of $\delta_1(t;N)$ to the right-hand side of (3·6) is

$$\ll x^{\frac{9}{8}}\sum_{n\leqslant N}c_n n^{-\frac{7}{8}} \ll x^{\frac{9}{8}}N^{\frac{1}{8}}.$$

On the other hand, the contribution of $E_N(t)$ is, after termwise integration by parts (which is justified by uniform convergence),

$$\ll H^{-1}x^{\frac{15}{8}}\sum_{n>N}c_n n^{-\frac{9}{8}} + x^{\frac{7}{8}} \ll H^{-1}x^{\frac{15}{8}}N^{-\frac{1}{8}} + x^{\frac{7}{8}}.$$

Therefore we obtain from (3·6) that

$$\Delta_1(x;\varphi) \ll x^{\frac{9}{8}}N^{\frac{1}{8}} + H^{-1}x^{\frac{15}{8}}N^{-\frac{1}{8}} + x^\beta H.$$

Choosing $N = x^{\frac{8}{3}\beta-1}$ and $H = x^{1-\frac{2}{3}\beta}$, we obtain (1·10). This completes the proof of Theorem 1.

The referee kindly remarked that one can deduce (1·8) from (1·7) without using Lemma 4, only using the positivity of c_n.

Remark. Formula (3·5) actually says that, in order to improve the estimate (1·3) for $\Delta_0(x;\varphi)$, the only necessary fact is the existence of the points a and b, satisfying $a \leqslant x \leqslant b$, $b-a = o(x^{\frac{3}{5}})$ and $|\Delta_1(b;\varphi)-\Delta_1(a;\varphi)| = o(x^{\frac{6}{5}})$. This is a problem of local behaviour in short intervals, which is probably more accessible than the global bound (1·7).

4. *An alternative proof of* (1·3)

It is possible to give a proof of (1·3) by applying the method of Landau [**12**] and Walfisz [**23**] to our case. The applicability of the Landau–Walfisz method to Rankin–Selberg series was first mentioned by Golubeva–Fomenko [**5**], in the course of the study of Siegel zeros of the twisted Rankin–Selberg series. To show (1·3), it is enough to modify the proof of theorem 4 in [**5**], combining with Shiu's estimate (Lemma 4 in Section 3). A necessary Voronoï formula, corresponding to (11) of [**5**], is given by (2·2).

However, by a somewhat different use of the double integral technique, we can give a simpler proof of (1·3). Namely, as an analogue of (3·5), by using Lemma 4, we have

$$\begin{aligned}\Delta_0(x;\varphi) &= \frac{1}{H^2}\int_0^H\int_0^H \Delta_0(x+t+u;\varphi)\,dt\,du + O(H)\\ &= \frac{1}{H^2}\int_0^H \left(\Delta_1(x+u+H;\varphi)-\Delta_1(x+u;\varphi)\right)\,du + O(H)\end{aligned}$$

for $x^\varepsilon \leqslant H \leqslant x$. Now we use Lemma 2 with $\rho = 1$ and $N = x$, so that the error terms in (2·5) are $\ll x^{1+\varepsilon}$. We split the sum in (2·5) at M $(1 < M < x)$. The terms $n \leqslant M$ of the sum are estimated trivially. The terms $n > M$ are estimated by the first derivative test. Hence we obtain

$$\Delta_0(x;\varphi) \ll H^{-1}\left(x^{\frac{9}{8}}M^{\frac{1}{8}} + H^{-1}x^{\frac{15}{8}}M^{-\frac{1}{8}} + H^2 + x^{1+\varepsilon}\right). \tag{4·1}$$

Therefore with $H = x^{\frac{3}{5}}, M = x^3H^{-4} = x^{\frac{3}{5}}$ it follows from (4·1) that

$$\Delta_0(x;\varphi) \ll x^{\frac{3}{5}},$$

which is (1·3).

5. *Proof of Theorem* 2

For the proof of Theorem 2 we need a truncated Voronoï formula of the type given by Meurman [**13**]. This is contained in

LEMMA 5. *For* $x > 1, M \geqslant 2x^{\frac{1}{3}}$ *and* x *not an integer, we have*

$$\Delta_1(x;\varphi) = \delta_1(x;M) + O(x^{\frac{3}{2}+\varepsilon}\|x\|^{-1}M^{-\frac{1}{2}} + x^{\frac{13}{8}}M^{-\frac{3}{8}+\varepsilon} + x^{\frac{9}{8}}M^{-\frac{11}{40}} + x^{\frac{7}{8}}), \tag{5·1}$$

where $\delta_1(x;M)$ *is the function defined by* (3·8) *and* $\|x\|$ *is the distance of* x *to the nearest integer.*

Proof. Suppose that $x > 1$ is not an integer. We have

$$E_M(x) = O(x^{\frac{9}{8}}(|S_1| + |S_2|)) + O(x^{\frac{7}{8}}), \tag{5·2}$$

where $E_M(x)$ is defined by (3·9),

$$S_1 = \sum_{n>M}(c_n - \tfrac{1}{6}\pi^2\kappa R_0)n^{-\frac{7}{8}}\sin\left(8\pi(nx)^{\frac{1}{4}} + \pi/4\right)$$

and

$$\begin{aligned}S_2 &= \tfrac{1}{6}\pi^2\kappa R_0\sum_{n>M} n^{-\frac{7}{8}}\sin\left(8\pi(nx)^{\frac{1}{4}} + \pi/4\right)\\ &= \tfrac{1}{6}\pi^2\kappa R_0\,\mathrm{Im}\left(e^{\pi i/4}\sum_{n>M} n^{-\frac{7}{8}}\exp\left(8\pi i(nx)^{\frac{1}{4}}\right)\right).\end{aligned}$$

First we consider S_2. By lemmas 4·2 and 4·8 of [**20**] we have

$$\sum_{M<n\leqslant y} e^{8\pi i(nx)^{\frac14}} \ll x^{-\frac14}y^{\frac34}$$

for $M \geqslant 2x^{\frac13}$. Hence by using partial summation we find that the series S_2 is convergent, so that the decomposition (5·2) is justified, and

$$S_2 \ll x^{-\frac14}M^{-\frac18}. \tag{5·3}$$

Next we consider S_1. Using partial summation, the relation

$$\sum_{n\leqslant y}(c_n - \tfrac16\pi^2\kappa R_0) = \Delta_0(y;\varphi) + O(1),$$

and Rankin's bound (1·3), we have

$$\begin{aligned} S_1 = &\int_M^\infty \Big(\Delta_0(y;\varphi) - \Delta_0(M;\varphi) + O(1)\Big) \\ &\times\big\{\tfrac78 y^{-\frac{15}{8}}\sin(8\pi(yx)^{\frac14}+\pi/4) - 2\pi x^{\frac14}y^{-\frac{13}{8}}\cos(8\pi(yx)^{\frac14}+\pi/4)\big\}dy \\ = &\int_M^\infty \Delta_0(y;\varphi)\big\{\tfrac78 y^{-\frac{15}{8}}\sin(8\pi(yx)^{\frac14}+\pi/4) - 2\pi x^{\frac14}y^{-\frac{13}{8}}\cos(8\pi(yx)^{\frac14}+\pi/4)\big\}dy \\ &+O(M^{-\frac{11}{40}} + x^{\frac14}M^{-\frac58}). \end{aligned}$$

The first term on the right-hand side can be expressed in terms of $\Delta_1(x;\varphi)$ by using (1·6) and integration by parts. Then we apply (1·11) of the Corollary to obtain

$$\begin{aligned} S_1 &= -4\pi^2 x^{\frac12}\int_M^\infty \Delta_1(y;\varphi)y^{-\frac{19}{8}}\sin(8\pi(yx)^{\frac14}+\pi/4)dy + O(x^{\frac14}M^{-\frac{17}{40}} + M^{-\frac{11}{40}}) \\ &= -4\pi^2 x^{\frac12}\sum_{k=0}^\infty \mathrm{Im}\left(e^{\pi i/4}I(2^kM)\right) + O(x^{\frac14}M^{-\frac{17}{40}} + M^{-\frac{11}{40}}), \end{aligned} \tag{5·4}$$

where

$$I(Y) := \int_Y^{2Y} \Delta_1(y;\varphi)y^{-\frac{19}{8}}\exp(8\pi i(yx)^{\frac14})dy.$$

Using the case $\rho = 1$ of Lemma 2 (with $N = Y$) we obtain

$$I(Y) \ll \sum_{n\leqslant Y} c_n n^{-\frac78}\left(|I_n^+(Y)| + |I_n^-(Y)|\right) + O(Y^{-\frac38+\varepsilon})$$

with

$$I_n^\pm(Y) = \int_Y^{2Y} y^{-\frac54}\exp(8\pi i(x^{\frac14}\pm n^{\frac14})y^{\frac14})dy.$$

Since

$$|I_n^\pm(Y)| \ll Y^{-\frac12}|x^{\frac14}\pm n^{\frac14}|^{-1},$$

the contribution of $I_n^+(Y)$ to $I(Y)$ is $O(Y^{-\frac12})$ and that of $I_n^-(Y)$ is

$$\begin{aligned} &\ll Y^{-\frac12}x^{-\frac18} + Y^{-\frac12}\sum_{\frac12 x<n<2x} c_n n^{-\frac78}\frac{x^{\frac34}}{|x-n|} \\ &\ll Y^{-\frac12}x^{-\frac18} + Y^{-\frac12}x^{-\frac18+\varepsilon}\left(\|x\|^{-1} + \log x\right). \end{aligned}$$

Therefore

$$I(Y) \ll Y^{-\frac{1}{2}} x^{-\frac{1}{8}+\varepsilon} \|x\|^{-1} + Y^{-\frac{3}{8}+\varepsilon}.$$

Thus by (5·4) we obtain the estimate

$$\left.\begin{aligned} S_1 &\ll x^{\frac{1}{2}} \left(M^{-\frac{1}{2}} x^{-\frac{1}{8}+\varepsilon} \|x\|^{-1} + M^{-\frac{3}{8}+\varepsilon} \right) + x^{\frac{1}{4}} M^{-\frac{17}{40}} + M^{-\frac{11}{40}} \\ &\ll x^{\frac{3}{8}+\varepsilon} \|x\|^{-1} M^{-\frac{1}{2}} + x^{\frac{1}{2}} M^{-\frac{3}{8}+\varepsilon} + M^{-\frac{11}{40}}. \end{aligned}\right\} \tag{5·5}$$

The assertion (5·1) follows immediately from (5·2), (5·3), (5·5) and (3·7).

Proof of Theorem 2. Let $X > 1$ and $M \gg X^{2+\varepsilon}$. Then, by the Cauchy–Schwarz inequality,

$$\begin{aligned} \int_X^{2X} \Delta_1(x;\varphi)^2 dx = \int_X^{2X} \delta_1(x;M)^2 dx \\ &+ O\left(\left(\int_X^{2X} \delta_1(x;M)^2 dx \right)^{\frac{1}{2}} \left(\int_X^{2X} E_M(x)^2 dx \right)^{\frac{1}{2}} \right) \\ &+ O\left(\int_X^{2X} E_M(x)^2 dx \right). \end{aligned} \tag{5·6}$$

By an argument similar to the proof of theorem 13·5 of [**8**], we obtain

$$\begin{aligned} \int_X^{2X} \delta_1(x;M)^2 dx = \tfrac{2}{13}(2\pi)^{-4} \left(\sum_{n=1}^{\infty} c_n^2 n^{-\frac{7}{4}} \right) \left((2X)^{\frac{13}{4}} - X^{\frac{13}{4}} \right) \\ + O(X^{\frac{13}{4}} M^{-\frac{3}{4}+\varepsilon}) + O(X^3 M^{\varepsilon}). \end{aligned} \tag{5·7}$$

As for $E_M(x)$, we have by Lemma 2 and Lemma 5,

$$E_M(x) \ll \begin{cases} x^{\frac{7}{8}} & \text{if } x^{\frac{5}{8}+\varepsilon} M^{-\frac{1}{2}} \ll \|x\|, \\ x M^{\varepsilon} & \text{otherwise.} \end{cases} \tag{5·8}$$

Hence we have

$$\left.\begin{aligned} \int_X^{2X} E_M(x)^2 dx &\ll \int\limits_{\substack{X \leqslant x \leqslant 2X \\ x^{\frac{5}{8}+\varepsilon} M^{-\frac{1}{2}} \ll \|x\|}} x^{\frac{7}{4}} dx + \int\limits_{\substack{X \leqslant x \leqslant 2X \\ x^{\frac{5}{8}+\varepsilon} M^{-\frac{1}{2}} \gg \|x\|}} (xM^{\varepsilon})^2 dx \\ &\ll \int_X^{2X} x^{\frac{7}{4}} dx + \sum_{X-1 \leqslant n \leqslant 2X+1} M^{\varepsilon} \int\limits_{|n-x| \ll x^{\frac{5}{8}+\varepsilon} M^{-\frac{1}{2}}} x^2 dx \\ &\ll X^{\frac{11}{4}} + X(X^{\frac{5}{8}+\varepsilon} M^{-\frac{1}{2}}) X^2 M^{\varepsilon} \\ &\ll X^{\frac{11}{4}}. \end{aligned}\right\} \tag{5·9}$$

Combining (5·6), (5·7) and (5·9), we obtain

$$\int_X^{2X} \Delta_1(x;M)^2 dx = \tfrac{2}{13}(2\pi)^{-4} \left(\sum_{n=1}^{\infty} c_n^2 n^{-\frac{7}{4}} \right) \left((2X)^{\frac{13}{4}} - X^{\frac{13}{4}} \right) + O(X^{3+\varepsilon}). \tag{5·10}$$

Now Theorem 2 follows on replacing X in (5·10) by $2^{-j}X$ $(j = 1, 2, \ldots)$ and adding all the resulting expressions.

6. *Proof of Theorem* 3

In this section we prove Theorem 3, which is an example of the principle 'One can improve the estimate of some quantity if one can refine the mean square formula of that quantity.' A simple method which embodies this principle is described on pp. 60–61 of the first author's Lecture Notes [**10**] (see the Remark at the end of Section 2). However, this method is not so useful in the case of $\Delta_1(x;\varphi)$. From the assumption (1·14), this method can only deduce

$$\Delta_1(x;\varphi) = O(x^{\max(\frac{9}{8},\,(1+\theta)/3)}),$$

which is non-trivial only if $\theta < \frac{13}{5}$. Therefore, for larger values of θ, some alternative approach is required.

Let us assume that

$$F_1(X;\varphi) = X^3 P(\log X) + O(X^\theta)$$

where $P(y)$ is a polynomial of y of degree ℓ. From (3·6) with $\beta = \frac{3}{5}$ we have

$$\Delta_1(x;\varphi) = \frac{1}{H}\int_x^{x+H} \Delta_1(t;\varphi)dt + O(x^{\frac{3}{5}}H) \quad (0 < H \leqslant x).$$

Hence by the Cauchy–Schwarz inequality, we obtain ($\ell \geqslant 0$)

$$\begin{aligned}\Delta_1(x;\varphi)^2 &\ll H^{-1}\int_x^{x+H} \Delta_1(t;\varphi)^2 dt + x^{\frac{6}{5}}H^2\\ &\ll x^{\frac{9}{4}} + H^{-1}(F_1(x+H;\varphi) - F_1(x;\varphi)) + x^{\frac{6}{5}}H^2\\ &\ll x^{\frac{9}{4}} + x^2(\log x)^\ell + H^{-1}x^\theta + x^{\frac{6}{5}}H^2\\ &\ll x^{\frac{9}{4}} + H^{-1}x^\theta + x^{\frac{6}{5}}H^2.\end{aligned}$$

Taking $H = x^{\frac{1}{3}\theta - \frac{2}{5}}$, we obtain

$$\Delta_1(x;\varphi) \ll x^{\max(\frac{9}{8},\,\frac{1}{3}\theta + \frac{1}{5})},$$

which completes the proof of Theorem 3.

7. *A consequence of Vorhauer's theorem*

Finally we study the logarithmic Riesz means (1·16), which is a special case of the general Riesz means

$$F_f(x;\xi) := \sum_{n \leqslant x} f(n) \log^\xi\left(\frac{x}{n}\right) \qquad (\xi \geqslant 0),$$

for a given arithmetic function $f(n)$. As ξ increases, it often happens that the logarithmic weight smoothes out the irregularities of distribution of $f(n)$ and that with increasing ξ one can obtain progressively sharper estimates of the error term for $F_f(x;\xi)$. A general result for the above Riesz means, which holds for a wide class of Dirichlet series satisfying a functional equation of the type first considered by Richert [**17**], has been obtained recently by Vorhauer [**22**]. Applying Vorhauer's theorem to our case when $f(n) \equiv c_n$, we shall obtain

$$\tilde{D}_\xi(x;\varphi) = \tfrac{1}{6}\pi^2\kappa R_0\, x + V_\xi(x,N) + R_\xi(x,N) \qquad (7·1)$$

for any integer $N \geqslant 1$ and any fixed $\xi \geqslant 0$, with

$$V_\xi(x,N) = (2\pi)^{-1-\xi}x^{(3-2\xi)/8}\sum_{n\leqslant N} c_n n^{-(5+2\xi)/8}\cos(8\pi(xn)^{\frac14} + (\tfrac14 - \xi/2)\pi)$$

and

$$R_\xi(x,N) = O_\varepsilon((xN)^\varepsilon(x^{(3-\xi)/4}N^{-(1+\xi)/4} + (xN)^{(1-\xi)/4} + x^{(1-\xi)/4})).$$

Note that the formula (7·1) for $\xi = 0$ reduces to Lemma 2 with $\rho = 0$.

Suppose that $0 \leqslant \xi < \frac32$; then by trivial estimation we obtain $V_\xi(x,N) \ll (xN)^{(3-2\xi)/8}$. Hence choosing $N = x^{\frac35}$ it follows that

$$\tilde{D}_\xi(x;\varphi) = \tfrac16\pi^2\kappa R_0 x + O(x^{(3-2\xi)/5+\varepsilon}),$$

and the error term here is $O(x^{\frac38+\varepsilon})$ for $\frac{9}{16} \leqslant \xi < \frac32$. One can easily see that

$$\tilde{D}_\xi(x;\varphi) = \tfrac16\pi^2\kappa R_0\, x + O(x^{\frac38+\varepsilon}) \tag{7·2}$$

holds also for $\xi \geqslant \frac32$. Note that the error term in (7·2) is the same as the conjectured bound in (1·4) for the most interesting case $\xi = 0$.

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REFERENCES

[**1**] E. Carletti, G. Monti Bragadin and A. Perelli. On general L-functions. *Acta Arith.* **66** (1994), 147–179.

[**2**] K. Chandrasekharan and R. Narasimhan. The approximate functional equation for a class of zeta-functions. *Math. Ann.* **152** (1963), 30–64.

[**3**] K. Chandrasekharan and R. Narasimhan. On the mean value of the error term for a class of arithmetical functions. *Acta Math.* **112** (1964), 41–67.

[**4**] P. Deligne. La conjecture de Weil I. *Inst. Hautes Études Sci. Publ. Math.* **43** (1974), 273–307.

[**5**] E. P. Golubeva and O. M. Fomenko. Values of Dirichlet series associated with modular forms at the points $s = \frac12, 1$. *Zap. Nauchn. Sem. LOMI* **134** (1984), 117–137 (English translation *J. Soviet Math.* **36** (1987), 79–93).

[**6**] J. L. Hafner. On the representation of the summatory functions of a class of arithmetical functions. In *Analytic Number Theory* (ed. M. I. Knopp). Lecture Notes in Math., vol. 899, (Springer-Verlag, 1981), pp. 148–165.

[**7**] Y. Ichihara. The Siegel–Walfisz theorem for Rankin–Selberg L-functions associated with two cusp forms. Preprint.

[**8**] A. Ivić. *The Riemann zeta-function* (John Wiley & Sons, 1985).

[**9**] A. Ivić. Large values of certain number-theoretic error terms. *Acta Arith.* **56** (1990), 135–159.

[**10**] A. Ivić. *Mean values of the Riemann zeta function*. Lectures on Math. and Phys., vol. 82 (Tata Inst. Fund. Res., Springer, 1991).

[**11**] M. Jutila. *A Method in the theory of exponential sums*. Lectures on Math. and Phys., vol. 80 (Tata Inst. Fund. Res., Springer, 1987).

[**12**] E. Landau. Zur analytischen Zahlentheorie der definiten quadratischen Formen (Über die Gitterpunkte in einem mehrdimensionalen Ellipsoid). *Sitzungsberichte der Kgl. Preuß. Akad. Wiss. Berlin* **31** (1915), 458–476 (Collected Works, Vol. 6, Thales Verlag, pp. 200–218).

[**13**] T. Meurman. On the mean square of the Riemann zeta-function. *Quart. J. Math. Oxford* (2) **38** (1987), 337–343.

[**14**] A. Perelli. On the prime number theorem for the coefficients of certain modular forms. In *Elementary and analytic theory of numbers* (ed. H. Iwaniec). Banach Center Publications, vol. 17 (PWN-Polish Scientific Publishers, 1985), pp. 405–410.

[**15**] R. A. Rankin. Contributions to the theory of Ramanujan's function $\tau(n)$ and similar arithmetical functions II. The order of the Fourier coefficients of integral modular forms. *Proc. Cambridge Phil. Soc.* **35** (1939), 357–372.

[**16**] R. A. Rankin. An Ω-result for the coefficients of cusp forms. *Math. Ann.* **203** (1973), 239–250.

[**17**] H.-E. Richert. Beiträge zur Summierbarkeit Dirichletscher Reihen mit Anwendungen auf die Zahlentheorie. *Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. IIa*, (1956), 77–125.

[**18**] A. Selberg. Bemerkungen über eine Dirichletsche Reihe, die mit der Theorie der Modulformen nahe verbunden ist. *Arch. Math. Naturvid.* **43** (1940), 47–50.

[**19**] P. Shiu. A Brun–Titchmarsh theorem for multiplicative functions. *J. reine angew. Math.* **31** (1980), 161–170.

[**20**] E. C. Titchmarsh. *The theory of the Riemann zeta-function* (Oxford, 1951).

[**21**] K.-C. Tong. On divisor problems III. *Acta Math. Sinica* **6** (1956), 515–541. (Chinese, English summary.)

[**22**] U. Vorhauer. Three two-dimensional Weyl steps in the circle problem II. The Riesz mean. To appear in the Proceedings of the Zakopane Conference 1997.

[**23**] A. Walfisz. Über die Koeffizientensummen einiger Modulformen. *Math. Ann.* **108** (1933), 75–90.