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Functional equations for double zeta-functions

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Abstract

As the first step of research on functional equations for multiple zeta-functions, we present a candidate of the functional equation for a class of two variable double zeta-functions of the Hurwitz–Lerch type, which includes the classical Euler sum as a special case.

1. *Introduction*

Let $u_1, \ldots, u_r$ be complex variables. The r-variable Euler–Zagier sum is a kind of multiple zeta-function defined by the series

$$\zeta_r(u_1, \ldots, u_r)$$
$$\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\cdots\sum_{m_r=1}^{\infty} m_1^{-u_1}(m_1+m_2)^{-u_2}\cdots(m_1+\cdots+m_r)^{-u_r} \qquad (1{\cdot}1)$$

which is convergent absolutely when $\Re(u_{r-k+1}+\cdots+u_r) > k\,(1 \leqslant k \leqslant r)$. The analytic continuation of (1·1) as an r-variable meromorphic function has recently been established (Arakawa and Kaneko [**2**], Zhao [**11**], Akiyama, Egami and Tanigawa [**1**] and the author [**8**, **9**]). A problem that follows naturally is to search for the functional equation(s), this has not yet been successful.

The aim of this paper is to propose a candidate of the functional equation for the simplest case $r = 2$, that is the classical Euler sum. Hereafter we write it as

$$\zeta_2(u, v) = \sum_{m=1}^{\infty} m^{-u} \sum_{n=1}^{\infty} (m+n)^{-v}. \qquad (1{\cdot}2)$$

Let $\Gamma(u), \zeta(u)$ be the gamma function and the Riemann zeta-function, respectively. Define

$$g(u, v) = \zeta_2(u, v) - \frac{\Gamma(1-u)}{\Gamma(v)}\Gamma(u+v-1)\zeta(u+v-1).$$

We use the notation $\sigma_\ell(k) = \sum_{d|k} d^\ell$. Let

$$\Psi(a, c; x) = \frac{1}{\Gamma(a)}\int_0^{\infty e^{i\phi}} e^{-xy}y^{a-1}(1+y)^{c-a-1}dy \qquad (1{\cdot}3)$$

be the confluent hypergeometric function, where $\Re a > 0$, $-\pi < \phi < \pi$, $|\phi + \arg x| < \pi/2$. Then our functional equation can be formulated, in terms of $g(u, v)$, as follows:

THEOREM 1. *We have*

$$\frac{g(u,v)}{(2\pi)^{u+v-1}\Gamma(1-u)} = \frac{g(1-v,1-u)}{i^{u+v-1}\Gamma(v)} + 2i\sin\Big(\frac{\pi}{2}(u+v-1)\Big) F_{+}(u,v), \tag{1·4}$$

where $i = \sqrt{-1} = \exp(\pi i/2)$ *and* $F_{+}(u,v)$ *is the series defined by*

$$F_{+}(u,v) = \sum_{k=1}^{\infty} \sigma_{u+v-1}(k)\Psi(v, u+v; 2\pi ik). \tag{1·5}$$

The series (1·5) *is covergent only in the region* $\Re u < 0$, $\Re v > 1$, *but it can be continued meromorphically to the whole* $\mathbb{C}^2$ *space.*

Remark. The function $F_{+}(u,v)$ itself satisfies a nice functional equation. See Proposition 2 in Section 3.

In the following sections we will prove the functional equation of a more general double zeta-function, which will show the duality more clearly. The main result (Theorem 2) will be stated in the last section.

2. *Double Hurwitz–Lerch zeta-functions*

When $r = 1$, the series (1·1) is nothing but the Riemann zeta-function, whose functional equation is well known. A classical generalization of the Riemann zeta-function is the Hurwitz zeta-function $\zeta(s,\alpha) = \sum_{n=0}^{\infty}(n+\alpha)^{-s}$, where $\alpha > 0$. The functional equation of $\zeta(s,\alpha)$ is of the form

$$\zeta(s,\alpha) = \frac{\Gamma(1-s)}{i(2\pi)^{1-s}}\left(e^{\pi is/2}\phi(1-s,\alpha) - e^{-\pi is/2}\phi(1-s,-\alpha)\right), \tag{2·1}$$

where $\phi(s,\alpha) = \sum_{n=1}^{\infty}\exp(2\pi in\alpha)n^{-s}$ is the Lerch zeta-function. (See [**10**, 2·17·3].) More generally, define the Hurwitz–Lerch zeta-function by

$$\zeta(s,\alpha,\beta) = \sum_{n=0}^{\infty}\frac{\exp(2\pi in\beta)}{(n+\alpha)^{s}}. \tag{2·2}$$

When $0 < \beta < 1$, the functional equation of $\zeta(s,\alpha,\beta)$ is given by

$$\zeta(s,\alpha,\beta) = \frac{\Gamma(1-s)}{(2\pi)^{1-s}}\Big(e^{\pi i(1-s)/2}e^{-2\pi i\alpha\beta}\zeta(1-s,\beta,-\alpha) + e^{-\pi i(1-s)/2}e^{2\pi i\alpha(1-\beta)}\zeta(1-s,1-\beta,\alpha)\Big) \tag{2·3}$$

which expresses the transparent duality between, not only s and $1-s$, but also α and β [**4**, section 1·8], see also [**5**, theorem 2*].

In [**7**], the author introduced the following generalized double zeta-function:

$$\zeta_2(u,v;\alpha,w) = \sum_{m=0}^{\infty}(\alpha+m)^{-u}\sum_{n=1}^{\infty}(\alpha+m+nw)^{-v}, \tag{2·4}$$

where $\alpha > 0$ and $w > 0$. This double series was further studied in [**8**]. The reason for introducing the weight w is to include the Barnes double zeta-function as a special case $u = 0$. The above (2·4) is a kind of two-variable generalization of the Hurwitz zeta-function. In view of (2·3), however, it is better to introduce a more general double series with some exponential factor, when we consider the subject of functional equations. In the present paper we consider the following two-variable double series of the Hurwitz–Lerch type:

$$\zeta_2(u, v; \alpha, \beta, w) = \sum_{m=0}^{\infty} (\alpha + m)^{-u} \sum_{n=1}^{\infty} e^{2\pi i n\beta} (\alpha + m + nw)^{-v}, \tag{2·5}$$

where $0 < \alpha \leqslant 1, 0 \leqslant \beta \leqslant 1$, and $w > 0$. This series is convergent absolutely when $\Re u > 1, \Re v > 1$.

In this section we prove a certain infinite series expression of $\zeta_2(u, v; \alpha, \beta, w)$. The argument is similar to that developed in [**7**], hence we only give a brief sketch.

First, similarly to [**7**, 3·4], we can show

$$\zeta_2(u, v; \alpha, \beta, w) = \frac{1}{\Gamma(u)\Gamma(v)} \int_0^{\infty} \frac{y^{v-1}}{e^{-2\pi i\beta} e^{wy} - 1} \int_0^{\infty} \frac{e^{(1-\alpha)(x+y)} x^{u-1}}{e^{x+y} - 1} \, dx \, dy. \tag{2·6}$$

The right-hand side is convergent when $\Re u > 0, \Re v > 1$, and $\Re(u + v) > 2$. Let

$$h(z; \alpha) = \frac{e^{(1-\alpha)z}}{e^z - 1} - \frac{1}{z},$$

and divide the right-hand side of (2·6) as

$$\begin{aligned} &\frac{1}{\Gamma(u)\Gamma(v)} \int_0^{\infty} \frac{y^{v-1}}{e^{-2\pi i\beta} e^{wy} - 1} \int_0^{\infty} \frac{x^{u-1}}{x + y} \, dx \, dy \\ &\quad + \frac{1}{\Gamma(u)\Gamma(v)} \int_0^{\infty} \frac{y^{v-1}}{e^{-2\pi i\beta} e^{wy} - 1} \int_0^{\infty} h(x + y; \alpha) x^{u-1} \, dx \, dy \\ &= g_0(u, v; \alpha, \beta, w) + g(u, v; \alpha, \beta, w), \end{aligned} \tag{2·7}$$

say. We can show

$$g_0(u, v; \alpha, \beta, w) = \frac{\Gamma(1 - u)}{\Gamma(v)} \Gamma(u + v - 1) \phi(u + v - 1, \beta) w^{1-u-v}.$$

Let $\mathcal{C}$ be the contour which consists of the half-line on the positive real axis from infinity to a small positive number, a small circle counterclockwise round the origin, and the other half-line on the positive real axis back to infinity. Deforming the path to the contour $\mathcal{C}$, we have

$$\begin{aligned} g(u, v; \alpha, \beta, w) = &\frac{1}{\Gamma(u)\Gamma(v)(e^{2\pi i u} - 1)(e^{2\pi i v} - 1)} \\ &\times \int_{\mathcal{C}} \frac{y^{v-1}}{e^{-2\pi i\beta} e^{wy} - 1} \int_{\mathcal{C}} h(x + y; \alpha) x^{u-1} \, dx \, dy. \end{aligned} \tag{2·8}$$

Since the right-hand side of (2·8) is convergent absolutely for $\Re u < 1$ and any v, in the same region we obtain

$$\begin{aligned} \zeta_2(u, v; \alpha, \beta, w) = &\frac{\Gamma(1 - u)}{\Gamma(v)} \Gamma(u + v - 1) \phi(u + v - 1, \beta) w^{1-u-v} \\ &+ g(u, v; \alpha, \beta, w). \end{aligned} \tag{2·9}$$

This is a generalization of [**7**, 3·11].

Since $0 < \alpha \leqslant 1$, we can apply the method in [**7**, section 5]. Let $\Re u < 0, \Re v > 1$. Changing the path of the inner integral on the right-hand side of (2·8) by a circle of large radius R, counting the residues of relevant poles, and letting $R \to \infty$, we obtain

$$g(u, v; \alpha, \beta, w) = \frac{-2\pi i}{\Gamma(u)\Gamma(v)(e^{2\pi i u} - 1)(e^{2\pi i v} - 1)} \sum_{n \neq 0} e^{-2\pi i n \alpha} I_n, \tag{2·10}$$

where

$$I_n = \int_{\mathcal{C}} \frac{y^{v-1}}{e^{-2\pi i \beta} e^{wy} - 1} (-y + 2\pi i n)^{u-1} \, dy. \tag{2·11}$$

The integral I_n is similar to $I_h(\tau)$ appearing in [**6**, section 5]. Applying the method in [**6**, pp. 35–37], we obtain the expression of $g(u, v; \alpha, \beta, w)$ analogous to [**7**, 5·4] which can be written, in terms of confluent hypergeometric functions (1·3), as follows. In the statement, we use the notation

$$F_{\pm}(u, v; \alpha, \beta, w) = \sum_{k=1}^{\infty} \sigma_{u+v-1}(k; \alpha, \beta) \Psi(v, u + v; \pm 2\pi i k w), \tag{2·12}$$

where

$$\sigma_{\ell}(k; \alpha, \beta) = \sum_{d | k} e^{2\pi i d \alpha} e^{2\pi i (k/d) \beta} d^{\ell}. \tag{2·13}$$

PROPOSITION 1. *We have*

$$g(u, v; \alpha, \beta, w) = (2\pi)^{u+v-1} \Gamma(1 - u) \times \{e^{\pi i (u+v-1)/2} F_{+}(u, v; \alpha, \beta, w) + e^{\pi i (1-u-v)/2} F_{-}(u, v; -\alpha, \beta, w)\} \tag{2·14}$$

in the region $\Re u < 0, \Re v > 1$.

This is a generalization of [**7**, 5·5].

Remark. Since $\sigma_{\ell}(k; \alpha, \beta)$ and $F_{\pm}(u, v; \alpha, \beta, w)$ are periodic (of period 1) with respect to α and also with respect to β, we can extend the definition of $g(u, v; \alpha, \beta, w)$ for any real α and β by this periodicity.

3. *Properties of* $F_{\pm}(u, v; \alpha, \beta, w)$

In this section we discuss the basic properties of the functions $F_{\pm}(u, v; \alpha, \beta, w)$. First recall well-known properties of $\Psi(a, c; x)$. They are the transformation formula

$$\Psi(a, c; x) = x^{1-c} \Psi(a - c + 1, 2 - c; x), \tag{3·1}$$

[**3**, formula 6·5(6)], and the asymptotic expansion

$$\Psi(a, c; x) = \sum_{k=0}^{N-1} \frac{(-1)^k (a - c + 1)_k (a)_k}{k!} x^{-a-k} + \rho_N(a, c; x), \tag{3·2}$$

[**3**, formula 6·13·1(1)], where N is an arbitrary non-negative integer, $(a)_k = \Gamma(a + k)/\Gamma(a)$ and $\rho_N(a, c; x)$ is the remainder term which can be explicitly written as

$$\rho_N(a,c;x) = \frac{(-1)^N(a-c+1)_N}{\Gamma(a)} \int_0^{\infty e^{i\phi}} e^{-xy} y^{a+N-1} \times \int_0^1 \frac{(1-\tau)^{N-1}}{(N-1)!}(1+\tau y)^{c-a-N-1}\, d\tau\, dy. \tag{3·3}$$

First we assume $\Re u < 0$ and $\Re v > 1$. From (2·12) and (3·1) we have

$$F_\pm(u,v;\alpha,\beta,w) = \sum_{k=1}^\infty \sigma_{u+v-1}(k;\alpha,\beta)(\pm 2\pi i k w)^{1-u-v} \times \Psi(1-u,2-u-v;\pm 2\pi i k w). \tag{3·4}$$

Applying the fact

$$\sigma_{u+v-1}(k;\alpha,\beta)k^{1-u-v} = \sigma_{1-u-v}(k;\beta,\alpha) \tag{3·5}$$

to (3·4), we find

$$\begin{aligned} F_\pm(u,v;\alpha,\beta,w) &= (\pm 2\pi i w)^{1-u-v} \sum_{k=1}^\infty \sigma_{1-u-v}(k;\beta,\alpha) \\ &\quad \times \Psi(1-u,2-u-v;\pm 2\pi i k w) \\ &= (\pm 2\pi i w)^{1-u-v} F_\pm(1-v,1-u;\beta,\alpha,w), \end{aligned} \tag{3·6}$$

which is the functional equation for $F_\pm(u,v;\alpha,\beta,w)$.

Applying (3·2) to (3·4), we obtain

$$\begin{aligned} &F_\pm(u,v;\alpha,\beta,w) \\ &= \sum_{j=0}^{N-1} \frac{(-1)^j}{j!}(v)_j(1-u)_j(\pm 2\pi i w)^{-v-j} \sum_{k=1}^\infty \frac{\sigma_{u+v-1}(k;\alpha,\beta)}{k^{j+v}} \\ &\quad + (\pm 2\pi i w)^{1-u-v} \sum_{k=1}^\infty \sigma_{u+v-1}(k;\alpha,\beta)k^{1-u-v} \\ &\quad \times \rho_N(1-u,2-u-v;\pm 2\pi i k w) \\ &= \sum_{j=0}^{N-1} \frac{(-1)^j}{j!}(v)_j(1-u)_j(\pm 2\pi i w)^{-v-j} \\ &\quad \times \phi(j-u+1,\alpha)\phi(j+v,\beta) \\ &\quad + (\pm 2\pi i w)^{1-u-v} \sum_{k=1}^\infty \sigma_{u+v-1}(k;\alpha,\beta)k^{1-u-v} \\ &\quad \times \rho_N(1-u,2-u-v;\pm 2\pi i k w). \end{aligned} \tag{3·7}$$

Noting

$$|\sigma_{u+v-1}(k;\alpha,\beta)| \leqslant \sigma_{\Re(u+v)-1}(k),$$

similarly to [**7**, 6·3] we find that the second term on the right-hand side of (3·7) is convergent absolutely for $\Re u < N$ and $\Re v > -N+1$. Therefore, since N is arbitrary, (3·7) implies that $F_\pm(u,v;\alpha,\beta,w)$ can be continued meromorphically to the whole $\mathbb{C}^2$ space. Summarizing the above argument, we now obtain

PROPOSITION 2. *The function* $F_{\pm}(u,v;\alpha,\beta,w)$ *can be continued meromorphically to the whole* $(u,\, v)$ *space, and satisfies the functional equation*

$$F_{\pm}(1-v,1-u;\beta,\alpha,w) = (\pm 2\pi i w)^{u+v-1} F_{\pm}(u,v;\alpha,\beta,w). \tag{3·8}$$

4. *The main result*

Now it is easy to complete the proof of our main result, that is the functional equation of $\zeta_2(u,v;\alpha,\beta,w)$, which is written in terms of $g(u,v;\alpha,\beta,w)$ as follows.

THEOREM 2. *Let* $0<\alpha\leqslant 1$, $0\leqslant\beta\leqslant 1$, *and* $w>0$. *Then we have the functional equation*

$$\frac{g(u,v;\alpha,\beta,w)}{(2\pi)^{u+v-1}\Gamma(1-u)} = \frac{g(1-v,1-u;1-\beta,1-\alpha,w)}{(iw)^{u+v-1}\Gamma(v)} + e^{\pi i(u+v-1)/2}F_{+}(u,v;\alpha,\beta,w) - e^{\pi i(1-u-v)/2}F_{+}(u,v;1-\alpha,1-\beta,w). \tag{4·1}$$

When $\beta=1$, we define $g(1-v,1-u;0,1-\alpha,w)=g(1-v,1-u;1,1-\alpha,w)$ (see the remark at the end of Section 2). In particular, when $\alpha=\beta=1$, from (4.1) we obtain

$$\frac{g(u,v;1,1,w)}{(2\pi)^{u+v-1}\Gamma(1-u)} = \frac{g(1-v,1-u;1,1,w)}{(iw)^{u+v-1}\Gamma(v)} + 2i\sin\left(\frac{\pi}{2}(u+v-1)\right)F_{+}(u,v;1,1,w). \tag{4·2}$$

Theorem 1 is a special case of this formula.

Proof of Theorem 2. First of all we note that, in view of Proposition 2, we now know that (2·14) gives the meromorphic continuation of $g(u,v;\alpha,\beta,w)$ to the whole (u,v) space. Changing u,v,α,β by $1-v,1-u,1-\beta,1-\alpha$ in (2·14), we have

$$g(1-v,1-u;1-\beta,1-\alpha,w) = (2\pi)^{1-u-v}\Gamma(v)\big\{e^{\pi i(1-u-v)/2}F_{+} \times(1-v,1-u;1-\beta,1-\alpha,w) + e^{\pi i(u+v-1)/2}F_{-}(1-v,1-u;\beta-1,1-\alpha,w)\big\}.$$

Substituting (3·8) into the right-hand side of the above, we obtain

$$\begin{aligned} &g(1-v,1-u;1-\beta,1-\alpha,w)\\ &\quad=\Gamma(v)w^{u+v-1}\{F_{+}(u,v;1-\alpha,1-\beta,w)+F_{-}(u,v;1-\alpha,\beta-1,w)\}\\ &\quad=\Gamma(v)w^{u+v-1}\{F_{+}(u,v;1-\alpha,1-\beta,w)+F_{-}(u,v;-\alpha,\beta,w)\}. \end{aligned}\tag{4·3}$$

From (2·14) and (4.3), by eliminating the terms $F_{-}(u,v;-\alpha,\beta,w)$, we arrive at the desired assertion.

It is highly desirable to extend our result to multiple zeta-functions (1·1) for $r\geqslant 3$. We can show the integral expression

$$\zeta_r(u_1,\ldots,u_r) = \frac{1}{\Gamma(u_1)\cdots\Gamma(u_r)}\int_0^\infty\cdots\int_0^\infty x_1^{u_1-1}\cdots x_r^{u_r-1} \times\frac{1}{e^{x_1+\cdots+x_r}-1}\,\frac{1}{e^{x_2+\cdots+x_r}-1}\cdots\frac{1}{e^{x_r}-1}\,dx_1\cdots dx_r \tag{4·4}$$

(cf. [**2**, theorem 3(i)]). Similarly to (2·7), we can divide this into 2^{r-1} integrals. By using the beta integral formula, we can see that the simplest integral among them, corresponding to $g_0(u,v;\alpha,\beta,w)$ in (2·7), can be expressed as a product of Γ-factors and a zeta-function. However it does not seem easy to find transformation formulas for remaining integrals.

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