

LIFTINGS AND MEAN VALUE THEOREMS FOR AUTOMORPHIC L -FUNCTIONS

KOHI MATSUMOTO

1. Introduction

Let a_n ($n = 1, 2, 3, \dots$) be complex numbers, and consider the Dirichlet series

$$\varphi(s) = \sum_{n=1}^{\infty} a_n n^{-s}, \quad (1.1)$$

where $s = \sigma + it$ is a complex variable. If

$$a_n = O(n^\varepsilon), \quad (1.2)$$

where ε is (here and in what follows) an arbitrarily small positive number, then (1.1) is convergent absolutely for $\sigma > 1$. When $\varphi(s)$ can be continued meromorphically to the left of the line $\sigma = 1$, it is an interesting and important problem to evaluate the mean value

$$\int_1^T |\varphi(\sigma + it)|^2 dt$$

for $\sigma \leq 1$. If

$$\int_1^T |\varphi(\sigma_0 + it)|^2 dt = O(T^{1+\varepsilon}) \quad (1.3)$$

holds for some $\sigma_0 \leq 1$, then there is a simple argument that can be used to deduce, from (1.3), the asymptotic formula of the form

$$\int_1^T |\varphi(\sigma + it)|^2 dt = \left(\sum_{n=1}^{\infty} |a_n|^2 n^{-2\sigma} \right) T + O(T^{\beta(\sigma)+\varepsilon}) \quad (1.4)$$

for $\sigma_0 < \sigma < 1$, where $\frac{1}{2} < \beta(\sigma) < 1$. This argument can be obtained by a direct generalization of Lemma 8.4 of Ivić [16]; see [22, Theorem 1(iii)].

One unsatisfactory point of formula (1.4) is that the exponent $\beta(\sigma)$ obtained by the above argument is greater than $\frac{1}{2}$, and hence does not tend to 0 when $\sigma \rightarrow 1$. Since we can easily show that

$$\int_1^T |\varphi(\sigma + it)|^2 dt = \left(\sum_{n=1}^{\infty} |a_n|^2 n^{-2\sigma} \right) T + O(1) \quad (1.5)$$

for any $\sigma > 1$ (by using the usual mean value theorem for Dirichlet polynomials recalled in (5.6) below), it is desirable to refine (1.4) to obtain a formula with the exponent tending to 0 when $\sigma \rightarrow 1$.

The author [26] has published a method of obtaining such a sharper formula, when $\varphi(s)$ is a Rankin–Selberg L -function $L(s, f \otimes f)$ attached to a holomorphic normalized Hecke-eigen cusp form

$$f(z) = \sum_{n=1}^{\infty} a(n)e^{2\pi inz}$$

of weight κ with respect to the full modular group $SL(2, \mathbb{Z})$. The definition of $L(s, f \otimes f)$ is

$$L(s, f \otimes f) = \zeta(2s) \sum_{n=1}^{\infty} a(n)^2 n^{1-\kappa-s},$$

which is convergent absolutely for $\sigma > 1$, where $\zeta(s)$ denotes the Riemann zeta-function. If we rewrite the right-hand side of the above as $\sum c_n n^{-s}$, then we have

$$c_n = n^{1-\kappa} \sum_{m^2|n} m^{2(\kappa-1)} a(n/m^2)^2,$$

and one of the author’s results in [26] is

$$\int_1^T |L(\sigma + it, f \otimes f)|^2 dt = \left(\sum_{n=1}^{\infty} c_n^2 n^{-2\sigma} \right) T + O(T^{\theta(\sigma)+\varepsilon}) \tag{1.6}$$

for $31/40 \leq \sigma \leq 1$, with

$$\theta(\sigma) = \frac{60(1 - \sigma)}{29 - 20\sigma}. \tag{1.7}$$

Though in [26] only the case of Rankin–Selberg L -functions is discussed, the author’s method presented in [26] is of a general nature. Kanemitsu, Sankaranarayanan and Tanigawa [22] generalized the author’s argument to a fairly general class of L -functions (Theorems 3 and 4 of their paper).

Ivić [20] then refined the methods in [26] and [22] to obtain better estimates of the error terms. One of the advantages of Ivić’s argument is that he avoids using the Cauchy–Schwarz inequality on which [26] and [22] depend. As for Rankin–Selberg L -functions, he improved (1.7) to obtain the exponent

$$\theta(\sigma) = 4(1 - \sigma) \tag{1.8}$$

for $\frac{3}{4} < \sigma \leq 1$. Ivić’s argument is also of a general nature, and he discussed the case of higher moments of the Riemann zeta-function in [20]. In fact we can develop Ivić’s method in a general frame to obtain the asymptotic formula of the form

$$\int_1^T |\varphi(\sigma + it)|^2 dt = \left(\sum_{n=1}^{\infty} |a_n|^2 n^{-2\sigma} \right) T + O(T^{\theta(\sigma)+\varepsilon}), \tag{1.9}$$

where $\theta(\sigma)$ tends to 0 when $\sigma \rightarrow 1$. The rigorous statement will be given as Theorem 2 in the next section.

An important feature of the methods developed in [26], [22] and [20], as well as in the proof of our Theorem 2, is that they depend on the estimate of the error term $R_\varphi(x)$ in the asymptotic formula for the summatory function of the coefficients, defined by

$$\sum'_{n \leq x} a_n = xP_\varphi(\log x) + R_\varphi(x),$$

where $x > 1$, the symbol \sum' means that the last term in the sum is to be halved if x is

an integer, and $xP_\varphi(\log x)$ is the ‘main’ term coming from the residue(s) of $\varphi(s)x^s s^{-1}$. Actually in the present paper, following [26] and [20], we use the estimate of the mean square of $R_\varphi(x)$.

On the other hand, on and near the critical line $\sigma = \frac{1}{2}$, it is sometimes very difficult to obtain an asymptotic formula for the mean square of $\varphi(s)$. In such a case, a more accessible problem is to obtain an upper bound of the mean square that is as sharp as possible. One useful general method is to apply the reflection principle. This principle shows, in the case of $L(s, f \otimes f)$, that

$$\int_1^T |L(\frac{1}{2} + it, f \otimes f)|^2 dt = O(T^{2+\varepsilon}) \tag{1.10}$$

(cf. Ivić [17]). The author [26] improved this estimate slightly to get the bound $O(T^2(\log T)^{1+\varepsilon})$. Recently, Sankaranarayanan [33] obtained the following improvement:

$$\int_1^T |L(\frac{1}{2} + it, f \otimes f)|^2 dt = O(T^{11/6+\varepsilon}) \tag{1.11}$$

(or even slightly better than that) unconditionally, and, if the Lindelöf hypothesis for $\zeta(s)$ is true, then

$$\int_1^T |L(\frac{1}{2} + it, f \otimes f)|^2 dt = O(T^{3/2+\varepsilon}). \tag{1.12}$$

The idea of Sankaranarayanan [33] is to use the decomposition

$$L(s, f \otimes f) = \zeta(s)L(s + \kappa - 1, \text{sym}^2 f) \tag{1.13}$$

of Shimura [34], where

$$L(s, \text{sym}^2 f) = \zeta(2s - 2\kappa + 2) \sum_{n=1}^\infty a(n^2)n^{-s} \tag{1.14}$$

is the symmetric square L -function attached to f .

Generally speaking, the upper bound estimate obtained by the reflection principle can usually be improved if one knows a decomposition similar to (1.13). In §3, we will discuss such improvements for L -functions attached to the images of Doi–Naganuma lifts [5] and of Ikeda lifts [15]. In particular, in the case of L -functions attached to Ikeda lifts, we can get reasonably exact information on the order of mean square values (Theorem 5).

An interesting general feature is that, if the upper bound of the mean square of $\varphi(s)$ were improved, then the estimate of the mean square of $R_\varphi(x)$ would also be improved (Theorem 1 below). Then, via Theorem 2, we could further improve the estimate of the error term in (1.9). Such refinements will be discussed in §3 for the aforementioned cases related to Rankin–Selberg L -functions, and Doi–Naganuma and Ikeda lifts (Theorems 3, 4 and 6).

Acknowledgements. The author expresses his sincere gratitude to Professor H. Yoshida for useful information, to Professor D. Blasius for sending his preprint [2] to the author before publication, and to Professor A. Sankaranarayanan and the referee for useful comments.

2. Statement of general results

Let $\varphi(s)$ be the Dirichlet series defined by (1.1), satisfying (1.2). We assume that $\varphi(s)$ can be continued meromorphically to the half-plane $\sigma > \rho$, where $-\infty \leq \rho < 1$, and is holomorphic in this half-plane except for a pole of order $k(\geq 0)$ at $s = 1$.

Let $\gamma = \gamma(\varphi)$ be the infimum of σ for which

$$\int_{-\infty}^{\infty} |\varphi(\sigma + it)|^2 \frac{dt}{|\sigma + it|^2} < +\infty, \tag{2.1}$$

and assume $\rho \leq \gamma < 1$.

Let $c > 1$. By Perron's formula we have

$$\sum'_{n \leq x} a_n = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \varphi(s) \frac{x^s}{s} ds \tag{2.2}$$

for any $x > 0$. Let $\gamma < \sigma_1 < 1$, and shift the path of integration on the right-hand side of (2.2) to $\Re s = \sigma_1$. In §4 we will show that this shifting is possible. The residue of $\varphi(s)x^s s^{-1}$ at $s = 1$ is of the form $xP_\varphi(\log x)$, where P_φ is a polynomial of degree at most $k - 1$. (If $k = 0$, then $P_\varphi \equiv 0$.) Hence we have

$$\sum'_{n \leq x} a_n = xP_\varphi(\log x) + R_\varphi(x), \tag{2.3}$$

where

$$R_\varphi(x) = \frac{1}{2\pi i} \int_{\sigma_1-i\infty}^{\sigma_1+i\infty} \varphi(s) \frac{x^s}{s} ds. \tag{2.4}$$

Let $\beta = \beta(\varphi)$ be the infimum of real numbers b for which

$$\int_X^{2X} |R_\varphi(x)|^2 dx = O(X^{1+2b+\varepsilon}) \tag{2.5}$$

holds for any $X > 1$. Then the following holds.

THEOREM 1. *We have $\beta = \gamma$.*

This is a generalization of Theorem 12.5 of Titchmarsh's book [36] (which is also Lemma 13.1 of Ivić [16]). We will prove this theorem in §4.

Now we state our general mean value theorem.

THEOREM 2. *Let $T > 1$. We have*

$$\int_1^T |\varphi(\sigma + it)|^2 dt = A(\varphi)T + O(T^{2(1-\sigma)/(1-\beta_1)+\varepsilon}) \tag{2.6}$$

for any σ satisfying $\max\{\gamma, \frac{1}{2}\} < \sigma \leq 1$, where

$$A(\varphi) = \sum_{n=1}^{\infty} |a_n|^2 n^{-2\sigma}$$

and $\beta_1 = \beta_1(\varphi) = \max\{\beta, 0\} = \max\{\gamma, 0\}$.

This theorem gives an improvement of a result of Kanemitsu, Sankaranarayanan and Tanigawa [22]. We will prove this theorem in §5 by generalizing the argument

of Ivić [20]. We remark that if $\sigma \leq 1 - \frac{1}{2}(1 - \beta_1)$, then the exponent of the remainder term in (2.6) is larger than 1; hence (2.6) just implies that

$$\int_1^T |\varphi(\sigma + it)|^2 dt = O(T^{2(1-\sigma)/(1-\beta_1)+\varepsilon}). \tag{2.7}$$

Here we mention several simple consequences of Theorem 2. Graham [10] proved that

$$\int_1^T |\zeta(\frac{5}{7} + it)|^{196} dt = O(T^{14+\varepsilon}), \tag{2.8}$$

which implies (by the convexity of mean square estimates) $\beta(\zeta^{98}) = \gamma(\zeta^{98}) \leq 89/91$. Hence from Theorem 2 we have the following.

COROLLARY 1. *For $89/91 < \sigma \leq 1$ we have*

$$\int_1^T |\zeta(\sigma + it)|^{196} dt = A(\zeta^{98})T + O(T^{91(1-\sigma)+\varepsilon}).$$

Graham’s proof of (2.8) is a variant of Heath-Brown’s method [12] in which

$$\int_1^T |\zeta(\frac{1}{2} + it)|^{12} dt = O(T^{2+\varepsilon}) \tag{2.9}$$

has been shown. This estimate (2.9) with Theorem 2 gives

$$\int_1^T |\zeta(\sigma + it)|^{12} dt = A(\zeta^6)T + O(T^{4(1-\sigma)+\varepsilon}), \tag{2.10}$$

which has already been proved by Ivić in [20] as a part of his Corollary 1.

Let $f(z)$ be the same as in § 1, and

$$L(s, f) = \sum_{n=1}^{\infty} a(n)n^{-s} \tag{2.11}$$

be the L -function attached to $f(z)$. Ivić [18, 20] discussed the mean square and the fourth power moment of $\tilde{L}(s, f) = L(s + \frac{1}{2}(\kappa - 1), f)$. As for the sixth power moment, Jutila [21] proved, as an analogue of (2.9), that

$$\int_1^T |\tilde{L}(\frac{1}{2} + it, f)|^6 dt = O(T^{2+\varepsilon})$$

Hence our Theorem 2 implies the following corollary.

COROLLARY 2. *For $\frac{1}{2} < \sigma \leq 1$ we have*

$$\int_1^T |\tilde{L}(\sigma + it, f)|^6 dt = A(\tilde{L}^3)T + O(T^{4(1-\sigma)+\varepsilon}).$$

3. Examples in the theory of automorphic L -functions

In what follows, the symbol $A \ll B$ means $A = O(B)$, and $A \asymp B$ means $B \ll A \ll B$. The implied constants may depend on σ and ε .

We begin by quoting some general mean value estimates, which will be used repeatedly in this section. Besides (1.2), assume that:

- (i) $\varphi(s)$ can be continued meromorphically to the whole plane, and all possible poles are real;
- (ii) the estimate $\varphi(s) = O(e^{c|t|})$ holds uniformly in any fixed strip $\sigma_1 \leq \sigma \leq \sigma_2$ ($\sigma_2 \geq 1$), with a certain constant $c = c(\sigma_1, \sigma_2)$;
- (iii) $\varphi(s)$ satisfies the functional equation

$$\Delta(s)\varphi(s) = B_1 B_2^{-s} \Delta(1-s)\varphi(1-s), \tag{3.1}$$

where B_1 and B_2 are constants, $B_2 > 0$,

$$\Delta(s) = \prod_{j=1}^{\mu} \Gamma(\alpha_j + \beta_j s),$$

α_j is real and $\beta_j > 0$ ($1 \leq j \leq \mu$), with the condition

$$\eta = \sum_{j=1}^{\mu} \beta_j \geq 1.$$

The above is a special case of the situation considered by Kanemitsu, Sankaranarayanan and Tanigawa [22]. Under the assumptions (i), (ii) and (iii), their Theorem 1 implies that

$$\int_1^T |\varphi(\sigma + it)|^2 dt \ll T^{2(1-2\sigma)\eta+1+\varepsilon} \tag{3.2}$$

for $0 < \sigma < (2\eta)^{-1}$, and

$$\int_1^T |\varphi(\sigma + it)|^2 dt \ll T^{2(1-\sigma)\eta+\varepsilon} \tag{3.3}$$

for $(2\eta)^{-1} \leq \sigma \leq 1 - (2\eta)^{-1}$.

EXAMPLE 1. *Rankin–Selberg L-functions.* Our first example is the Rankin–Selberg L -function $L(s, f \otimes f)$ defined in §1. As in Sankaranarayanan’s paper [33], we first study the mean square of the symmetric square L -function $L(s, \text{sym}^2 f)$. Shimura [34] proved that $L(s, \text{sym}^2 f)$ can be continued meromorphically to the whole plane, holomorphic except for the possible poles at $s = \kappa$ and $s = \kappa - 1$, and satisfies the functional equation $R(2\kappa - 1 - s) = R(s)$, where

$$R(s) = \pi^{-3s/2} \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s+1}{2}\right) \Gamma\left(\frac{s-\kappa+2}{2}\right) L(s, \text{sym}^2 f).$$

Then $\tilde{R}(1-s) = \tilde{R}(s)$, where $\tilde{R}(s) = R(s + \kappa - 1)$. This can be rewritten as

$$\Delta(s, \text{sym}^2 f) \tilde{L}(s, \text{sym}^2 f) = \pi^{-(3/2)+3s} \Delta(1-s, \text{sym}^2 f) \tilde{L}(1-s, \text{sym}^2 f),$$

where

$$\Delta(s, \text{sym}^2 f) = \Gamma\left(\frac{s+\kappa-1}{2}\right) \Gamma\left(\frac{s+\kappa}{2}\right) \Gamma\left(\frac{s+1}{2}\right)$$

and

$$\tilde{L}(s, \text{sym}^2 f) = L(s + \kappa - 1, \text{sym}^2 f).$$

From the expression (0.1) of Shimura [34] and Deligne’s estimate on Fourier coefficients of cusp forms, we see that $\tilde{L}(s, \text{sym}^2 f)$ satisfies (1.2). Hence we can

apply (3.2) and (3.3) with $\eta = \frac{3}{2}$ to $\tilde{L}(s, \text{sym}^2 f)$ to obtain

$$\int_1^T |\tilde{L}(\sigma + it, \text{sym}^2 f)|^2 dt \ll \begin{cases} T^{3(1-2\sigma)+1+\varepsilon} & \text{for } 0 < \sigma < \frac{1}{3}, \\ T^{3(1-\sigma)+\varepsilon} & \text{for } \frac{1}{3} \leq \sigma \leq \frac{2}{3}. \end{cases} \tag{3.4}$$

Let θ_1 be any non-negative number for which

$$\zeta\left(\frac{1}{2} + it\right) \ll (|t| + 2)^{\theta_1 + \varepsilon} \tag{3.5}$$

holds for any real t . The convexity estimate is $\theta_1 = \frac{1}{4}$, and the classical estimate of Hardy and Littlewood is $\theta_1 = \frac{1}{6}$. The best known result at present is $\theta_1 = 32/205$ due to Huxley [14]. The Lindelöf hypothesis would yield $\theta_1 = 0$.

From (3.5) and the fact $\zeta(it) \ll (|t| + 2)^{1/2+\varepsilon}$ (Ivić [16, Theorem 1.9]), we have

$$\zeta(\sigma + it) \ll \begin{cases} (|t| + 2)^{1/2 - (1-2\theta_1)\sigma + \varepsilon} & \text{for } 0 \leq \sigma \leq \frac{1}{2}, \\ (|t| + 2)^{2\theta_1(1-\sigma) + \varepsilon} & \text{for } \frac{1}{2} < \sigma \leq 1, \end{cases} \tag{3.6}$$

by the convexity argument. Using (1.13), (3.4) and (3.6), we have

$$\begin{aligned} \int_1^T |L(\sigma + it, f \otimes f)|^2 dt &\leq \left(\max_{1 \leq t \leq T} |\zeta(\sigma + it)|^2 \right) \int_1^T |\tilde{L}(\sigma + it, \text{sym}^2 f)|^2 dt \\ &\ll \begin{cases} T^{5-4(2-\theta_1)\sigma + \varepsilon} & \text{for } 0 < \sigma < \frac{1}{3}, \\ T^{4-(5-4\theta_1)\sigma + \varepsilon} & \text{for } \frac{1}{3} \leq \sigma \leq \frac{1}{2}. \end{cases} \end{aligned} \tag{3.7}$$

Note that when $\sigma = \frac{1}{2}$, (3.7) with $\theta_1 = \frac{1}{6}$ implies (1.11) and with $\theta_1 = 0$ it implies (1.12).

From (3.7) we see that

$$\gamma(L(\cdot, f \otimes f)) \leq \frac{2}{5 - 4\theta_1} \left(\leq \frac{1}{2} \right) \tag{3.8}$$

for $0 \leq \theta \leq \frac{1}{4}$, where the symbol $\gamma(\cdot)$ is defined at the beginning of §2. Therefore Theorem 2 implies the following.

THEOREM 3. *We have*

$$\int_1^T |L(\sigma + it, f \otimes f)|^2 dt = A(L(\cdot, f \otimes f))T + O(T^{c_1(\theta_1)(1-\sigma)+\varepsilon}) \tag{3.9}$$

for $\frac{1}{2} < \sigma \leq 1$, where $c_1(\theta_1) = 2(5 - 4\theta_1)/(3 - 4\theta_1)$.

The estimate (1.8) of Ivić [20] is just the case $\theta_1 = \frac{1}{4}$ in the above. The cases $c_1(\frac{1}{6}) = 26/7 = 3.714\dots$ and $c_1(32/205) = 1794/487 = 3.683\dots$ clearly improve (1.8). The Lindelöf hypothesis would yield $c_1(0) = 10/3 = 3.333\dots$

REMARK. As for the symmetric square L -function, from (3.4) we see that $\gamma(\tilde{L}(\cdot, \text{sym}^2 f)) \leq \frac{1}{3}$. Hence from Theorem 2 we have

$$\int_1^T |\tilde{L}(\sigma + it, \text{sym}^2 f)|^2 dt = A(\tilde{L}(\cdot, \text{sym}^2 f))T + O(T^{3(1-\sigma)+\varepsilon}) \tag{3.10}$$

for $\frac{1}{2} < \sigma \leq 1$.

EXAMPLE 2. *Doi-Naganuma lifts.* Let K be a totally real number field of degree $m (\geq 2)$, \mathfrak{o}_K its ring of integers, \mathfrak{d}_K its different, d_K its discriminant, and

$\Gamma_K = \text{SL}(2, \mathfrak{o}_K)$ the Hilbert modular group. We assume that the narrow class number of K is 1. Let κ be a positive even integer. The L -function attached to a Hilbert cusp form h of weight κ with respect to Γ_K is defined by

$$L(s, h) = \sum_{\mathfrak{a}} c(\mathfrak{a})N(\mathfrak{a})^{-s}, \tag{3.11}$$

where \mathfrak{a} runs over all integral ideals of \mathfrak{o}_K , $N(\mathfrak{a})$ is the norm of \mathfrak{a} , and $c(\mathfrak{a}) = c(\alpha\delta^{-1})$, the $\alpha\delta^{-1}$ th Fourier coefficient of h , where α and δ are totally positive generators of \mathfrak{a} and \mathfrak{d}_K , respectively. Then (3.11) can be rewritten as in the form (1.1), with

$$a_n = \sum_{N(\mathfrak{a})=n} c(\mathfrak{a}).$$

The Ramanujan conjecture for h was recently settled by Blasius [2]; hence we have $a_n = O(n^{(\kappa-1)/2+\varepsilon})$. Therefore $\tilde{L}(s, h) = L(s + \frac{1}{2}(\kappa - 1), h)$ is convergent absolutely for $\sigma > 1$, and satisfies (1.2). Moreover the functional equation

$$\tilde{\Lambda}(1 - s, h) = (-1)^{m\kappa/2} \tilde{\Lambda}(s, h)$$

holds (Herrmann [13]; see also Bump [4, §1.7]), where

$$\tilde{\Lambda}(s, h) = (2\pi)^{-m(s+(\kappa-1)/2)} d_K^{s+(\kappa-1)/2} \Gamma\left(s + \frac{\kappa - 1}{2}\right)^m \tilde{L}(s, h).$$

Hence we can apply (3.3) with $\eta = m$ to obtain

$$\int_1^T |\tilde{L}(\sigma + it, h)|^2 dt \ll T^{2m(1-\sigma)+\varepsilon} \tag{3.12}$$

for $(2m)^{-1} \leq \sigma \leq 1 - (2m)^{-1}$. Hence $\gamma(\tilde{L}(\cdot, h)) \leq 1 - m^{-1}$, and therefore, from Theorem 2, we obtain

$$\int_1^T |\tilde{L}(\sigma + it, h)|^2 dt = A(\tilde{L}(\cdot, h))T + O(T^{2m(1-\sigma)+\varepsilon}) \tag{3.13}$$

for $1 - m^{-1} < \sigma \leq 1$.

These results can be improved if h is included in the image of the Doi–Naganuma lift. Here we restrict ourselves to the case that K is a real quadratic field. Let f be a holomorphic normalized Hecke-eigen cusp form of weight κ with respect to $\text{SL}(2, \mathbb{Z})$. Then there exists a Hecke-eigen Hilbert cusp form h_0 (the Doi–Naganuma lift of f) with respect to Γ_K such that

$$L(s, h_0) = L(s, f)L(s, f, \chi_K), \tag{3.14}$$

where $\chi_K = (\frac{d_K}{\cdot})$, $L(s, f)$ is the L -function (2.11) attached to f , and $L(s, f, \chi_K)$ its twist by χ_K . Hence

$$\int_1^T |\tilde{L}(\sigma + it, h_0)|^2 dt \leq \max_{1 \leq t \leq T} |\tilde{L}(\sigma + it, f)|^2 \int_1^T |\tilde{L}(\sigma + it, f, \chi_K)|^2 dt, \tag{3.15}$$

where $\tilde{L}(s, f) = L(s + \frac{1}{2}(\kappa - 1), f)$ and $\tilde{L}(s, f, \chi_K) = L(s + \frac{1}{2}(\kappa - 1), f, \chi_K)$. The functional equation of $\tilde{L}(s, f, \chi_K)$ is of the form

$$\Gamma\left(s + \frac{\kappa - 1}{2}\right) \tilde{L}(s, f, \chi_K) = B_1 B_2^{-s} \Gamma\left(\frac{\kappa + 1}{2} - s\right) \tilde{L}(1 - s, f, \chi_K),$$

where B_1 and B_2 are constants depending on K and κ . Hence, applying (3.2) and (3.3) with $\eta = 1$, we have

$$\int_1^T |\tilde{L}(\sigma + it, f, \chi_K)|^2 dt \ll T^{3-4\sigma+\varepsilon} \tag{3.16}$$

for $0 < \sigma \leq \frac{1}{2}$.

Let θ_2 be any non-negative number for which

$$\tilde{L}(\frac{1}{2} + it, f) \ll (|t| + 2)^{\theta_2+\varepsilon} \tag{3.17}$$

holds. Then

$$\tilde{L}(\sigma + it, f) \ll (|t| + 2)^{1-2(1-\theta_2)\sigma+\varepsilon} \quad (0 \leq \sigma \leq \frac{1}{2}). \tag{3.18}$$

Hence from (3.15) and (3.16) we obtain the first part of the following theorem.

THEOREM 4. *Let h_0 be a Doi–Naganuma lift as above. Then*

$$\int_1^T |\tilde{L}(\sigma + it, h_0)|^2 dt = O(T^{5-(8-4\theta_2)\sigma+\varepsilon}) \tag{3.19}$$

for $0 < \sigma \leq \frac{1}{2}$, and

$$\int_1^T |\tilde{L}(\sigma + it, h_0)|^2 dt = A(\tilde{L}(\cdot, h_0))T + O(T^{c_2(\theta_2)(1-\sigma)+\varepsilon}) \tag{3.20}$$

for $\frac{1}{2} < \sigma \leq 1$, where $c_2(\theta_2) = 2(8 - 4\theta_2)/(5 - 4\theta_2)$.

The second part of the theorem easily follows from the first part by applying Theorem 2, because from (3.19) we see that

$$\gamma(\tilde{L}(\cdot, h_0)) \leq \frac{3}{8 - 4\theta_2} \quad (\leq \frac{1}{2}).$$

Good [9] proved that we can take $\theta_2 = \frac{1}{3}$. (An alternative proof was given by Jutila [21].) Using this value, (3.19) implies that

$$\int_1^T |\tilde{L}(\sigma + it, h_0)|^2 dt = O(T^{5-(20/3)\sigma+\varepsilon}), \tag{3.21}$$

which improves (3.12) for $m = 2$ in the region $\frac{3}{8} < \sigma \leq \frac{1}{2}$. Also, (3.20) with $c_2(\frac{1}{3}) = 40/11$ improves (3.13) for $m = 2$.

It is also to be noted that from (3.19) and the convexity argument we have

$$\int_1^T |\tilde{L}(\sigma + it, h_0)|^2 dt = O(T^{4\theta_2(1-\sigma)+1+\varepsilon})$$

for $\frac{1}{2} < \sigma \leq 1$, which is better than (3.20) for some range of σ .

We can extend the above argument to a more general situation (such as the case treated by Saito [32]) without difficulty.

EXAMPLE 3. *Ikeda lifts.* Let F be a holomorphic Hecke-eigen Siegel cusp form of weight κ with respect to the Siegel modular group $\text{Sp}(g, \mathbb{Z})$ of degree g , which is a subgroup of $\text{GL}(2g, \mathbb{Z})$. The standard L -function attached to F is defined by

$$L(s, F, \text{st}) = \prod_p \left\{ (1 - p^{-s}) \prod_{j=1}^g (1 - \alpha_j(p)p^{-s})(1 - \alpha_j(p)^{-1}p^{-s}) \right\}^{-1}, \tag{3.22}$$

where p runs over all primes and the $\alpha_j(p)$ are Satake parameters. This is absolutely convergent for $\sigma > g + 1$. It is known that $L(s, F, \text{st})$ can be continued meromorphically to the whole plane, and satisfies the functional equation

$$\Lambda(s, F, \text{st}) = \Lambda(1 - s, F, \text{st}), \tag{3.23}$$

where

$$\Lambda(s, F, \text{st}) = \pi^{-(s+\omega)/2} \Gamma\left(\frac{s+\omega}{2}\right) \prod_{j=1}^g \left(\frac{2}{(2\pi)^{s+\kappa-j}} \Gamma(s+\kappa-j)\right) L(s, F, \text{st})$$

and $\omega = 0$ or 1 according as g is even or odd (Andrianov and Kalinin [1], Böcherer [3]). From (3.22) we have $L(s, F, \text{st}) = O(1)$ for $\sigma > g + 1$, while if $\sigma < -g$, then by (3.23) and Stirling’s formula we find that

$$L(s, F, \text{st}) \ll (|t| + 2)^{(g+1/2)(1-2\sigma)}.$$

Hence by the convexity argument we have

$$\int_1^T |L(\sigma + it, F, \text{st})|^2 dt \ll T^{(2g+1)(g+1-\sigma)+1+\varepsilon} \tag{3.24}$$

for $-g \leq \sigma \leq g + 1$.

(The referee kindly pointed out that, by using a result of Duke, Howe and Li [6], we can improve (3.24). In [6] it is shown that (3.22) is convergent absolutely for $\sigma > \frac{2}{3}g + 1$ in general, and $\sigma > \frac{1}{2}g + 1$ if g is a power of 2. Hence the factor $g + 1 - \sigma$ in the exponent on the right-hand side of (3.24) can be replaced by $\frac{2}{3}g + 1 - \sigma$, or by $\frac{1}{2}g + 1 - \sigma$ if g is a power of 2. For further discussion, see [25].)

We can improve this estimate drastically when F is an Ikeda lift. Let f be a holomorphic normalized Hecke-eigen cusp form of weight 2κ with respect to $\text{SL}(2, \mathbb{Z})$, and ν be a positive integer with $\nu \equiv \kappa \pmod{2}$. Then Ikeda [15] proved the existence of a Hecke-eigen Siegel cusp form F_0 of weight $\kappa + \nu$ with respect to $\text{Sp}(2\nu, \mathbb{Z})$, which is called the Ikeda lift of f , such that

$$L(s, F_0, \text{st}) = \zeta(s) \prod_{j=1}^{2\nu} L(s + \kappa + \nu - j, f). \tag{3.25}$$

When $\nu = 1$, the Ikeda lifting map is just the Saito–Kurokawa lifting map. Hence its image is the Maass space (see Eichler and Zagier [7]). As for the case $\nu \geq 2$, there is a conjecture of Kohnen [23] on the image of the Ikeda lifting map which has been partially solved by Kohnen and Kojima [24].

The critical strip for $L(s, f)$ is $\kappa - \frac{1}{2} \leq \sigma \leq \kappa + \frac{1}{2}$, and its functional equation is

$$(2\pi)^{-s} \Gamma(s) L(s, f) = (-1)^\kappa (2\pi)^{s-2\kappa} \Gamma(2\kappa - s) L(2\kappa - s, f). \tag{3.26}$$

Hence the critical strip for $L(s + \kappa + \nu - j, f)$ is

$$-\nu + j - \frac{1}{2} \leq \sigma \leq -\nu + j + \frac{1}{2}$$

for $1 \leq j \leq 2\nu$, which implies from (3.25) that the critical strip for $L(s, F_0, \text{st})$ is $-\nu + \frac{1}{2} \leq \sigma \leq \nu + \frac{1}{2}$. We study the mean square of $L(s, F_0, \text{st})$ in this strip. Let

$$I(\sigma; T) = \int_1^T |L(\sigma + it, F_0, \text{st})|^2 dt.$$

In the case $\frac{1}{2} \leq \sigma \leq \nu + \frac{1}{2}$, our results are as follows.

THEOREM 5. *Let $T \geq 2$, and ℓ be a positive integer.*

(i) *For $\nu + 1 \leq \ell \leq 2\nu$, we have*

$$I(\sigma; T) \asymp T^{2(2\nu-\ell)(\ell+1-2\sigma)+1} \quad (-\nu + \ell < \sigma < -\nu + \ell + \frac{1}{2}), \tag{3.27}$$

$$T^{2(2\nu-\ell)^2+1}(\log T)^{-4} \ll I(-\nu + \ell + \frac{1}{2}; T) \ll T^{2(2\nu-\ell)^2+1}(\log T)^4. \tag{3.28}$$

(ii) *For $\nu + 2 \leq \ell \leq 2\nu$, we have*

$$I(\sigma; T) \asymp T^{2(2\nu-\ell)(\ell+1-2\sigma)+4(\ell-\nu-\sigma)+1} \quad (-\nu + \ell - \frac{1}{2} < \sigma < -\nu + \ell), \tag{3.29}$$

$$I(-\nu + \ell; T) \asymp T^{2(2\nu-\ell)(2\nu+1-\ell)+1} \log T. \tag{3.30}$$

(iii) *We have*

$$\begin{aligned} T^{2\nu(\nu-1)+1}(\log T)^{-1/3}(\log \log T)^{-2/3} &\ll I(1; T) \\ &\ll T^{2\nu(\nu-1)+1}(\log T)^{7/3}. \end{aligned} \tag{3.31}$$

(iv) *We have*

$$I(\sigma; T) \ll T^{2\nu(\nu+1-2\sigma)+\min\{4\theta_1(1-\sigma), 2\theta_2(2\sigma-1)\}+1+\varepsilon} \quad (\frac{1}{2} < \sigma < 1), \tag{3.32}$$

where θ_1 and θ_2 are defined by (3.5) and (3.17), respectively.

(v) *We have*

$$T^{2\nu^2+1}(\log T)^{-7} \ll I(\frac{1}{2}; T) \ll T^{2\nu^2+1}(\log T)^9. \tag{3.33}$$

REMARK 1. The case $-\nu + \frac{1}{2} \leq \sigma < \frac{1}{2}$ can be treated similarly; or, the results in that case can be immediately deduced from the above theorem and the functional equation (3.23). Consequently, we can determine the order of magnitude of $I(\sigma; T)$ (sometimes up to log-factors) except for the cases $0 < \sigma < \frac{1}{2}$ and $\frac{1}{2} < \sigma < 1$.

REMARK 2. The result (3.33) implies that

$$L(\frac{1}{2} + it, F_0, \text{st}) = \Omega(t^{\nu^2}(\log t)^{-7/2})$$

for $t \geq 2$. This shows that the analogue of the Lindelöf hypothesis for $L(s, F_0, \text{st})$ is false on the line $\sigma = \frac{1}{2}$, which is the ‘critical line’ (the axis of symmetry of the functional equation) for $L(s, F_0, \text{st})$. This is a natural consequence of the decomposition (3.25), because the L -functions on the right-hand side of (3.25) have critical lines different from each other.

REMARK 3. In the case $\frac{1}{2} < \sigma < 1$, we cannot prove any lower bound. However, since

$$\zeta(\sigma + it) = \Omega(\exp(C(\sigma)(\log t)^{1-\sigma}(\log \log t)^{-\sigma})),$$

where $C(\sigma)$ is a constant depending on σ (Montgomery [27]), by using (3.52) below we obtain

$$I(\sigma; T) = \Omega(T^{2\nu(\nu+1-2\sigma)+1} \exp(2C(\sigma)(\log T)^{1-\sigma}(\log \log T)^{-\sigma}))$$

for $\frac{1}{2} < \sigma < 1$. Note that we can take a $C(\sigma)$ that tends to infinity when $\sigma \rightarrow 1$ (Ramachandra and Sankaranarayanan [31]).

Proof of Theorem 5. First of all, we note that it is enough to prove the statements for

$$I(\sigma; T, 2T) = \int_T^{2T} |L(\sigma + it, F_0, \text{st})|^2 dt$$

instead of $I(\sigma; T)$, because, then, summing up the results for $T, T/2, T/2^2, \dots$, we obtain the theorem.

Since f is Hecke-eigen, the Euler product expansion

$$L(s, f) = \prod_p (1 - a(p)p^{-s} + p^{2\kappa-1-2s})^{-1}$$

holds for $\sigma > \kappa + \frac{1}{2}$. Hence, using Deligne's estimate, we have

$$\begin{aligned} |\log |L(s, f)|| &= |\Re \log L(s, f)| = \left| \Re \left(\sum_p a(p)p^{-s} + O(1) \right) \right| \\ &\leq 2 \sum_p p^{\kappa-\sigma-1/2} + C_0 \leq 2 \log \zeta(\sigma - \kappa + \frac{1}{2}) + C'_0 \end{aligned} \tag{3.34}$$

for $\sigma > \kappa + \frac{1}{2}$, where C_0 and C'_0 are constants independent of σ . This implies that

$$|L(s, f)| \asymp 1 \quad (\sigma > \kappa + \frac{1}{2}), \tag{3.35}$$

and similarly

$$|\zeta(s)| \asymp 1 \quad (\sigma > 1). \tag{3.36}$$

Let $\nu + 1 \leq \ell \leq 2\nu$, and consider the strip

$$\mathcal{I}(\ell) = \{ \sigma \mid -\nu + \ell - \frac{1}{2} < \sigma < -\nu + \ell + \frac{1}{2} \}.$$

Then from (3.35) we find that

$$|L(s + \kappa + \nu - j, f)| \asymp 1 \quad (1 \leq j \leq \ell - 1) \tag{3.37}$$

for $\sigma \in \mathcal{I}(\ell)$. On the other hand, by the functional equation (3.26) we have

$$|L(s + \kappa + \nu - j, f)| \asymp (|t| + 2)^{2(j-\nu-\sigma)} |L(\kappa - \nu + j - s, f)| \tag{3.38}$$

(for any s), and hence

$$|L(s + \kappa + \nu - j, f)| \asymp (|t| + 2)^{2(j-\nu-\sigma)} \quad (\ell + 1 \leq j \leq 2\nu) \tag{3.39}$$

for $\sigma \in \mathcal{I}(\ell)$. Hence, if $\sigma \in \mathcal{I}(\ell)$ and $\sigma > 1$, with (3.36) we obtain

$$\begin{aligned} I(\sigma; T, 2T) &\asymp \left(\prod_{j=\ell+1}^{2\nu} T^{4(j-\nu-\sigma)} \right) \int_T^{2T} |L(\sigma + it + \kappa + \nu - \ell, f)|^2 dt \\ &= T^{2(2\nu-\ell)(\ell+1-2\sigma)} \int_T^{2T} |L(\sigma + it + \kappa + \nu - \ell, f)|^2 dt. \end{aligned} \tag{3.40}$$

The integral on the right-hand side can be evaluated as

$$\asymp \begin{cases} T & \text{if } \sigma > -\nu + \ell, \\ T \log T & \text{if } \sigma = -\nu + \ell, \\ T^{4(\ell-\nu-\sigma)+1} & \text{if } -\nu + \ell - \frac{1}{2} < \sigma < -\nu + \ell. \end{cases} \tag{3.41}$$

The first result is due to Potter [30], the second is due to Good [8], and the third follows from the first result and (3.38). Therefore we obtain (3.27), (3.29) and (3.30) (for $I(\sigma; T, 2T)$). Note that the condition $\ell \geq \nu + 2$ is necessary for (3.29) and (3.30) to satisfy $\sigma > 1$.

Next we prove the inequalities

$$(\log(|t| + 2))^{-2} \ll |L(\kappa + \frac{1}{2} + it, f)| \ll (\log(|t| + 2))^2. \tag{3.42}$$

The following proof of (3.42) is an analogue of the proof of Titchmarsh [36, Theorem 3.11].

Let

$$\tilde{L}(s, f) = L(s + \kappa - \frac{1}{2}, f).$$

It is known that $\tilde{L}(s, f) \neq 0$ for $\sigma \geq 1 - C_1(\log(|t| + 2))^{-1}$, where C_1 is a certain positive constant (Moreno [28, 29]; see also Grupp [11]). Put $\phi(t) = C_2 \log(|t| + 2)$, where C_2 is chosen so large as to satisfy

$$\tilde{L}(s, f) = O(e^{\phi(t)}) \tag{3.43}$$

for $\frac{3}{4} < \sigma < 2$. Let $t_0 \geq 2$, and $s_0 = 1 + C_3\phi(t_0 + 1)^{-1} + it_0$, where C_3 is a small positive constant. From (3.34) we have

$$|\tilde{L}(s_0, f)| \gg \exp(-2 \log \zeta(1 + C_3\phi(t_0 + 1)^{-1}) + O(1)) \gg \phi(t_0 + 1)^{-2}.$$

Hence, with (3.43), we have

$$\left| \frac{\tilde{L}(s, f)}{\tilde{L}(s_0, f)} \right| \ll \phi(t_0 + 1)^2 e^{\phi(t)} \ll e^{C_4\phi(t_0+1)} \tag{3.44}$$

for $|s - s_0| \leq \frac{1}{4}$. Next, differentiating both sides of

$$\log \tilde{L}(s, f) = - \sum_p \log(1 - a(p)p^{-s-\kappa+1/2} + p^{-2s})$$

we can see that

$$\begin{aligned} \left| \frac{\tilde{L}'(s_0, f)}{\tilde{L}(s_0, f)} \right| &\ll \sum_p |a(p)| p^{-\Re s_0 - \kappa + 1/2} \log p + 1 \\ &\ll \sum_p p^{-1 - C_3\phi(t_0+1)^{-1}} \log p + 1 \\ &\ll \int_2^\infty \left(\sum_{p \leq y} \log p \right) y^{-2 - C_3\phi(t_0+1)^{-1}} dy + 1 \\ &\ll \int_2^\infty y^{-1 - C_3\phi(t_0+1)^{-1}} dy + 1 \ll \phi(t_0 + 1) \end{aligned} \tag{3.45}$$

by partial summation.

Lemma γ in §3.9 of Titchmarsh [36] asserts that if the function $f(s)$ satisfies the conditions

- (i) $f(s)$ is holomorphic and $|f(s)/f(s_0)| < e^M$ ($M > 1$) in $|s - s_0| \leq r$,
- (ii) $|f'(s_0)/f(s_0)| < M/r$, and
- (iii) there exists an r' , satisfying $0 < r' < \frac{1}{4}r$, such that $f(s) \neq 0$ for $|s - s_0| \leq r$ and $\sigma \geq \Re s_0 - 2r'$,

then $|f'(s)/f(s)| \ll M/r$ for $|s - s_0| \leq r'$.

We use this lemma with $r = \frac{1}{4}$ and $M = C_4\phi(t_0 + 1)$. Conditions (i) and (ii) are satisfied by (3.44) and (3.45). Also, since $\tilde{L}(s, f) \neq 0$ for $\sigma \geq 1 - C_1C_2\phi(t)^{-1}$, putting $r' = 2C_3\phi(t_0 + 1)^{-1}$ and choosing C_3 small enough to satisfy $r' < \frac{1}{4}r$ and $C_1C_2 > 3C_3$, we find that condition (iii) holds. Hence from the above lemma we obtain

$$\left| \frac{\tilde{L}'(s, f)}{\tilde{L}(s, f)} \right| \ll \phi(t_0 + 1) \tag{3.46}$$

for $|s - s_0| \leq r'$.

Let $s = \sigma + it_0$, with $1 - \frac{1}{2}r' \leq \sigma \leq 1 + \frac{1}{2}r' = 1 + C_3\phi(t_0 + 1)^{-1}$. Then

$$\begin{aligned} \log |\tilde{L}(s, f)| &= \Re \log \tilde{L}(s, f) \\ &= \Re \left(\log \tilde{L}(s_0, f) + \int_{1+(r'/2)}^{\sigma} \frac{\tilde{L}'(u + it_0, f)}{\tilde{L}(u + it_0, f)} du \right) \\ &= \Re \log \tilde{L}(s_0, f) + O(1) \end{aligned}$$

by (3.46). Using (3.34) we obtain

$$|\Re \log \tilde{L}(s_0, f)| \leq 2 \log \zeta(1 + \frac{1}{2}r') + C'_0.$$

Hence

$$\phi(t_0 + 1)^{-2} \ll |\tilde{L}(s, f)| \ll \phi(t_0 + 1)^2,$$

which especially implies (3.42).

From (3.38) and (3.42) we have

$$\frac{|t| + 2}{(\log(|t| + 2))^2} \ll |L(\kappa - \frac{1}{2} + it, f)| \ll (|t| + 2)(\log(|t| + 2))^2. \quad (3.47)$$

When $\sigma = -\nu + \ell + \frac{1}{2}$, (3.37) is still valid, and (3.39) is valid only for $\ell + 2 \leq j \leq 2\nu$. Hence we have

$$\begin{aligned} &\left(\min_{T \leq t \leq 2T} |L(\kappa - \frac{1}{2} + it, f)|^2 \right) \prod_{j=\ell+2}^{2\nu} T^{4(j-\ell-1/2)} \int_T^{2T} |L(\kappa + \frac{1}{2} + it, f)|^2 dt \\ &\ll I(-\nu + \ell + \frac{1}{2}; T, 2T) \\ &\ll \left(\max_{T \leq t \leq 2T} |L(\kappa - \frac{1}{2} + it, f)|^2 \right) \\ &\quad \times \prod_{j=\ell+2}^{2\nu} T^{4(j-\ell-1/2)} \int_T^{2T} |L(\kappa + \frac{1}{2} + it, f)|^2 dt. \end{aligned} \quad (3.48)$$

Applying (3.47), the first result of (3.41), and the fact

$$\prod_{j=\ell+2}^{2\nu} T^{4(j-\ell-1/2)} = T^{2(2\nu-\ell)^2-2}$$

to (3.48), we obtain (3.28).

Now we evaluate $I(\sigma; T, 2T)$ in the remaining strip $\frac{1}{2} \leq \sigma \leq 1$. In this strip, by (3.35) and (3.38) we have

$$\prod_{j \neq \nu, \nu+1} |L(\sigma + it + \kappa + \nu - j, f)|^2 \asymp \prod_{j=\nu+2}^{2\nu} T^{4(j-\nu-\sigma)} = T^{2\nu(\nu+1-2\sigma)-4(1-\sigma)}$$

for $T \leq t \leq 2T$. Hence

$$\begin{aligned} & \left(\min_{T \leq t \leq 2T} |L(\sigma + it + \kappa, f)L(\sigma + it + \kappa - 1, f)|^2 \right) \\ & \quad \times T^{2\nu(\nu+1-2\sigma)-4(1-\sigma)} \int_T^{2T} |\zeta(\sigma + it)|^2 dt \\ & \ll I(\sigma; T, 2T) \\ & \ll \left(\max_{T \leq t \leq 2T} |L(\sigma + it + \kappa, f)L(\sigma + it + \kappa - 1, f)|^2 \right) \\ & \quad \times T^{2\nu(\nu+1-2\sigma)-4(1-\sigma)} \int_T^{2T} |\zeta(\sigma + it)|^2 dt, \end{aligned} \tag{3.49}$$

and it is known (Titchmarsh [36, Theorems 7.2 and 7.4]) that

$$\int_T^{2T} |\zeta(\sigma + it)|^2 dt \asymp \begin{cases} T \log T & \text{if } \sigma = \frac{1}{2}, \\ T & \text{if } \frac{1}{2} < \sigma < 1. \end{cases} \tag{3.50}$$

When $\sigma = \frac{1}{2}$, substituting (3.42), (3.47) and (3.50) into (3.49), we obtain (3.33). When $\frac{1}{2} < \sigma < 1$, the factor $|L(\sigma + it + \kappa, f)|$ can be evaluated by (3.35). Also from (3.18) we have

$$L(\sigma + it + \kappa - 1, f) \ll (|t| + 2)^{1-(1-\theta_2)(2\sigma-1)+\varepsilon}.$$

Applying these results and (3.50) to the second inequality of (3.49), we obtain

$$I(\sigma; T, 2T) \ll T^{2\nu(\nu+1-2\sigma)+2\theta_2(2\sigma-1)+1+\varepsilon} \quad \left(\frac{1}{2} < \sigma < 1\right). \tag{3.51}$$

On the other hand, if $\frac{1}{2} < \sigma \leq 1$, we have

$$\begin{aligned} & \left(\min_{T \leq t \leq 2T} |\zeta(\sigma + it)|^2 \right) T^{2\nu(\nu+1-2\sigma)-4(1-\sigma)} \int_T^{2T} |L(\sigma + it + \kappa - 1, f)|^2 dt \\ & \ll I(\sigma; T, 2T) \\ & \ll \left(\max_{T \leq t \leq 2T} |\zeta(\sigma + it)|^2 \right) T^{2\nu(\nu+1-2\sigma)-4(1-\sigma)} \int_T^{2T} |L(\sigma + it + \kappa - 1, f)|^2 dt. \end{aligned} \tag{3.52}$$

When $\frac{1}{2} < \sigma < 1$, we apply (3.6) and the third formula of (3.41) to the second inequality of (3.52) to obtain

$$I(\sigma; T, 2T) \ll T^{2\nu(\nu+1-2\sigma)+4\theta_1(1-\sigma)+1+\varepsilon}.$$

This estimate with (3.51) yields (3.32). Finally, when $\sigma = 1$, we use the bounds

$$\zeta(1 + it) = O((\log t)^{2/3})$$

(Ivić [16, (6.7)]) and

$$\frac{1}{\zeta(1 + it)} = O((\log t)^{2/3}(\log \log t)^{1/3})$$

(Titchmarsh [36, §6.19]), both of which are obtained by the method of Vinogradov and Korobov. Applying these bounds and the second formula of (3.41) to (3.52), we obtain (3.31). This completes the proof of Theorem 5. \square

From Theorem 5 (the case $\ell = 2\nu$ of (3.29)) we see that $\gamma(L(\cdot, F_0, st)) \leq \nu - \frac{1}{4}$. Hence the function $\tilde{L}(s, F_0, st) = L(s + \nu - \frac{1}{2}, F_0, st)$ satisfies (1.2) and $\gamma(\tilde{L}(\cdot, F_0, st)) \leq \frac{1}{4}$. Therefore from Theorem 2 the following result holds.

THEOREM 6. *We have*

$$\int_1^T |L(\sigma + it, F_0, st)|^2 dt = A(L(\cdot, F_0, st))T + O(T^{(8/3)(\nu+1/2-\sigma)+\varepsilon}) \tag{3.53}$$

for $\nu < \sigma \leq \nu + \frac{1}{2}$.

4. Proof of Theorem 1

In this section we prove Theorem 1 by a method which is a generalization of the proof of Titchmarsh [36, Theorem 12.5].

Let $\gamma < \sigma_0 < 1$. Then (2.1) implies that

$$\int_1^\infty |\varphi(\sigma + it)|^2 \frac{dt}{|\sigma + it|^2} = C(\sigma),$$

where $C(\sigma)$ is a non-negative constant, for any $\sigma \geq \sigma_0$. Let

$$h(t) = \int_{\sigma_0}^{4-\sigma_0} |\varphi(\sigma + it)|^2 \frac{d\sigma}{|\sigma + it|^2}.$$

Then $h(t) \geq 0$ and

$$\int_1^\infty h(t) dt = \int_{\sigma_0}^{4-\sigma_0} \int_1^\infty |\varphi(\sigma + it)|^2 \frac{dt}{|\sigma + it|^2} d\sigma = \int_{\sigma_0}^{4-\sigma_0} C(\sigma) d\sigma < +\infty.$$

Hence, for any $\varepsilon > 0$, there exists a sufficiently large $T = T(\varepsilon) > 0$ for which

$$\int_T^\infty h(t) dt < \varepsilon \tag{4.1}$$

holds.

Now we quote the following lemma (Titchmarsh [36, § 11.9]).

LEMMA 1. *If $f(z)$ is holomorphic in $|z - z_0| \leq R$ and*

$$\iint_{|z-z_0| \leq R} |f(z)|^2 dx dy \leq H,$$

then

$$|f(z)| \leq \frac{(H/\pi)^{1/2}}{R - R'}$$

for $|z - z_0| \leq R' < R$.

Let $\sigma_0 < \sigma_1 < 1$, and apply this lemma with $R = 2 - \sigma_0$, $R' = 2 - \sigma_1$, $z_0 = 2 + it$ ($t \geq T + 2 - \sigma_0$), $f(z) = \varphi(z)/z$ and $H = \varepsilon$. We obtain

$$\left| \frac{\varphi(z)}{z} \right| \leq \frac{1}{\sigma_1 - \sigma_0} \left(\frac{\varepsilon}{\pi} \right)^{1/2} \tag{4.2}$$

for $|z - z_0| \leq R'$, especially for $z = \sigma + it$, $\sigma_1 \leq \sigma \leq 4 - \sigma_1$, $t \geq T(\varepsilon) + 2 - \sigma_0$. We

may choose c in (2.2) that satisfies $1 < c < 4 - \sigma_1$. Then (4.2) implies that

$$\left| \int_{\sigma_1+it}^{c+it} \varphi(s) \frac{x^s}{s} ds \right| \rightarrow 0$$

as $t \rightarrow \infty$. Hence we can shift the path of integration on the right-hand side of (2.2) to $\Re s = \sigma_1$, to obtain (2.3) with (2.4). Putting $y = x^{-1}$, we can rewrite (2.4) as

$$R_\varphi\left(\frac{1}{y}\right) = \frac{1}{2\pi i} \int_{\sigma_1-i\infty}^{\sigma_1+i\infty} \varphi(s) y^{-s} \frac{ds}{s}. \tag{4.3}$$

The next lemma is a version of Parseval’s identity for Mellin transforms. Hence it is essentially included in Titchmarsh’s book [35], but is not stated explicitly in the following form, so we supply a proof.

LEMMA 2. (i) Let $\mathcal{G}(\sigma_1 + it) \in L^2(-\infty, \infty)$ as a function in t . Then, as $a \rightarrow \infty$, the function

$$g(x, a) = \frac{1}{2\pi i} \int_{\sigma_1-ia}^{\sigma_1+ia} \mathcal{G}(s) x^{-s} ds \quad (x > 0) \tag{4.4}$$

converges in mean to a function $g(x)$, and

$$\int_0^\infty |g(x)|^2 x^{2\sigma_1-1} dx = \frac{1}{2\pi} \int_{-\infty}^\infty |\mathcal{G}(\sigma_1 + it)|^2 dt. \tag{4.5}$$

Moreover,

$$\mathcal{G}(s, a) = \int_{1/a}^a g(x) x^{s-1} dx \quad (\Re s = \sigma_1) \tag{4.6}$$

converges in mean to $\mathcal{G}(s)$ as $a \rightarrow \infty$.

(ii) If $g(x)$ is a function defined on $(0, \infty)$, for which the left-hand side of (4.5) is finite, then $\mathcal{G}(s, a)$, defined by (4.6), converges in mean square to a function $\mathcal{G}(s)$, and formula (4.5) holds.

Proof. To prove the first part, define

$$G(\xi, a) = \frac{1}{\sqrt{2\pi}} \int_{-a}^a \mathcal{G}(\sigma_1 + it) e^{i\xi t} dt.$$

By [35, Theorem 48], as $a \rightarrow \infty$, $G(\xi, a)$ converges in mean to a function $G(\xi) \in L^2(-\infty, \infty)$ and

$$\widehat{g}(t, b) = \frac{1}{\sqrt{2\pi}} \int_{-b}^b G(\xi) e^{-it\xi} d\xi$$

converges in mean to $\mathcal{G}(\sigma_1 + it)$ as $b \rightarrow \infty$. Moreover, from [35, Theorem 49],

$$\int_{-\infty}^\infty |\mathcal{G}(\sigma_1 + it)|^2 dt = \int_{-\infty}^\infty |G(\xi)|^2 d\xi. \tag{4.7}$$

Putting $x = e^{-\xi}$ on the right-hand side of (4.4), we obtain

$$g(x, a) = \frac{1}{2\pi} \int_{-a}^a \mathcal{G}(\sigma_1 + it) e^{\xi(\sigma_1+it)} dt = \frac{1}{\sqrt{2\pi}} e^{\xi\sigma_1} G(\xi, a).$$

Hence, as $a \rightarrow \infty$, $g(x, a)$ converges in mean to

$$g(x) = \frac{1}{\sqrt{2\pi}} e^{\xi\sigma_1} G(\xi) = \frac{1}{\sqrt{2\pi}} x^{-\sigma_1} G(\xi).$$

Substituting this into (4.7), we obtain (4.5). Also, putting $x = e^{-\xi}$ and $a = e^b$ we find that $\widehat{g}(t, b) = \mathcal{G}(\sigma_1 + it, a)$; hence $\mathcal{G}(\sigma_1 + it, a)$ converges in mean to $\mathcal{G}(\sigma_1 + it)$. The second part of the lemma is [35, Theorem 71]. \square

Let $\gamma < \sigma_1 < 1$. Formula (4.3) implies that $R_\varphi(1/y) = g(y)$ if we put $\mathcal{G}(s) = \varphi(s)/s$. Hence from the first part of Lemma 2 we have

$$\begin{aligned} \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \frac{\varphi(\sigma_1 + it)}{\sigma_1 + it} \right|^2 dt &= \int_0^{\infty} \left| R_\varphi\left(\frac{1}{y}\right) \right|^2 y^{2\sigma_1-1} dy \\ &= \int_0^{\infty} |R_\varphi(x)|^2 x^{-2\sigma_1-1} dx, \end{aligned} \tag{4.8}$$

and also

$$\int_0^{\infty} R_\varphi(x) x^{-s-1} dx = \frac{\varphi(s)}{s} \quad (\Re s = \sigma_1) \tag{4.9}$$

in the mean value sense. Since the left-hand side of (4.8) is finite by assumption (2.1), we have

$$\int_X^{2X} |R_\varphi(x)|^2 x^{-2\sigma_1-1} dx = O(1)$$

for any $X > 1$; hence

$$\int_X^{2X} |R_\varphi(x)|^2 dx = O(X^{1+2\sigma_1}).$$

This implies that $\beta \leq \sigma_1$ for any $\sigma_1 > \gamma$; hence we obtain $\beta \leq \gamma$.

On the other hand, (4.9) is primarily valid only in the mean value sense for $\gamma < \sigma_1 < 1$, but actually the left-hand side of (4.9) is convergent absolutely for any $\sigma_2 \in (\beta, 1)$. In fact, the left-hand side is

$$\int_0^1 + \sum_{j=1}^{\infty} \int_{2^{j-1}}^{2^j} = I_0 + \sum_{j=1}^{\infty} I_j,$$

say. From (2.3), $R_\varphi(x) = -xP_\varphi(\log x)$ for $0 < x < 1$, and hence I_0 is convergent because $\sigma_2 < 1$. Using the Cauchy–Schwarz inequality and (2.5), we have

$$\begin{aligned} \int_X^{2X} |R_\varphi(x)| x^{-\sigma_2-1} dx &\leq \left(\int_X^{2X} |R_\varphi(x)|^2 x^{-2\sigma_2-1} dx \right)^{1/2} \left(\int_X^{2X} \frac{dx}{x} \right)^{1/2} \\ &\ll \left(X^{-2\sigma_2-1} \int_X^{2X} |R_\varphi(x)|^2 dx \right)^{1/2} \ll X^{\beta-\sigma_2+\varepsilon}. \end{aligned} \tag{4.10}$$

Hence $\sum_{j=1}^{\infty} I_j$ is convergent for $\sigma_2 > \beta$. Therefore by analytic continuation, (4.9) is valid for $\beta < \Re s < 1$. Moreover, the above argument also implies that

$$\int_0^{\infty} |R_\varphi(x)|^2 x^{-2\sigma_2-1} dx = O(1)$$

for $\beta < \sigma_2 < 1$. Hence we can apply the second part of Lemma 2 with $g(x) = R_\varphi(1/x)$. Then (4.9) implies $\mathcal{G}(s) = \varphi(s)/s$, and from (4.5) we find that

$$\int_{-\infty}^{\infty} \left| \frac{\varphi(\sigma_2 + it)}{\sigma_2 + it} \right|^2 dt = O(1),$$

which yields $\sigma_2 \geq \gamma$ (for any $\sigma_2 > \beta$). Hence $\beta \geq \gamma$, and the proof of Theorem 1 is complete.

5. Proof of Theorem 2

The purpose of this section is to prove Theorem 2. At first we assume $\sigma > 1$. Let $X > 1$, which is not an integer. Then

$$\begin{aligned} \varphi(s) &= \sum_{n \leq X} a_n n^{-s} + \sum_{n > X} a_n n^{-s} \\ &= \sum_{n \leq X} a_n n^{-s} + \int_X^\infty \left(\sum_{X < n \leq x} a_n \right) s x^{-s-1} dx \end{aligned}$$

by partial summation. Using (2.3), we see that the second term on the right-hand side is

$$s \int_X^\infty \{x P_\varphi(\log x) + R_\varphi(x)\} x^{-s-1} dx - \{X P_\varphi(\log X) + R_\varphi(X)\} X^{-s}.$$

Since P_φ is a polynomial of order at most $k - 1$, by integration by parts ($(k - 1)$ -times) we have

$$\int_X^\infty x P_\varphi(\log x) x^{-s-1} dx = \sum_{j=1}^k \frac{(-1)^j}{(1-s)^j} P_\varphi^{(j-1)}(\log X) X^{1-s}.$$

Hence

$$\begin{aligned} \varphi(s) &= \sum_{n \leq X} a_n n^{-s} + s X^{1-s} \sum_{j=1}^k \frac{(-1)^j}{(1-s)^j} P_\varphi^{(j-1)}(\log X) \\ &\quad + s \int_X^\infty R_\varphi(x) x^{-s-1} dx - \{X P_\varphi(\log X) + R_\varphi(X)\} X^{-s}. \end{aligned} \tag{5.1}$$

By using the Cauchy-Schwarz inequality and (2.5), we see that

$$\begin{aligned} \left| \int_X^{2X} R_\varphi(x) x^{-s-1} dx \right| &\leq \int_X^{2X} |R_\varphi(x)| x^{-\sigma-1} dx \\ &\leq \left(\int_X^{2X} |R_\varphi(x)|^2 dx \right)^{1/2} \left(\int_X^{2X} x^{-2\sigma-2} dx \right)^{1/2} \\ &\ll (X^{1+2\gamma+\varepsilon})^{1/2} (X^{-2\sigma-1})^{1/2} \ll X^{\gamma-\sigma+\varepsilon}. \end{aligned} \tag{5.2}$$

Hence the integral on the right-hand side of (5.1) is convergent absolutely if $\sigma > \gamma$. Therefore (5.1) gives the meromorphic continuation of $\varphi(s)$ to the region $\sigma > \gamma$. Note that (5.1) is a generalization of a previous result of the author [26, (6.5)].

For any $X > 1$, from (2.5) we can find an $X_0 \in [X, 2X]$ such that $R_\varphi(X_0) = O(X_0^{\gamma+\varepsilon})$. Hence we may assume, and hereafter we do assume, that

$$R_\varphi(X) = O(X^{\gamma+\varepsilon}). \tag{5.3}$$

Then, in the region $\sigma > \gamma, T \leq t \leq 2T (T > 1)$, from (5.1) we obtain

$$\varphi(s) = \sum_{n \leq X} a_n n^{-s} + s \int_X^\infty R_\varphi(x) x^{-s-1} dx + O(T^{-1} X^{1-\sigma+\varepsilon} + X^{\gamma-\sigma+\varepsilon}). \tag{5.4}$$

In fact, we see that

$$\begin{aligned} & sX^{1-s} \sum_{j=1}^k \frac{(-1)^j}{(1-s)^j} P_\varphi^{(j-1)}(\log X) - \{XP_\varphi(\log X) + R_\varphi(X)\}X^{-s} \\ &= \frac{1}{s-1} X^{1-s} P_\varphi(\log X) + sX^{1-s} \sum_{j=2}^k \frac{(-1)^j}{(1-s)^j} P_\varphi^{(j-1)}(\log X) - R_\varphi(X)X^{-s} \\ &\ll T^{-1}X^{1-\sigma+\varepsilon} + TX^{1-\sigma+\varepsilon}T^{-2} + X^{\gamma-\sigma+\varepsilon} \end{aligned}$$

by (5.3), and hence (5.4) follows.

A fundamental idea of Ivić [20] is to divide the right-hand side of (5.4) as

$$\varphi(s) = D(s) + E(s),$$

where

$$D(s) = \sum_{n \leq X/2} a_n n^{-s}$$

and $E(s)$ is the sum of remaining terms. Then

$$\begin{aligned} \int_T^{2T} |\varphi(\sigma + it)|^2 dt &= \int_T^{2T} |D(\sigma + it)|^2 dt + \int_T^{2T} |E(\sigma + it)|^2 dt \\ &\quad + 2\Re \int_T^{2T} D(\sigma + it) \overline{E(\sigma + it)} dt \end{aligned} \tag{5.5}$$

for $\sigma > \gamma$.

The following mean value theorem for Dirichlet polynomials is well known: for any complex numbers a_1, \dots, a_N , we have

$$\int_0^T \left| \sum_{n \leq N} a_n n^{it} \right|^2 dt = T \sum_{n \leq N} |a_n|^2 + O\left(\sum_{n \leq N} n |a_n|^2 \right) \tag{5.6}$$

(see [16, Theorem 5.2]). Hereafter we assume that $\max\{\gamma, \frac{1}{2}\} < \sigma \leq 1$. Then, using formula (5.6), we have

$$\begin{aligned} \int_T^{2T} |D(\sigma + it)|^2 dt &= T \sum_{n \leq X/2} |a_n|^2 n^{-2\sigma} + O\left(\sum_{n \leq X/2} |a_n|^2 n^{1-2\sigma} \right) \\ &= T \left(\sum_{n=1}^{\infty} - \sum_{n > X/2} \right) |a_n|^2 n^{-2\sigma} + O\left(\sum_{n \leq X/2} |a_n|^2 n^{1-2\sigma} \right) \\ &= T \sum_{n=1}^{\infty} |a_n|^2 n^{-2\sigma} + O(TX^{1-2\sigma+\varepsilon} + X^{2-2\sigma+\varepsilon}). \end{aligned} \tag{5.7}$$

Next, we have

$$\begin{aligned} \int_T^{2T} |E(\sigma + it)|^2 dt &\ll \int_T^{2T} \left| \sum_{X/2 < m \leq X} a_m m^{-s} \right|^2 dt \\ &\quad + \int_T^{2T} \left| s \int_X^\infty R_\varphi(x) x^{-s-1} dx \right|^2 dt \\ &\quad + \int_T^{2T} (T^{-2} X^{2-2\sigma+\varepsilon} + X^{2\gamma-2\sigma+\varepsilon}) dt. \end{aligned} \tag{5.8}$$

LEMMA 3. Let $[a, b] \subset [2, +\infty)$ (b can be $+\infty$), and $h(x)$ be a complex-valued function defined and integrable on $[a, b]$. Then

$$\int_0^T \left| \int_a^b h(x)x^{-s} dx \right|^2 dt \leq 4\pi \int_a^b |h(x)|^2 x^{1-2\sigma} dx. \tag{5.9}$$

Proof. When $h(x)$ is real-valued, the inequality

$$\int_0^T \left| \int_a^b h(x)x^{-s} dx \right|^2 dt \leq 2\pi \int_a^b h(x)^2 x^{1-2\sigma} dx \tag{5.10}$$

has been proved by Ivić [19, Lemma 4]. When $h(x)$ is complex-valued, we apply (5.10) to $\Re h(x)$ and $\Im h(x)$, and use the inequality

$$\begin{aligned} & \int_0^T \left| \int_a^b (\Re h(x))x^{-s} dx + i \int_a^b (\Im h(x))x^{-s} dx \right|^2 dt \\ & \leq 2 \int_0^T \left\{ \left| \int_a^b (\Re h(x))x^{-s} dx \right|^2 + \left| \int_a^b (\Im h(x))x^{-s} dx \right|^2 \right\} dt \end{aligned}$$

to obtain (5.9). □

Using Lemma 3 and (2.5) we find that the second term on the right-hand side of (5.8) is

$$\begin{aligned} & \ll T^2 \int_T^{2T} \left| \int_X^\infty R_\varphi(x)x^{-s-1} dx \right|^2 dt \\ & \ll T^2 \int_X^\infty |R_\varphi(x)|^2 x^{-1-2\sigma} dx \\ & \ll T^2 \sum_{j=1}^\infty (2^j X)^{-1-2\sigma} \int_{2^{j-1}X}^{2^j X} |R_\varphi(x)|^2 dx \\ & \ll T^2 \sum_{j=1}^\infty (2^j X)^{2\gamma-2\sigma+\varepsilon} \ll T^2 X^{2\gamma-2\sigma+\varepsilon}. \end{aligned}$$

Applying (5.6) we see that the first term on the right-hand side of (5.8) is

$$\ll T \sum_{X/2 < m \leq X} a_m^2 m^{-2\sigma} + \sum_{X/2 < m \leq X} a_m^2 m^{1-2\sigma} \ll TX^{1-2\sigma+\varepsilon} + X^{2-2\sigma+\varepsilon}.$$

Hence we obtain

$$\int_T^{2T} |E(\sigma + it)|^2 dt \ll TX^{1-2\sigma+\varepsilon} + T^2 X^{2\gamma-2\sigma+\varepsilon} + X^{2-2\sigma+\varepsilon}. \tag{5.11}$$

Lastly,

$$\begin{aligned} & \int_T^{2T} D(\sigma + it) \overline{E(\sigma + it)} dt \\ &= \sum_{n \leq X/2} a_n n^{-\sigma} \int_T^{2T} n^{-it} \sum_{X/2 < m \leq X} \bar{a}_m m^{-\sigma+it} dt \\ &+ \sum_{n \leq X/2} a_n n^{-\sigma} \int_T^{2T} n^{-it} (\sigma - it) \int_X^\infty \overline{R_\varphi(x)} x^{-\sigma+it-1} dx dt \\ &+ \sum_{n \leq X/2} a_n n^{-\sigma} \int_T^{2T} n^{-it} O(T^{-1} X^{1-\sigma+\varepsilon} + X^{\gamma-\sigma+\varepsilon}) dt \\ &= S_1 + S_2 + S_3, \end{aligned}$$

say. It is easy to see that

$$S_3 \ll TX^{1+\gamma-2\sigma+\varepsilon} + X^{2-2\sigma+\varepsilon}. \tag{5.12}$$

Since the inequality

$$\frac{1}{\log(1+x)} \leq 1 + \frac{1}{x} \tag{5.13}$$

holds for any $x > 0$, we have

$$\begin{aligned} S_1 &= \sum_{n \leq X/2} \sum_{X/2 < m \leq X} a_n \bar{a}_m (mn)^{-\sigma} \int_T^{2T} \left(\frac{m}{n}\right)^{it} dt \\ &\ll \sum_{n \leq X/2} \sum_{X/2 < m \leq X} (mn)^{-\sigma+\varepsilon} \frac{1}{\log(m/n)} \\ &\leq \sum_{n \leq X/2} \sum_{X/2 < m \leq X} (mn)^{-\sigma+\varepsilon} \frac{m}{m-n} \\ &\ll X^{1-\sigma+\varepsilon} \sum_{n \leq X/2} n^{-\sigma+\varepsilon} \sum_{X/2 < m \leq X} \frac{1}{m-n} \ll X^{2-2\sigma+\varepsilon}. \end{aligned} \tag{5.14}$$

As for S_2 , we have

$$\begin{aligned} S_2 &= \sum_{n \leq X/2} a_n n^{-\sigma} \int_X^\infty \overline{R_\varphi(x)} x^{-\sigma-1} \int_T^{2T} (\sigma - it) \left(\frac{x}{n}\right)^{it} dt dx \\ &\ll T \sum_{n \leq X/2} n^{-\sigma+\varepsilon} \int_X^\infty \overline{R_\varphi(x)} x^{-\sigma-1} \frac{dx}{\log(x/n)}, \end{aligned}$$

because

$$\int_T^{2T} t \left(\frac{x}{n}\right)^{it} dt \ll \frac{T}{\log(x/n)}$$

by the first derivative test (Titchmarsh [36, Lemma 4.3]). Using (5.13) again and noting that $x \ll x - n \ll x$, we have

$$\frac{1}{\log(x/n)} \leq \frac{x}{x-n} \ll 1,$$

and so, with (5.2), we have

$$S_2 \ll TX^{\gamma-\sigma+\varepsilon} \sum_{n \leq X/2} n^{-\sigma+\varepsilon} \ll TX^{1+\gamma-2\sigma+\varepsilon}. \tag{5.15}$$

From (5.12), (5.14) and (5.15) we obtain

$$\int_T^{2T} D(\sigma+it) \overline{E(\sigma+it)} dt \ll TX^{1+\gamma-2\sigma+\varepsilon} + X^{2-2\sigma+\varepsilon}. \tag{5.16}$$

Collecting (5.5), (5.7), (5.11) and (5.16), we now arrive at

$$\begin{aligned} & \int_T^{2T} |\varphi(\sigma+it)|^2 dt \\ &= T \sum_{n=1}^{\infty} a_n^2 n^{-2\sigma} \\ & \quad + O(TX^{1-2\sigma+\varepsilon} + T^2 X^{2\gamma-2\sigma+\varepsilon} + TX^{1+\gamma-2\sigma+\varepsilon} + X^{2-2\sigma+\varepsilon}) \end{aligned} \tag{5.17}$$

for $\max\{\gamma, \frac{1}{2}\} < \sigma \leq 1$.

If $\gamma < 0$, then the error term on the right-hand side is

$$O(TX^{1-2\sigma+\varepsilon} + T^2 X^{2\gamma-2\sigma+\varepsilon} + X^{2-2\sigma+\varepsilon}),$$

which is $O(T^{2-2\sigma+\varepsilon})$ under the choice $X = cT$, where c is a constant. The condition (5.3) can be satisfied if we choose c suitably.

If $\gamma \geq 0$, then

$$TX^{1-2\sigma+\varepsilon} \leq TX^{1+\gamma-2\sigma+\varepsilon} \ll T^2 X^{2\gamma-2\sigma+\varepsilon} + X^{2-2\sigma+\varepsilon}.$$

Hence the error term on the right-hand side of (5.17) is $O(T^2 X^{2\gamma-2\sigma+\varepsilon} + X^{2-2\sigma+\varepsilon})$, which is $O(T^{2(1-\sigma)/(1-\gamma)+\varepsilon})$ under the choice $X = cT^{1/(1-\gamma)}$. The constant c is again suitably chosen so as to satisfy (5.3). Thus we have completed the proof of Theorem 2.

References

1. A. N. ANDRIANOV and V. L. KALININ, ‘On the analytic properties of standard zeta functions of Siegel modular forms’, *Mat. Sb.* 106(148) (1978) 323–339 (Russian), *Math. USSR Sb.* 35 (1979) 1–17 (English).
2. D. BLASIUS, ‘Hilbert modular forms and the Ramanujan conjecture’, Preprint, 2002, www.math.ucla.edu/classes/faculty/blsius.html.
3. S. BÖCHERER, ‘Über die Funktionalgleichung automorpher L -Funktionen zur Siegelschen Modulgruppe’, *J. Reine Angew. Math.* 362 (1985) 146–168.
4. D. BUMP, *Automorphic forms and representations* (Cambridge University Press, 1997).
5. K. DOI and H. NAGANUMA, ‘On the functional equation of certain Dirichlet series’, *Invent. Math.* 9 (1969) 1–14.
6. W. DUKE, R. HOWE and J.-S. LI, ‘Estimating Hecke eigenvalues of Siegel modular forms’, *Duke Math. J.* 67 (1992) 219–240.
7. M. EICHLER and D. ZAGIER, *The theory of Jacobi forms*, Progress in Mathematics 55 (Birkhäuser, Basel, 1985).
8. A. GOOD, ‘Ein Mittelwertsatz für Dirichletreihen, die Modulformen assoziiert sind’, *Comment. Math. Helv.* 49 (1974) 35–47.
9. A. GOOD, ‘The square mean of Dirichlet series associated with cusp forms’, *Mathematika* 29 (1982) 278–295.
10. S. W. GRAHAM, ‘Large values of the Riemann zeta-function’, *Topics in analytic number theory* (ed. S. W. Graham and J. D. Vaaler, University of Texas Press, Austin, 1985) 98–126.
11. F. GRUPP, ‘On Dirichlet series attached to cusp forms and the Siegel-zero’, *Glasgow Math. J.* 25 (1984) 107–119.

12. D. R. HEATH-BROWN, 'The twelfth power moment of the Riemann-function', *Quart. J. Math. Oxford* (2) 29 (1978) 443–462.
13. O. HERRMANN, 'Über Hilbertsche Modulfunktionen und die Dirichletschen Reihen mit Eulerscher Produktentwicklung', *Math. Ann.* 127 (1954) 357–400.
14. M. N. HUXLEY, 'Integer points, exponential sums and the Riemann zeta function', *Number theory for the millennium II*, Proceedings of the Millennial Conference on Number Theory, May 2000 (ed. M. A. Bennett *et al.*, A. K. Peters, Wellesley, MA, 2002) 275–290.
15. T. IKEDA, 'On the lifting of elliptic modular forms to Siegel cusp forms of degree $2n$ ', *Ann. of Math.* (2) 154 (2001) 641–681.
16. A. IVIĆ, *The Riemann zeta-function* (Wiley, New York, 1985).
17. A. IVIĆ, 'Large values of number-theoretic error terms', *Acta Arith.* 56 (1990) 135–159.
18. A. IVIĆ, 'On zeta-functions associated with Fourier coefficients of cusp forms', *Proceedings of the Amalfi Conference on Analytic Number Theory*, Maiori, Amalfi, Italy, 1989 (ed. E. Bombieri *et al.*, Università di Salerno, 1992) 231–246.
19. A. IVIĆ, 'On some conjectures and results for the Riemann zeta-function and Hecke series', *Acta Arith.* 99 (2001) 115–145.
20. A. IVIĆ, 'On mean values of some zeta-functions in the critical strip', *J. Théor. Nombres Bordeaux* 15 (2003) 151–166.
21. M. JUTILA, *Lectures on a method in the theory of exponential sums*, Lectures in Mathematics and Physics 80 (Tata Institute for Fundamental Research, Bombay, and Springer, Berlin, 1987).
22. S. KANEMITSU, A. SANKARANARAYANAN and Y. TANIGAWA, 'A mean value theorem for Dirichlet series and a general divisor problem', *Monatsh. Math.* 136 (2002) 17–34.
23. W. KOHNEN, 'Lifting modular forms of half-integral weight to Siegel modular forms of even genus', *Math. Ann.* 322 (2002) 787–809.
24. W. KOHNEN and H. KOJIMA, 'A Maass space in higher genus', *Compositio Math.* 141 (2005) 313–322.
25. W. KOHNEN, A. SANKARANARAYANAN and J. SENGUPTA, 'The quadratic mean of automorphic L -functions', Preprint, Tata Institute for Fundamental Research, 2004.
26. K. MATSUMOTO, 'The mean values and the universality of Rankin–Selberg L -functions', *Number theory*, Proceedings of the Turku Symposium in Memory of K. Inkeri, May/June 1999 (ed. M. Jutila and T. Metsänkylä, Walter de Gruyter, Berlin, 2001) 201–221.
27. H. L. MONTGOMERY, 'Extreme values of the Riemann zeta function', *Comment. Math. Helv.* 52 (1977) 511–518.
28. C. J. MORENO, 'Prime number theorems for the coefficients of modular forms', *Bull. Amer. Math. Soc.* 78 (1972) 796–798.
29. C. J. MORENO, 'Explicit formulas in the theory of automorphic forms', *Number theory day* (ed. M. B. Nathanson), Lecture Notes in Mathematics 626 (Springer, Berlin, 1977) 73–216.
30. H. S. A. POTTER, 'The mean values of certain Dirichlet series Γ ', *Proc. London Math. Soc.* (2) 46 (1940) 467–478.
31. K. RAMACHANDRA and A. SANKARANARAYANAN, 'Note on a paper by H. L. Montgomery (Omega theorems for the Riemann zeta-function)', *Publ. Inst. Math. Beograd*, NS 50(64) (1991) 51–59.
32. H. SAITO, *Automorphic forms and algebraic extensions of number fields*, Lectures in Mathematics 8 (Kyoto University, Kinokuniya, Tokyo, 1975).
33. A. SANKARANARAYANAN, 'Fundamental properties of symmetric square L -functions Γ ', *Illinois J. Math.* 46 (2002) 23–43.
34. G. SHIMURA, 'On the holomorphy of certain Dirichlet series', *Proc. London Math. Soc.* (3) 31 (1975) 79–98.
35. E. C. TITCHMARSH, *Introduction to the theory of Fourier integrals*, 2nd edn (Oxford University Press, 1948).
36. E. C. TITCHMARSH, *The theory of the Riemann zeta-function*, 2nd edn, revised by D. R. Heath-Brown (Oxford University Press, 1986).

Kohji Matsumoto
 Graduate School of Mathematics
 Nagoya University
 Chikusa-ku
 Nagoya 464–8602
 Japan
 kohjimat@math.nagoya-u.ac.jp