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ASYMPTOTIC EXPANSIONS OF MULTIPLE ZETA FUNCTIONS AND POWER MEAN VALUES OF HURWITZ ZETA FUNCTIONS

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Abstract

Let $\zeta(s, \alpha)$ be the Hurwitz zeta function with parameter α. Power mean values of the form $\sum_{a=1}^{q} \zeta(s, a/q)^h$ or $\sum_{a=1}^{q} |\zeta(s, a/q)|^{2h}$ are studied, where q and h are positive integers. These mean values can be written as linear combinations of $\sum_{a=1}^{q} \zeta_r(s_1, \dots, s_r; a/q)$, where $\zeta_r(s_1, \dots, s_r; \alpha)$ is a generalization of Euler–Zagier multiple zeta sums. The Mellin–Barnes integral formula is used to prove an asymptotic expansion of $\sum_{a=1}^{q} \zeta_r(s_1, \dots, s_r; a/q)$ with respect to q. Hence a general way of deducing asymptotic expansion formulas for $\sum_{a=1}^{q} \zeta(s, a/q)^h$ and $\sum_{a=1}^{q} |\zeta(s, a/q)|^{2h}$ is obtained. In particular, the asymptotic expansion of $\sum_{a=1}^{q} \zeta(1/2, a/q)^3$ with respect to q is written down.

1. *Introduction and statement of results*

Let r be a positive integer, let $s_1, \dots, s_r$ be complex variables, let $\alpha > 0$, and define

$$\zeta_r(s_1, \dots, s_r; \alpha) = \sum_{m_1=0}^{\infty} \sum_{m_2=1}^{\infty} \dots \sum_{m_r=1}^{\infty} (\alpha + m_1)^{-s_1} (\alpha + m_1 + m_2)^{-s_2} \times \dots \times (\alpha + m_1 + \dots + m_r)^{-s_r}. \tag{1.1}$$

This multiple series is convergent absolutely if $\Re(s_{r-k+1} + \dots + s_r) > k$ for $1 \leqslant k \leqslant r$. When $r = 1$, the series (1.1) is nothing but the classical Hurwitz zeta function $\zeta(s_1, \alpha)$. Katsurada and Matsumoto [**11**] proved that for $r = 2$, (1.1) can be continued meromorphically to the whole $\mathbb{C}^2$ space. The meromorphic continuation of (1.1) for general r was established by Akiyama and Ishikawa [**1**].

We remark that the case $\alpha = 1$ of (1.1) is the well known Euler–Zagier sum, which has been studied by Zagier [**19**] and many other people. When $r = 2$, it is the classical Euler sum, and its analytic continuation was first obtained by Atkinson [**3**]. The analytic continuation of the Euler–Zagier sum for any r was proved by Zhao [**20**], and proved independently by Akiyama, Egami and Tanigawa [**2**]. Paper [**1**] is a generalization of [**2**]. For further details on the history of the problem of analytic continuation of (1.1), the readers are referred to [**13**].

Let q be a positive integer. In this paper, we study the asymptotic behaviour of the sum

$$J_r(s_1, \dots, s_r; q) = \sum_{a=1}^{q} \zeta_r(s_1, \dots, s_r; a/q) \tag{1.2}$$

with respect to q. The result is embodied in the following Theorem 1. Throughout this paper, the empty sum is considered to be zero, and the empty product is considered to be 1. The letter ε denotes an arbitrarily small positive number that is

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not necessarily the same in each occurrence. By $\mathbb{N}_0$ and $\mathbb{C}$, we denote the sets of all non-negative integers and all complex numbers, respectively. Define

$$\mathscr{D}_r(N;\varepsilon)=\{(s_1,\ldots,s_r)\in\mathbb{C}^r \mid \Re s_1<N+1-\varepsilon, \Re(s_2+\ldots+s_r)>r-1-N+\varepsilon\}.$$

Then we have the following theorem.

THEOREM 1. *Let $r\geqslant 2$, and let N be a positive integer. Assume that s_1 is not a positive integer. Then, for any $\varepsilon>0$, the asymptotic expansion of the form*

$$\begin{aligned} J_r(s_1,\ldots,s_r;q) &= X_r(s_1,\ldots,s_r)q \\ &\quad+\sum_{n=0}^{N-1}\frac{(-1)^n}{n!}\zeta(s_1-n)Y_{r,n}(s_2,\ldots,s_r)q^{s_1-n} \\ &\quad+Z_{r,N}(s_1,\ldots,s_r;\varepsilon,q) \end{aligned} \tag{1.3}$$

holds in the region $\mathscr{D}_r(N;\varepsilon)$, where

(i) *$X_r(s_1,\ldots,s_r)$ is independent of q, and is meromorphic in the whole $\mathbb{C}^r$ space;*

(ii) *$Y_{r,n}(s_2,\ldots,s_r)$ is also independent of q, and actually*

$$\begin{aligned} Y_{r,n}(s_2,\ldots,s_r) &= \sum_{n_2+\ldots+n_r=n}\frac{n!}{n_2!\ldots n_r!} \\ &\quad\times\frac{\Gamma(s_2+n_2)\ldots\Gamma(s_r+n_r)}{\Gamma(s_2)\ldots\Gamma(s_r)} \\ &\quad\times\zeta_{r-1}(s_2+n_2,\ldots,s_r+n_r;1), \end{aligned} \tag{1.4}$$

where the summation runs over all non-negative integers $n_2,\ldots,n_r$ satisfying $n_2+\ldots+n_r=n$. Hence it is clearly meromorphic in the whole $\mathbb{C}^r$ space;

(iii) *$Z_{r,N}(s_1,\ldots,s_r;\varepsilon,q)$ is meromorphic in the region $\mathscr{D}_r(N;\varepsilon)$ and, except for the possible singularities, the estimate*

$$Z_{r,N}(s_1,\ldots,s_r;\varepsilon,q)=O(q^{\Re s_1-N+\varepsilon}) \tag{1.5}$$

holds for each $(s_1,\ldots,s_r)\in\mathscr{D}_r(N;\varepsilon)$.

REMARK 1. In Theorem 1, we exclude the case when s_1 is a positive integer. When $s_1=m+1$ $(m=0,1,\ldots,N-1)$, then, instead of (1.3), the asymptotic expansion

$$\begin{aligned} J_r(s_1,\ldots,s_r;q) &= \frac{(-1)^m}{m!}Y_{r,m}(s_2,\ldots,s_r)q\log q \\ &\quad+\frac{(-1)^m}{m!}\left\{\left(1+\frac{1}{2}+\ldots+\frac{1}{m}\right)Y_{r,m}(s_2,\ldots,s_r)\right. \\ &\quad\left.-Y^*_{r,m}(s_2,\ldots,s_r)\right\}q \\ &\quad+\sum_{\substack{n=0\\ n\neq m}}^{N-1}\frac{(-1)^n}{n!}\zeta(s_1-n)Y_{r,n}(s_2,\ldots,s_r)q^{s_1-n} \\ &\quad+Z_{r,N}(s_1,\ldots,s_r;\varepsilon,q) \end{aligned} \tag{1.6}$$

holds, where $Y^*_{r,m}(s_2,\ldots,s_r)$ is independent of q and is meromorphic in the whole $\mathbb{C}^r$ space.

REMARK 2. As for the location of the singularities of X_r and $Z_{r,N}$, we will show the following:

(i) The possible singularities of $X_2(s_1,s_2)$ are only on $s_1 = 1+n$ ($n \in \mathbb{N}_0$) and $s_1+s_2 = 2-n$ ($n \in \mathbb{N}_0$). For $r \geqslant 3$, the possible singularities of $X_r(s_1,\ldots,s_r)$ are only on the subsets of $\mathbb{C}^r$ defined by one of the following conditions:

$$\begin{aligned} s_1 &= n+1, && n \in \mathbb{N}_0, \\ s_r &= 1, && \\ s_k + s_{k+1} + \ldots + s_r &= r-k+1-n, && 3 \leqslant k \leqslant r-1,\ n \in \mathbb{N}_0, \\ s_1 + s_2 + \ldots + s_r &= r-n, && n \in \mathbb{N}_0. \end{aligned}$$

(ii) The function $Z_{2,N}(s_1,s_2)$ is holomorphic in the whole of $\mathscr{D}_2(N;\varepsilon)$. The possible singularities of $Z_{3,N}(s_1,s_2,s_3)$ in $\mathscr{D}_3(N;\varepsilon)$ are only on $s_3 = 1$. For $r \geqslant 4$, the possible singularities of $Z_{r,N}(s_1,\ldots,s_r)$ are only on the subsets of $\mathscr{D}_r(N;\varepsilon)$ defined by one of the following conditions:

$$\begin{aligned} s_r &= 1, && \\ s_k + s_{k+1} + \ldots + s_r &= r-k+1-n, && 3 \leqslant k \leqslant r-1,\ n \in \mathbb{N}_0. \end{aligned}$$

REMARK 3. The definitions of X_r, $Y_{r,n}$ and $Z_{r,N}$ are given by (3.5), (3.6) and (3.7), respectively.

In [**7**], Katsurada introduced a simple elegant method of studying (1.1) for $r = 2$ by using a kind of Mellin–Barnes type of integral. Inspired by this work [**7**] of Katsurada, the second author of this paper [**14**] studied (1.1) for general r by the method of a Mellin–Barnes type of integral, and in particular gave an alternative proof of analytic continuation of (1.1).

The basic tool of this paper is also the Mellin–Barnes type of integrals. The fundamental formula is (2.1) (which is also fundamental in [**7**] and [**14**]). Applying (2.1) repeatedly, we deduce a multiple integral expression of $\zeta_r(s_1,\ldots,s_r;\alpha)$ in Section 2. Then, in Section 3, shifting the path of integration suitably, we prove the asymptotic expansion in the region $\Re s_j > 1$ ($1 \leqslant j \leqslant r$). The analytic continuation of the terms appearing in the expansion is carried out in Sections 4–6.

Our initial motivation for this study was the consideration of the discrete power mean values of Hurwitz zeta functions. Let h be a positive integer, and define

$$U_h(s,q) = \sum_{a=1}^{q} \zeta(s,a/q)^h \tag{1.7}$$

and

$$\tilde{U}_h(s,q) = \sum_{a=1}^{q} |\zeta(s,a/q)|^{2h}. \tag{1.8}$$

In Section 7, we show that, by an elementary argument, it is possible to obtain asymptotic formulas for $U_h(s,q)$ and $\tilde{U}_h(s,q)$ with respect to q with the error term $O(q)$. However, when $h = 1$, we have much more precise information about the mean value $\tilde{U}_1(s,q) = \sum_{a=1}^{q} |\zeta(s,a/q)|^2$. Indeed, for $0 < \Re s < 1$, asymptotic expansions for

$\tilde{U}_1(s,q)$ with the error term $O(q^{\sigma-N})$, where N is an arbitrary positive integer, are proved by Katsurada and Matsumoto [**9**]. Therefore it is an interesting problem to obtain sharper asymptotic formulas for higher power moments.

In Section 8, we prove a general principle (Lemma 4) that reduces the problem to the evaluation of $J_r(s_1,\ldots,s_r;q)$. Combining Lemma 4 with (1.3), we obtain the following asymptotic expansion formulas. Define

$$\mathscr{E}_h(N;\varepsilon)=\left\{s\in\mathbb{C}\;\middle|\;1-\frac{N}{h-1}+\varepsilon<\Re s<\frac{N+1}{h-1}+\varepsilon\right\},$$

and denote by $\mathscr{S}_h$ (respectively $\tilde{\mathscr{S}}_h$) the set of all points $s\in\mathbb{C}$ at which the right-hand side of (1.9) (respectively (1.10)) has no singularity.

THEOREM 2. *Let $h\geqslant 2$, and let N be a positive integer. Then we have*

$$U_h(s,q)=\zeta(hs)q^{hs}+\sum_{r=2}^{h}\sum_{\substack{h_1+\ldots+h_r=h\\ h_j\geqslant 1\,(1\leqslant j\leqslant r)}}\frac{h!}{h_1!\ldots h_r!}\left\{X_r(h_1s,\ldots,h_rs)q\right.$$
$$+\sum_{n=0}^{N-1}\frac{(-1)^n}{n!}\zeta(h_1s-n)Y_{r,n}(h_2s,\ldots,h_rs)q^{h_1s-n}$$
$$\left.+Z_{r,N}(h_1s,\ldots,h_rs;\varepsilon,q)\right\} \tag{1.9}$$

for $s\in\mathscr{E}_h(N;\varepsilon)\cap\mathscr{S}_h$ and

$$\tilde{U}_h(s,q)=\zeta(2h\Re s)q^{2h\Re s}+\sum_{r=2}^{2h}\sum_{\substack{\alpha_1+\ldots+\alpha_r=h\\ \beta_1+\ldots+\beta_r=h\\ \alpha_j+\beta_j\geqslant 1(1\leqslant j\leqslant r)}}\frac{h!}{\alpha_1!\ldots\alpha_r!}\frac{h!}{\beta_1!\ldots\beta_r!}$$
$$\times\left\{X_r(\alpha_1s+\beta_1\bar{s},\ldots,\alpha_rs+\beta_r\bar{s})q\right.$$
$$+\sum_{n=0}^{N-1}\frac{(-1)^n}{n!}\zeta(\alpha_1s+\beta_1\bar{s}-n)Y_{r,n}(\alpha_2s+\beta_2\bar{s},\ldots,\alpha_rs+\beta_r\bar{s})q^{\alpha_1s+\beta_1\bar{s}-n}$$
$$\left.+Z_{r,N}(\alpha_1s+\beta_1\bar{s},\ldots,\alpha_rs+\beta_r\bar{s};\varepsilon,q)\right\} \tag{1.10}$$

for $s\in\mathscr{E}_h(N;\varepsilon)\cap\tilde{\mathscr{S}}_h$.

Since N is arbitrary, our problem is now completely solved for $s\in\mathscr{S}_h\cap\tilde{\mathscr{S}}_h$. Moreover, it is clear that almost all complex numbers (in the sense of Lebesgue measure) are elements of $\mathscr{S}_h\cap\tilde{\mathscr{S}}_h$. However, it is also true that, in many interesting cases, the values of s are actually included in the exceptional set $\mathbb{C}\backslash\mathscr{S}_h\cap\tilde{\mathscr{S}}_h$. In particular, the points on the critical line $\Re s=1/2$ are exceptional points. In those cases, several singularities appear on the right-hand side of (1.9) or (1.10), and it is necessary to evaluate the contribution of those singularities carefully to obtain asymptotic expansions for exceptional points. In this paper, as a simple example, we prove the asymptotic expansion of cubic mean at $s=1/2$.

THEOREM 3. *Let N be a positive integer. We have*

$$\begin{aligned} U_3\left(\frac{1}{2}, q\right) &= \sum_{a=1}^{q} \zeta\left(\frac{1}{2}, \frac{a}{q}\right)^3 \\ &= \zeta\left(\frac{3}{2}\right) q^{3/2} + 3\zeta\left(\frac{1}{2}\right) q \log q \\ &\quad + \left\{ 3\pi\zeta\left(\frac{1}{2}\right) + 3(\gamma + \log 4)\zeta\left(\frac{1}{2}\right) - 3\zeta'\left(\frac{1}{2}\right) + 6X_3\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \right\} q \\ &\quad + 3\zeta\left(\frac{1}{2}\right)^3 q^{1/2} + 3\sum_{n=1}^{N-1} \frac{(-1)^n}{n!} \zeta(1-n) Y_{2,n}\left(\frac{1}{2}\right) q^{1-n} \\ &\quad + 3\sum_{n=1}^{N-1} \frac{(-1)^n}{n!} \zeta\left(\frac{1}{2} - n\right) \left\{ Y_{2,n}(1) + 2Y_{3,n}\left(\frac{1}{2}, \frac{1}{2}\right) \right\} q^{1/2-n} \\ &\quad + O(q^{1-N+\varepsilon}), \end{aligned} \tag{1.11}$$

where γ is Euler's constant.

The proof of Theorem 3 is given in the last section. Asymptotic expansion formulas for higher power moments can be deduced similarly; the calculations are very complicated but straightforward.

REMARK 4. From (3.9), we find that $Y_{2,n}(1) = n!\zeta(n+1)$ and

$$Y_{2,n}(\tfrac{1}{2}) = \frac{1}{\sqrt{\pi}} \Gamma(n + \tfrac{1}{2})\zeta(n + \tfrac{1}{2}).$$

The explicit form of $Y_{3,n}(1/2, 1/2)$ can be found by (1.4) or (9.2).

REMARK 5. A similar problem is the evaluation of the mean value

$$\tilde{V}_h(s, q) = \sum_{\chi \bmod q} |L(s, \chi)|^{2h},$$

where χ denotes a Dirichlet character mod q and $L(s, \chi)$ denotes the associated Dirichlet L-function. The mean square value $\tilde{V}_1(s, q)$ where $h = 1$ has been studied extensively by many people, for example Heath-Brown [**6**], Motohashi [**16**] and Katsurada and Matsumoto [**8**, **10**], and this quantity is closely connected with $\tilde{U}_1(s, q)$. (See [**12**, Section 10].) However, for $h \geqslant 2$, it seems that there is no direct connection between $\tilde{U}_h(s, q)$ and $\tilde{V}_h(s, q)$.

2. *The multiple integral expression for $\zeta_r(s_1, \ldots, s_r; \alpha)$*

In the following sections, we use the notation

$$\mathbf{s}(j,k) = s_j + s_{j+1} + \ldots + s_k, \quad \mathbf{z}(j,k) = z_j + z_{j+1} + \ldots + z_k$$
$$\mathbf{c}(j,k) = c_j + c_{j+1} + \ldots + c_k, \quad \mathbf{n}(j,k) = n_j + n_{j+1} + \ldots + n_k$$

for brevity.

The starting point of our method is the formula

$$\Gamma(s)(1+\lambda)^{-s} = \frac{1}{2\pi i}\int_{(c)} \Gamma(s+z)\Gamma(-z)\lambda^z dz, \tag{2.1}$$

where s and λ are complex, $\Re s > 0$, $|\arg\lambda| < \pi$, $\lambda \neq 0$, $\Re s < 0 < 0$, and the path is the vertical line from $c - i\infty$ to $c + i\infty$. (See Whittaker and Watson [**18**, Section 14.51, Corollary, p. 289].)

Let $r \geqslant 2$, and let c_r be real with $-\Re s_r < c_r < -1$. We first assume that $\Re s_j > 1$ for $1 \leqslant j \leqslant r$. Putting $s = s_r$, $c = c_r$, $z = z_r$ and

$$\lambda = \frac{m_r}{\alpha + m_1 + m_2 + \ldots + m_{r-1}}$$

in (2.1), dividing both sides by

$$\Gamma(s_r)(\alpha+m_1)^{s_1}\ldots(\alpha+m_1+\ldots+m_{r-2})^{s_{r-2}}(\alpha+m_1+\ldots+m_{r-1})^{s_{r-1}+s_r},$$

and summing up with respect to $m_1, \ldots, m_r$, we obtain

$$\zeta_r(s_1,\ldots,s_r;\alpha) = \frac{1}{2\pi i}\int_{(c_r)} \frac{\Gamma(s_r+z_r)}{\Gamma(s_r)}\Gamma(-z_r)\zeta(-z_r) \times \zeta_{r-1}(s_1,\ldots,s_{r-2},s_{r-1}+s_r+z_r;\alpha)dz_r. \tag{2.2}$$

Note that, in [**14**], this formula was used essentially in the proof of analytic continuation of $\zeta_r(s_1,\ldots,s_r;\alpha)$.

By the same method, we have

$$\begin{aligned}&\zeta_{r-1}(s_1,\ldots,s_{r-2},s_{r-1}+s_r+z_r;\alpha)\\ &= \frac{1}{2\pi i}\int_{(c_{r-1})} \frac{\Gamma(s_{r-1}+s_r+z_r+z_{r-1})}{\Gamma(s_{r-1}+s_r+z_r)}\Gamma(-z_{r-1})\zeta(-z_{r-1})\\ &\times \zeta_{r-2}(s_1,\ldots,s_{r-3},s_{r-2}+s_{r-1}+s_r+z_r+z_{r-1};\alpha)dz_{r-1},\end{aligned} \tag{2.3}$$

where $-\Re(s_{r-1}+s_r) - c_r < c_{r-1} < -1$. Substituting (2.3) into (2.2), we obtain a double integral expression for $\zeta_r(s_1,\ldots,s_r;\alpha)$. Repeating this procedure, we finally arrive at

$$\begin{aligned}\zeta_r(s_1,\ldots,s_r;\alpha) &= \frac{1}{(2\pi i)^{r-1}}\int_{(c_r)}\cdots\int_{(c_2)}\left\{\prod_{k=2}^{r}\frac{\Gamma(\mathbf{s}(k,r)+\mathbf{z}(k,r))}{\Gamma(\mathbf{s}(k,r)+\mathbf{z}(k+1,r))}\right.\\ &\quad\left.\times\Gamma(-z_k)\zeta(-z_k)\right\}\zeta(\mathbf{s}(1,r)+\mathbf{z}(2,r);\alpha)dz_2\ldots dz_r\\ &= \frac{1}{\Gamma(s_r)(2\pi i)^{r-1}}\int_{(c_r)}\cdots\int_{(c_3)}\left\{\prod_{k=3}^{r}\frac{\Gamma(\mathbf{s}(k,r)+\mathbf{z}(k,r))}{\Gamma(\mathbf{s}(k-1,r)+\mathbf{z}(k,r))}\right.\\ &\quad\left.\times\Gamma(-z_k)\zeta(-z_k)\right\}dz_3\ldots dz_r\int_{(c_2)}\Gamma(\mathbf{s}(2,r)+\mathbf{z}(2,r))\\ &\quad\times\Gamma(-z_2)\zeta(-z_2)\zeta(\mathbf{s}(1,r)+\mathbf{z}(2,r);\alpha)dz_2\end{aligned} \tag{2.4}$$

for any $r \geqslant 2$, where c_k satisfies

$$-\Re(\mathbf{s}(k,r)) - \mathbf{c}(k+1,r) < c_k < -1, \qquad 2 \leqslant k \leqslant r. \tag{2.5}$$

Since the relation

$$\sum_{a=1}^{q} \zeta(z, a/q) = q^z \zeta(z) \tag{2.6}$$

holds, putting $\alpha = a/q$ in (2.4) and summing up with respect to a, we obtain

$$\begin{aligned} J_r(s_1, \dots, s_r; q) &= \frac{1}{\Gamma(s_r)(2\pi i)^{r-1}} \int_{(c_r)} \cdots \int_{(c_3)} \left\{ \prod_{k=3}^{r} \frac{\Gamma(\mathbf{s}(k,r) + \mathbf{z}(k,r))}{\Gamma(\mathbf{s}(k-1,r) + \mathbf{z}(k,r))} \right. \\ &\quad \left. \times \Gamma(-z_k)\zeta(-z_k) \right\} dz_3 \dots dz_r \int_{(c_2)} \Gamma(\mathbf{s}(2,r) + \mathbf{z}(2,r)) \\ &\quad \times \Gamma(-z_2)\zeta(-z_2)\zeta(\mathbf{s}(1,r) + \mathbf{z}(2,r)) q^{\mathbf{s}(1,r)+\mathbf{z}(2,r)} dz_2 \end{aligned} \tag{2.7}$$

for any $r \geqslant 2$.

3. *The asymptotic expansion*

Let N be a positive integer satisfying the condition

$$N > \Re s_1 - 1 + \varepsilon. \tag{3.1}$$

We shift the path of the innermost integration on the right-hand side of (2.7) to $\Re z_2 = c_2(N)$, where

$$c_2(N) = -\Re(\mathbf{s}(2,r)) - \mathbf{c}(3,r) - N + \varepsilon.$$

From (2.5) and (3.1), we find that the poles of the integrand, as a function in z_2, lying in the strip $c_2(N) < \Re z_2 < c_2$ are at

$$z_2 = 1 - (\mathbf{s}(1,r) + \mathbf{z}(3,r)) \tag{3.2}$$

and

$$z_2 = -n - (\mathbf{s}(2,r) + \mathbf{z}(3,r)), \qquad 0 \leqslant n \leqslant N-1. \tag{3.3}$$

Now we assume that s_1 is not a positive integer. Then all the above poles are simple. Hence, counting the residues of those poles, we obtain

$$\begin{aligned} &\frac{1}{2\pi i} \int_{(c_2)} \Gamma(\mathbf{s}(2,r) + \mathbf{z}(2,r))\Gamma(-z_2)\zeta(-z_2)\zeta(\mathbf{s}(1,r) + \mathbf{z}(2,r)) q^{\mathbf{s}(1,r)+\mathbf{z}(2,r)} dz_2 \\ &= \Gamma(1-s_1)\Gamma(-1 + \mathbf{s}(1,r) + \mathbf{z}(3,r))\zeta(-1 + \mathbf{s}(1,r) + \mathbf{z}(3,r)) q \\ &\quad + \sum_{n=0}^{N-1} \frac{(-1)^n}{n!} \Gamma(n + \mathbf{s}(2,r) + \mathbf{z}(3,r))\zeta(n + \mathbf{s}(2,r) + \mathbf{z}(3,r))\zeta(s_1 - n) q^{s_1 - n} \\ &\quad + \frac{1}{2\pi i} \int_{(c_2(N))} \Gamma(\mathbf{s}(2,r) + \mathbf{z}(2,r))\Gamma(-z_2)\zeta(-z_2)\zeta(\mathbf{s}(1,r) + \mathbf{z}(2,r)) q^{\mathbf{s}(1,r)+\mathbf{z}(2,r)} dz_2. \end{aligned}$$

Substituting this into (2.7), and putting

$$z_2' = \mathbf{s}(2,r) + \mathbf{z}(2,r) + N,$$

we obtain

$$\begin{aligned} J_r(s_1,\ldots,s_r;q) &= X_r(s_1,\ldots,s_r)q \\ &\quad + \sum_{n=0}^{N-1}\frac{(-1)^n}{n!}\zeta(s_1-n)Y_{r,n}(s_2,\ldots,s_r)q^{s_1-n} \\ &\quad + Z_{r,N}(s_1,\ldots,s_r;\varepsilon,q) \end{aligned} \tag{3.4}$$

for $r \geqslant 2$, where $\Re s_j > 1$ $(1 \leqslant j \leqslant r)$, s_1 is not a positive integer,

$$\begin{aligned} X_r(s_1,\ldots,s_r) &= \frac{\Gamma(1-s_1)}{(2\pi i)^{r-2}\Gamma(s_r)}\int_{(c_r)}\cdots\int_{(c_3)}\Bigg\{\prod_{k=3}^{r}\Gamma(-z_k)\zeta(-z_k) \\ &\quad \times \frac{\Gamma(\mathbf{s}(k,r)+\mathbf{z}(k,r))}{\Gamma(\mathbf{s}(k-1,r)+\mathbf{z}(k,r))}\Bigg\}\Gamma(-1+\mathbf{s}(1,r)+\mathbf{z}(3,r)) \\ &\quad \times \zeta(-1+\mathbf{s}(1,r)+\mathbf{z}(3,r))dz_3\ldots dz_r, \end{aligned} \tag{3.5}$$

$$\begin{aligned} Y_{r,n}(s_2,\ldots,s_r) &= \frac{1}{(2\pi i)^{r-2}\Gamma(s_r)}\int_{(c_r)}\cdots\int_{(c_3)}\Bigg\{\prod_{k=3}^{r}\Gamma(-z_k)\zeta(-z_k) \\ &\quad \times \frac{\Gamma(\mathbf{s}(k,r)+\mathbf{z}(k,r))}{\Gamma(\mathbf{s}(k-1,r)+\mathbf{z}(k,r))}\Bigg\}\Gamma(n+\mathbf{s}(2,r)+\mathbf{z}(3,r)) \\ &\quad \times \zeta(n+\mathbf{s}(2,r)+\mathbf{z}(3,r))dz_3\ldots dz_r, \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} Z_{r,N}(s_1,\ldots,s_r;\varepsilon,q) &= \frac{1}{(2\pi i)^{r-1}\Gamma(s_r)}\int_{(\varepsilon)}\Gamma(z_2'-N)\zeta(s_1+z_2'-N) \\ &\quad \times q^{s_1+z_2'-N}dz_2'\int_{(c_r)}\cdots\int_{(c_3)}\Bigg\{\prod_{k=3}^{r}\Gamma(-z_k)\zeta(-z_k) \\ &\quad \times \frac{\Gamma(\mathbf{s}(k,r)+\mathbf{z}(k,r))}{\Gamma(\mathbf{s}(k-1,r)+\mathbf{z}(k,r))}\Bigg\}\Gamma(-z_2'+\mathbf{s}(2,r)+\mathbf{z}(3,r)+N) \\ &\quad \times \zeta(-z_2'+\mathbf{s}(2,r)+\mathbf{z}(3,r)+N)dz_3\ldots dz_r. \end{aligned} \tag{3.7}$$

Note that the change of the order of integration in (3.7) can be easily verified by Fubini's theorem.

Formula (3.4) already gives expansion (1.3) in the region $\Re s_j > 1$ $(1 \leqslant j \leqslant r)$. Therefore the remaining task is to prove the analytic continuation of X_r, $Y_{r,n}$ and $Z_{r,N}$. This will be done by induction. As the first step, we remark here that the analytic continuation in the case $r = 2$ has already been shown.

In fact, when $r = 2$, (3.5)–(3.7) reduce to

$$X_2(s_1,s_2) = \frac{\Gamma(1-s_1)}{\Gamma(s_2)}\Gamma(-1+s_1+s_2)\zeta(-1+s_1+s_2), \tag{3.8}$$

$$Y_{2,n}(s_2) = \frac{1}{\Gamma(s_2)}\Gamma(n+s_2)\zeta(n+s_2), \tag{3.9}$$

and

$$Z_{2,N}(s_1,s_2;\varepsilon,q)=\frac{1}{2\pi i\Gamma(s_2)}\int_{(\varepsilon)}\Gamma(z_2'-N)\zeta(s_1+z_2'-N)$$
$$\times\Gamma(-z_2'+s_2+N)\zeta(-z_2'+s_2+N)q^{s_1+z_2'-N}dz_2'. \qquad (3.10)$$

Hence clearly X_2 is meromorphic in $\mathbb{C}^2$ and $Y_{2,n}$ is meromorphic in $\mathbb{C}$. As for $Z_{2,N}$, we see that the poles of the integrand on the right-hand side of (3.10), as a function in z_2', are located only at $z_2'=N-n$ $(n\in\mathbb{N}_0)$, s_2+N+n $(n\in\mathbb{N}_0)$, $1-s_1+N$ and $-1+s_2+N$. Hence $Z_{2,N}$ can be continued to (and holomorphic in) $\mathscr{D}_2(N;\varepsilon)$, because the above poles do not lie on the path of integration if $(s_1,s_2)\in\mathscr{D}_2(N;\varepsilon)$.

Lastly in this section, we consider the case when s_1 is a positive integer. Let $s_1=m+1$, $m=0,1,\dots,N-1$. We can replace the original assumption $\Re s_j>1$ $(1\leqslant j\leqslant r)$ by $\Re s_1>0$, $\Re s_2>2$ and $\Re s_j>1$ $(3\leqslant j\leqslant r)$, and hence we can include the case $m=0$ in our consideration. In this case, pole (3.2) coincides with pole (3.3) with $n=m$, and hence this is a double pole. Note that $\zeta(1+\delta)=\delta^{-1}+\gamma+O(\delta)$ and

$$\Gamma(-m+\delta)=\frac{(-1)^m}{m!}\left\{\frac{1}{\delta}+\left(1+\frac{1}{2}+\dots+\frac{1}{m}-\gamma\right)+O(\delta)\right\}$$

for small $\delta>0$, where γ is Euler's constant. Using these facts, we can easily see that the residue of the above double pole of the innermost integrand on the right-hand side of (2.7) is

$$\frac{(-1)^m}{m!}\left\{\left(1+\frac{1}{2}+\dots+\frac{1}{m}+\log q\right)\Gamma(u_m)\zeta(u_m)-\Gamma'(u_m)\zeta(u_m)-\Gamma(u_m)\zeta'(u_m)\right\}q,$$

where we temporarily write

$$u_m=m+\mathbf{s}(2,r)+\mathbf{z}(3,r)$$

for brevity. The contribution of this residue to $J_r(s_1,\dots,s_r;q)$ is

$$\frac{(-1)^m}{m!}\left(1+\frac{1}{2}+\dots+\frac{1}{m}+\log q\right)Y_{r,m}(s_2,\dots,s_r)q-\frac{(-1)^m}{m!}Y^*_{r,m}(s_2,\dots,s_r)q,$$

where

$$Y^*_{r,m}(s_2,\dots,s_r)=\frac{1}{(2\pi i)^{r-2}\Gamma(s_r)}\int_{(c_r)}\cdots\int_{(c_3)}$$
$$\times\left\{\prod_{k=3}^{r}\frac{\Gamma(\mathbf{s}(k,r)+\mathbf{z}(k,r))}{\Gamma(\mathbf{s}(k-1,r)+\mathbf{z}(k,r))}\Gamma(-z_k)\zeta(-z_k)\right\}\{\Gamma'(u_m)\zeta(u_m)$$
$$+\Gamma(u_m)\zeta'(u_m)\}dz_3\dots dz_r. \qquad (3.11)$$

This implies formula (1.6) in Remark 1. We omit the proof of the meromorphic continuation of $Y^*_{r,m}(s_2,\dots,s_r)$ to the whole $\mathbb{C}^{r-1}$ space because it can be done in much the same way as the proof of the continuation of X_r presented in the next section.

4. *Analytic continuation of* $X_r(s_1,\dots,s_r)$

In this section, we prove that $X_r(s_1,\dots,s_r)$, $r\geqslant 3$, can be continued meromorphically to the whole $\mathbb{C}^r$ space. First assume that $\Re s_j>1$ $(1\leqslant j\leqslant r)$ and that s_1 is not a positive integer. From (3.5), we have

$$X_r(s_1,\dots,s_r)=\frac{\Gamma(1-s_1)}{(2\pi i)^{r-2}\Gamma(s_r)}\int_{(c_{r-1})}\cdots\int_{(c_3)}\left\{\prod_{k=3}^{r-1}\Gamma(-z_k)\zeta(-z_k)\right\}dz_3\dots dz_{r-1}$$
$$\times\int_{(c_r)}\left\{\prod_{k=3}^{r}\frac{\Gamma(\mathbf{s}(k,r)+\mathbf{z}(k,r))}{\Gamma(\mathbf{s}(k-1,r)+\mathbf{z}(k,r))}\right\}\Gamma(-z_r)\zeta(-z_r)$$
$$\times\Gamma(-1+\mathbf{s}(1,r)+\mathbf{z}(3,r))\zeta(-1+\mathbf{s}(1,r)+\mathbf{z}(3,r))dz_r. \tag{4.1}$$

We shift the path of the innermost integral to $\Re z_r=M-\varepsilon$, where M is a positive integer. From (2.5) we have

$$-\Re(\mathbf{s}(k,r))-\mathbf{c}(k,r-1)<c_r,\qquad 2\leqslant k\leqslant r,$$

and hence we see that the poles

$$z_r=-n-\mathbf{s}(k,r)-\mathbf{z}(k,r-1),\qquad n\in\mathbb{N}_0,$$

and

$$z_r=2-n-\mathbf{s}(1,r)-\mathbf{z}(3,r-1),\qquad n\in\mathbb{N}_0,$$

of the integrand are all located on the left-hand side of the original path $\Re z_r=c_r$. (As for the latter, we use $\Re(2-n-s_1-s_2)<0$.) Therefore the only relevant poles are at $z_r=-1,0,1,\dots,M-1$. Counting the residues of those poles, we obtain

$$X_r(s_1,\dots,s_r)=X_r^{(1)}(-1)+\sum_{m=0}^{M-1}\frac{(-1)^m}{m!}\zeta(-m)X_r^{(1)}(m)+X_r^{(2)}(M), \tag{4.2}$$

where

$$X_r^{(1)}(m)=X_r^{(1)}(m;s_1,\dots,s_r)$$
$$=\frac{\Gamma(1-s_1)}{(2\pi i)^{r-3}\Gamma(s_r)}\int_{(c_{r-1})}\cdots\int_{(c_3)}\left\{\prod_{k=3}^{r-1}\Gamma(-z_k)\zeta(-z_k)\right\}$$
$$\times\left\{\prod_{k=3}^{r}\frac{\Gamma(\mathbf{s}(k,r)+\mathbf{z}(k,r-1)+m)}{\Gamma(\mathbf{s}(k-1,r)+\mathbf{z}(k,r-1)+m)}\right\}$$
$$\times\Gamma(-1+\mathbf{s}(1,r)+\mathbf{z}(3,r-1)+m)$$
$$\times\zeta(-1+\mathbf{s}(1,r)+\mathbf{z}(3,r-1)+m)dz_3\dots dz_{r-1} \tag{4.3}$$

for $-1\leqslant m\leqslant M-1$, and $X_r^{(2)}(M)=X_r^{(2)}(M;s_1,\dots,s_r)$ is almost the same as $X_r(s_1,\dots,s_r)$, but with the path (c_r) replaced by $(M-\varepsilon)$. Define

$$\mathscr{A}_r(M;\varepsilon)=\left\{(s_1,\dots,s_r)\in\mathbb{C}^r\;\middle|\;\begin{array}{ll}\Re(\mathbf{s}(k,r))>-\mathbf{c}(k,r-1)-M+\varepsilon, & 3\leqslant k\leqslant r,\\ \Re(\mathbf{s}(1,r))>2-\mathbf{c}(3,r-1)-M+\varepsilon & \end{array}\right\}.$$

Then we have the following lemma.

Lemma 1. *The function $X_r^{(2)}(M;s_1,\dots,s_r)$ can be continued meromorphically to $\mathscr{A}_r(M;\varepsilon)$, and singularities are only on $s_1=1+n$ $(n\in\mathbb{N}_0)$.*

This lemma easily follows from the fact that the poles of the innermost integrand of $X_r^{(2)}(M)$ are located only at

$$
\begin{aligned}
&z_r = -n - (\mathbf{s}(k,r) + \mathbf{z}(k,r-1)), && n \in \mathbb{N}_0,\ 3 \leqslant k \leqslant r,\\
&z_r = 2 - n - (\mathbf{s}(1,r) + \mathbf{z}(3,r-1)) && n \in \mathbb{N}_0,\\
&z_r = n && n \in \mathbb{N}_0,\\
&z_r = -1.
\end{aligned}
$$

The argument is similar to the proof of holomorphic continuation of $Z_{2,N}$ given in Section 3.

Now we prove Theorem 1(i) and Remark 2(i) by induction. First consider the case $r = 3$. Then (4.3) reduces to

$$
\begin{aligned}
X_3^{(1)}(m) &= \frac{\Gamma(1-s_1)\Gamma(s_3+m)}{\Gamma(s_3)\Gamma(s_2+s_3+m)}\Gamma(-1+s_1+s_2+s_3+m)\\
&\quad \times \zeta(-1+s_1+s_2+s_3+m), \qquad -1 \leqslant m \leqslant M-1, \qquad (4.4)
\end{aligned}
$$

which is clearly meromorphic in $\mathbb{C}^3$. Hence, combining with Lemma 1, we find that $X_3(s_1,s_2,s_3)$ is meromorphic in $\mathscr{A}_3(M;\varepsilon)$. Since M is arbitrary, this implies that $X_3(s_1,s_2,s_3)$ can be continued meromorphically to the whole $\mathbb{C}^3$. Moreover, we see that the possible singularities of $X_3(s_1,s_2,s_3)$ are only on $s_1 = 1+n$ ($n \in \mathbb{N}_0$), $s_3 = 1$, and $s_1+s_2+s_3 = 3-n$ ($n \in \mathbb{N}_0$). Hence the case $r = 3$ is established.

Now let $r \geqslant 4$. Putting $s_{r-1}^*(m) = s_{r-1}+s_r+m$ in (4.3), we have

$$
\begin{aligned}
&X_r^{(1)}(m;s_1,\dots,s_r)\\
&\quad = \frac{\Gamma(1-s_1)}{(2\pi i)^{r-3}\Gamma(s_r)}\int_{(c_{r-1})}\cdots\int_{(c_3)}\left\{\prod_{k=3}^{r-1}\Gamma(-z_k)\zeta(-z_k)\right\}\\
&\quad\quad \times\left\{\prod_{k=3}^{r-1}\frac{\Gamma(\mathbf{s}(k,r-2)+s_{r-1}^*(m)+\mathbf{z}(k,r-1))}{\Gamma(\mathbf{s}(k-1,r-2)+s_{r-1}^*(m)+\mathbf{z}(k,r-1))}\right\}\frac{\Gamma(s_r+m)}{\Gamma(s_{r-1}^*(m))}\\
&\quad\quad \times\Gamma(-1+\mathbf{s}(1,r-2)+s_{r-1}^*(m)+\mathbf{z}(3,r-1))\\
&\quad\quad \times\zeta(-1+\mathbf{s}(1,r-2)+s_{r-1}^*(m)+\mathbf{z}(3,r-1))dz_3\dots dz_{r-1}\\
&\quad = \frac{\Gamma(s_r+m)}{\Gamma(s_r)}X_{r-1}(s_1,\dots,s_{r-2},s_{r-1}^*(m)) \qquad -1 \leqslant m \leqslant M-1. \qquad (4.5)
\end{aligned}
$$

Since $X_{r-1}(s_1,\dots,s_{r-2},s_{r-1}^*(m))$ is meromorphic by the induction assumption, (4.5) implies the meromorphy of $X_r^{(1)}(m;s_1,\dots,s_r)$, $-1 \leqslant m \leqslant M-1$. From this, (4.2) and Lemma 1 it follows that $X_r(s_1,\dots,s_r)$ is meromorphic in $\mathscr{A}_r(M;\varepsilon)$ for any M, and hence is meromorphic in the whole $\mathbb{C}^r$ space. Moreover, by the induction assumption, the singularities of $X_{r-1}(s_1,\dots,s_{r-2},s_{r-1}^*(m))$ are located on one of the subsets defined by

$$
\begin{aligned}
s_1 &= n+1, && n \in \mathbb{N}_0,\\
s_{r-1}^*(m) &= 1,\\
\mathbf{s}(k,r-2)+s_{r-1}^*(m) &= r-k-n, && 3 \leqslant k \leqslant r-2,\ n \in \mathbb{N}_0,\\
\mathbf{s}(1,r-2)+s_{r-1}^*(m) &= r-1-n, && n \in \mathbb{N}_0.
\end{aligned}
$$

Therefore the assertion of Remark 2(i) for $X_r(s_1,\dots,s_r)$ follows from (4.2), (4.5) and Lemma 1.

5. *Analytic continuation of* $Z_{r,N}(s_1,\dots,s_r;\varepsilon,q)$

In this section, we prove the meromorphic continuation of $Z_{r,N}(s_1,\dots,s_r;\varepsilon,q)$, $r \geqslant 3$, to $\mathscr{D}_r(N;\varepsilon)$. The basic structure of the argument is similar to that in the preceding section.

First, in a similar way to (4.2) and (4.3), we have

$$Z_{r,N}(s_1,\dots,s_r;\varepsilon,q) = Z_{r,N}^{(1)}(-1) + \sum_{m=0}^{M-1} \frac{(-1)^m}{m!}\zeta(-m)Z_{r,N}^{(1)}(m) + Z_{r,N}^{(2)}(M), \tag{5.1}$$

where

$$\begin{aligned} Z_{r,N}^{(1)}(m) &= Z_{r,N}^{(1)}(m;s_1,\dots,s_r;\varepsilon,q) \\ &= \frac{1}{(2\pi i)^{r-2}\Gamma(s_r)}\int_{(\varepsilon)}\Gamma(z_2'-N)\zeta(s_1+z_2'-N)q^{s_1+z_2'-N}dz_2' \\ &\quad \times \int_{(c_{r-1})}\cdots\int_{(c_3)}\left\{\prod_{k=3}^{r-1}\Gamma(-z_k)\,\zeta(-z_k)\right\} \\ &\quad \times \left\{\prod_{k=3}^{r}\frac{\Gamma(\mathbf{s}(k,r)+\mathbf{z}(k,r-1)+m)}{\Gamma(\mathbf{s}(k-1,r)+\mathbf{z}(k,r-1)+m)}\right\} \\ &\quad \times \Gamma(-z_2'+\mathbf{s}(2,r)+\mathbf{z}(3,r-1)+m+N) \\ &\quad \times \zeta(-z_2'+\mathbf{s}(2,r)+\mathbf{z}(3,r-1)+m+N)dz_3\dots dz_{r-1} \end{aligned} \tag{5.2}$$

for $-1 \leqslant m \leqslant M-1$, and $Z_{r,N}^{(2)}(M) = Z_{r,N}^{(2)}(M;s_1,\dots,s_r;\varepsilon,q)$ is almost the same as $Z_{r,N}(s_1,\dots,s_r;\varepsilon,q)$, but with the path (c_r) replaced by $(M-\varepsilon)$. Let $\mathscr{B}_r(M,N;\varepsilon)$ be the subset of $\mathbf{C}^r$ that consists of all $(s_1,\dots,s_r)$ satisfying the conditions

$$\Re s_1 < N+1-\varepsilon,$$

$$\Re(\mathbf{s}(k,r)) > -\mathbf{c}(k,r-1)-M+\varepsilon, \qquad 3 \leqslant k \leqslant r,$$

and

$$\Re(\mathbf{s}(2,r)) > -\mathbf{c}(3,r-1)-M-N+1+\varepsilon.$$

Then we have the following lemma.

LEMMA 2. *The function* $Z_{r,N}^{(2)}(M;s_1,\dots,s_r;\varepsilon,q)$ *can be continued holomorphically to* $\mathscr{B}_r(M,N;\varepsilon)$, *and the estimate*

$$Z_{r,N}^{(2)}(M;s_1,\dots,s_r;\varepsilon,q) = O(q^{\Re s_1 - N+\varepsilon}) \tag{5.3}$$

holds in that region.

Indeed, the continuation can be shown in a similar way to Lemma 1, and estimate (5.3) is clear.

Now we prove Theorem 1(iii) and Remark 2(ii) by induction. In a similar way to (4.5), we have

$$Z^{(1)}_{r,N}(m; s_1, \ldots, s_r; \varepsilon, q) = \frac{\Gamma(s_r + m)}{\Gamma(s_r)} Z_{r-1,N}(s_1, \ldots, s_{r-2}, s^*_{r-1}(m)), \qquad -1 \leqslant m \leqslant M-1, \tag{5.4}$$

for $r \geqslant 3$. In particular, since we have already shown in Section 3 that the function $Z_{2,N}(s_1, s_2; \varepsilon, q)$ is holomorphic in $\mathscr{D}_2(N; \varepsilon)$, from (5.4) we see that

(i) $Z^{(1)}_{3,N}(-1)$ is holomorphic except for the singularities on $s_3 = 1$ in the region $\Re s_1 < N + 1 - \varepsilon$, $\Re(s_2 + s_3) > 2 - N + \varepsilon$;

(ii) $Z^{(1)}_{3,N}(m)$ $(0 \leqslant m \leqslant M-1)$ is holomorphic in the region $\Re s_1 < N + 1 - \varepsilon$, $\Re(s_2 + s_3) > 1 - m - N + \varepsilon$.

Moreover, from (3.10), we have $Z_{2,N} = O(q^{\Re s_1 - N + \varepsilon})$, and hence, using (5.4) we find that

$$Z^{(1)}_{3,N}(m) = O(q^{\Re s_1 - N + \varepsilon}), \qquad -1 \leqslant m \leqslant M-1,$$

except for the singularities. Therefore, combining this with the case $r = 3$ of Lemma 2, we can conclude that, in the region

$$\left\{ (s_1, s_2, s_3) \in \mathbb{C}^r \;\middle|\; \begin{array}{l} \Re s_1 < N + 1 - \varepsilon \\ \Re(s_2 + s_3) > 2 - N + \varepsilon \\ \Re s_3 > -M + \varepsilon \end{array} \right\},$$

the function $Z_{3,N}$ is holomorphic and the estimate (1.5) holds, except for the singularities on $s_3 = 1$. Since M is arbitrary, the desired assertions follow for $r = 3$. Then, using Lemma 2, (5.4) and the above information about the case $r = 3$, we find that $Z_{4,N}$ can be continued meromorphically to $\mathscr{D}_4(N; \varepsilon)$, and the singularities are located only on $s_4 = 1$ and $s_3 + s_4 = 2 - n$ $(n \in \mathbb{N}_0)$, as desired. Similarly, we can prove the assertions for all $r \geqslant 5$ inductively.

6. *The explicit form of* $Y_{r,n}(s_2, \ldots, s_r)$

The purpose of this section is to prove relation (1.4). The case $r = 2$ has already been dealt with by (3.9), and hence we can assume that $r \geqslant 3$. Therefore we can use expression (2.4) for ζ_{r-1}. Using it and putting $\alpha = 1$ and replacing c_k, s_k, z_k by c_{k+1}, $s_{k+1} + n_{k+1}$, z_{k+1}, respectively $(1 \leqslant k \leqslant r-1)$, we find that the right-hand side of (1.4) is

$$\begin{aligned} &\sum_{n_2 + \ldots + n_r = n} \frac{n!}{n_2! \ldots n_r!} \frac{\Gamma(s_2 + n_2) \ldots \Gamma(s_{r-1} + n_{r-1})}{\Gamma(s_2) \ldots \Gamma(s_{r-1})} \\ &\times \frac{1}{(2\pi i)^{r-2} \Gamma(s_r)} \int_{(c_r)} \cdots \int_{(c_4)} \left\{ \prod_{k=4}^{r} \frac{\Gamma(\mathbf{s}(k,r) + \mathbf{n}(k,r) + \mathbf{z}(k,r))}{\Gamma(\mathbf{s}(k-1,r) + \mathbf{n}(k-1,r) + \mathbf{z}(k,r))} \right. \\ &\left. \times \Gamma(-z_k)\zeta(-z_k) \right\} dz_4 \ldots dz_r \int_{(c_3)} \Gamma(\mathbf{s}(3,r) + \mathbf{n}(3,r) + \mathbf{z}(3,r)) \\ &\times \Gamma(-z_3)\zeta(-z_3)\zeta(\mathbf{s}(2,r) + n + \mathbf{z}(3,r)) dz_3. \end{aligned}$$

Comparing this with expression (3.6) of the left-hand side of (1.4), we find that, if we can show that

$$\left\{\prod_{k=3}^{r}\frac{\Gamma(\mathbf{s}(k,r)+\mathbf{z}(k,r))}{\Gamma(\mathbf{s}(k-1,r)+\mathbf{z}(k,r))}\right\}\Gamma(n+\mathbf{s}(2,r)+\mathbf{z}(3,r))$$
$$=\sum_{n_2+\ldots+n_r=n}\frac{n!}{n_2!\ldots n_r!}\frac{\Gamma(s_2+n_2)\ldots\Gamma(s_{r-1}+n_{r-1})}{\Gamma(s_2)\ldots\Gamma(s_{r-1})}$$
$$\times\left\{\prod_{k=4}^{r}\frac{\Gamma(\mathbf{s}(k,r)+\mathbf{n}(k,r)+\mathbf{z}(k,r))}{\Gamma(\mathbf{s}(k-1,r)+\mathbf{n}(k-1,r)+\mathbf{z}(k,r))}\right\}$$
$$\times\Gamma(\mathbf{s}(3,r)+\mathbf{n}(3,r)+\mathbf{z}(3,r)), \tag{6.1}$$

then (1.4) will follow.

We use the Pochhammer symbol $(a)_n=\Gamma(a+n)/\Gamma(a)$ for any number a and any $n\in\mathbb{N}_0$. Equation (6.1) is equivalent to the following.

LEMMA 3. *The identity*

$$\sum_{n_2+\ldots+n_r=n}\frac{n!}{n_2!\ldots n_r!}(s_2)_{n_2}\ldots(s_{r-1})_{n_{r-1}}\prod_{k=3}^{r}\frac{(\mathbf{s}(k,r)+\mathbf{z}(k,r))_{\mathbf{n}(k,r)}}{(\mathbf{s}(k-1,r)+\mathbf{z}(k,r))_{\mathbf{n}(k-1,r)}}=1 \tag{6.2}$$

is valid for any $r\geqslant 3$ and any $n\in\mathbb{N}_0$, where the summation runs over all non-negative integers $n_2,\ldots,n_r$ satisfying $n_2+\ldots+n_r=n$.

The formula (6.2) can be rewritten as

$$(\mathbf{s}(2,r)+\mathbf{z}(3,r))_n=\sum_{n_2+\ldots+n_r=n}\frac{n!}{n_2!\ldots n_r!}(s_2)_{n_2}(\mathbf{s}(3,r)+\mathbf{z}(3,r))_{\mathbf{n}(3,r)}$$
$$\times\prod_{k=4}^{r}\frac{(s_{k-1})_{n_{k-1}}(\mathbf{s}(k,r)+\mathbf{z}(k,r))_{\mathbf{n}(k,r)}}{(\mathbf{s}(k-1,r)+\mathbf{z}(k,r))_{\mathbf{n}(k-1,r)}}. \tag{6.3}$$

We prove this by induction. First we note that the following binomial theorem for the Pochhammer symbol holds:

$$(a+b)_n=\sum_{m=0}^{n}\binom{n}{m}(a)_{n-m}(b)_m \tag{6.4}$$

for any numbers a, b and any $n\in\mathbb{N}_0$. This is a classical formula of Nörlund (cf. Berge [**4**, Section 1, Chapter 3]), or it can easily be shown by induction. Putting $a=s_2$ and $b=s_3+z_3$ in (6.4), we find that the case $r=3$ of (6.3) immediately follows.

Now let $r\geqslant 4$. Putting $a=s_2$ and $b=\mathbf{s}(3,r)+\mathbf{z}(3,r)$ in (6.4), we have

$$(\mathbf{s}(2,r)+\mathbf{z}(3,r))_n=\sum_{n_2+m=n}\frac{n!}{n_2!m!}(s_2)_{n_2}(\mathbf{s}(3,r)+\mathbf{z}(3,r))_m. \tag{6.5}$$

By the induction assumption, (6.2) holds for $r-1$, that is,

$$1 = \sum_{n_3+\ldots+n_r=m} \frac{m!}{n_3!\ldots n_r!}(s_3)_{n_3}\ldots(s_{r-1})_{n_{r-1}}$$
$$\times \prod_{k=4}^{r} \frac{(\mathbf{s}(k,r)+\mathbf{z}(k,r))_{\mathbf{n}(k,r)}}{(\mathbf{s}(k-1,r)+\mathbf{z}(k,r))_{\mathbf{n}(k-1,r)}}. \tag{6.6}$$

Multiplying (6.5) by (6.6), we obtain (6.3) for r. This completes the proof of Lemma 3, and hence that of (1.4). All the assertions of Theorem 1, Remark 1 and Remark 2 are now proved.

REMARK 6. We have proved (1.4) by induction, but the authors first guessed the correct form of the relation through a different observation. The first author of this paper [**5**] studied $J_3(s_1,s_2,s_3;q)$ by a different method, which naturally led to expression (1.4) (for $r=3$) in the domain of absolute convergence. This result guided the authors to the form (1.4) for general r. The method in [**5**] is based on a triple integral expression of $J_3(s_1,s_2,s_3;q)$, and this triple integral is treated in [**5**] by an argument analogous to that developed in Motohashi [**16**] and Katsurada and Matsumoto [**8**] in the double zeta case.

7. *An elementary argument for the evaluation of $U_h(s,q)$ and $\tilde{U}_h(s,q)$*

The rest of this paper is devoted to the study of power mean values $U_h(s,q)$ and $\tilde{U}_h(s,q)$ for Hurwitz zeta functions, defined by (1.7) and (1.8).

In this section, we show that asymptotic formulas for $U_h(s,q)$ and $\tilde{U}_h(s,q)$ with respect to q with the error term $O(q)$ can easily be proved by an elementary argument.

For $n \geqslant 2$, we have $|\alpha/n| \leqslant 1/2$ $(0<\alpha\leqslant 1)$, and hence

$$\left(1+\frac{\alpha}{n}\right)^{-s} = \sum_{m=0}^{M} \binom{-s}{m}\left(\frac{\alpha}{n}\right)^{m} + O\left(\left(\frac{\alpha}{n}\right)^{M+1}\right)$$

for any positive integer M, where the implied constant does not depend on α. Therefore

$$\zeta(s,\alpha) = \alpha^{-s} + (1+\alpha)^{-s} + \sum_{n\geqslant 2} n^{-s}\left(1+\frac{\alpha}{n}\right)^{-s}$$
$$= \alpha^{-s} + (1+\alpha)^{-s} + \sum_{m=0}^{M}\binom{-s}{m}(\zeta(s+m)-1)\alpha^{m} + O(\alpha^{M+1}) \tag{7.1}$$

uniformly for $0<\alpha\leqslant 1$, which is valid when $\sigma = \Re s > -M$.

Hereafter in this section, we assume that $\sigma \geqslant 0$ for simplicity. From (7.1), we have

$$U_h(s,q) = \sum_{a=1}^{q}\left\{\left(\frac{a}{q}\right)^{-s} + \left(1+\frac{a}{q}\right)^{-s}\right.$$
$$\left. + \sum_{m=0}^{M}\binom{-s}{m}(\zeta(s+m)-1)\left(\frac{a}{q}\right)^{m} + O\left(\left(\frac{a}{q}\right)^{M+1}\right)\right\}^{h}. \tag{7.2}$$

Note that

$$\sum_{a=1}^{q}\left(\frac{a}{q}\right)^{A}=q^{-A}\sum_{a=1}^{q}a^{A}\ll q \tag{7.3}$$

for any $A>-1$. The error term on the right-hand side of (7.2) contributes

$$\begin{aligned}&\ll\sum_{a=1}^{q}\sum_{\ell=0}^{h-1}\left|\left(\frac{a}{q}\right)^{-s}+\left(1+\frac{a}{q}\right)^{-s}\right.\\&\qquad\left.+\sum_{m=0}^{M}\binom{-s}{m}(\zeta(s+m)-1)\left(\frac{a}{q}\right)^{m}\right|^{\ell}\left(\frac{a}{q}\right)^{(M+1)(h-\ell)}\\&\ll\sum_{a=1}^{q}\left(\frac{a}{q}\right)^{-\sigma(h-1)+(M+1)},\end{aligned}$$

which is $O(q)$ in view of (7.3) if M is sufficiently large. Hence we have

$$\begin{aligned}U_h(s,q)=\sum_{a=1}^{q}\Bigg\{&\left(\frac{a}{q}\right)^{-s}+\left(1+\frac{a}{q}\right)^{-s}\\&+\sum_{m=0}^{M}\binom{-s}{m}(\zeta(s+m)-1)\left(\frac{a}{q}\right)^{m}\Bigg\}^{h}+O(q).\end{aligned} \tag{7.4}$$

By expanding the right-hand side of (7.4), we find that the typical term we should consider is of the form

$$\sum_{a=1}^{q}\left(\frac{a}{q}\right)^{-B}\left(1+\frac{a}{q}\right)^{-C}, \tag{7.5}$$

where B, C are complex numbers with $\Re C\geqslant 0$.

Since $(1+a/q)^{-C}=O(1)$, if $\Re B<1$ then (7.5) is $O(q)$ in view of (7.3). Consider the case $\Re B\geqslant 1$. Then

$$\begin{aligned}&\sum_{a=1}^{q}\left(\frac{a}{q}\right)^{-B}\left(1+\frac{a}{q}\right)^{-C}\\&\quad=\sum_{1\leqslant a\leqslant q/2}\left(\frac{a}{q}\right)^{-B}\left\{\sum_{j=0}^{J}\binom{-C}{j}\left(\frac{a}{q}\right)^{j}+O\left(\left(\frac{a}{q}\right)^{J+1}\right)\right\}\\&\qquad+O\left(\sum_{q/2<a\leqslant q}\left(\frac{a}{q}\right)^{-\Re B}\right)\end{aligned}$$

for sufficiently large J, and hence

$$=\sum_{0\leqslant j\leqslant \Re B-1}\binom{-C}{j}\sum_{1\leqslant a\leqslant q/2}\left(\frac{a}{q}\right)^{j-B}+O(q) \tag{7.6}$$

by (7.3). Now we use the formulas

$$\sum_{1 \leqslant a \leqslant q/2} a^{j-B} = \zeta(B-j) + O(q^{j-\Re B+1}), \qquad 0 \leqslant j < \Re B - 1,$$

$$\sum_{1 \leqslant a \leqslant q/2} a^{-1} = \log q + O(1),$$

and

$$\sum_{1 \leqslant a \leqslant q/2} a^{-1-i\Im B} = \zeta(1+i\Im B) + O(1) = O(1) \qquad \Im B \neq 0.$$

As for the proof of the last one, see the formula stated at the end of Titchmarsh [**17**, Section 3.5]. Applying these formulas to (7.6), we obtain an asymptotic formula for the term (7.5) with the error term $O(q)$. This completes the proof of our assertion for $U_h(s,q)$. The case of $\tilde{U}_h(s,q)$ is similar.

It seems difficult to refine the error term $O(q)$ using the method of this section. This is why we introduce the method of using multiple zeta functions in this paper.

8. *Proof of Theorem* 2

In this section we prove the following Lemma 4. We can easily show Theorem 2 from Theorem 1 and Lemma 4 by noting (2.6).

LEMMA 4. *We have*

$$U_h(s,q) = \sum_{r=1}^{h} \sum_{\substack{h_1+\ldots+h_r=h \\ h_j \geqslant 1\,(1 \leqslant j \leqslant r)}} \frac{h!}{h_1!\ldots h_r!} J_r(h_1 s, \ldots, h_r s; q) \tag{8.1}$$

and

$$\tilde{U}_h(s,q) = \sum_{r=1}^{2h} \sum_{\substack{\alpha_1+\ldots+\alpha_r=h \\ \beta_1+\ldots+\beta_r=h \\ \alpha_j+\beta_j \geqslant 1 (1 \leqslant j \leqslant r)}} \frac{h!}{\alpha_1!\ldots\alpha_r!} \frac{h!}{\beta_1!\ldots\beta_r!}$$

$$\times J_r(\alpha_1 s + \beta_1 \bar{s}, \ldots, \alpha_r s + \beta_r \bar{s}; q). \tag{8.2}$$

Proof. It is sufficient to prove these relations when $\Re s > 1$, because the remaining case then follows by analytic continuation.

When $\Re s > 1$, it follows from (1.7) that

$$U_h(s,q) = \sum_{a=1}^{q} \sum_{n_1=0}^{\infty} \ldots \sum_{n_h=0}^{\infty} \left(\frac{a}{q} + n_1\right)^{-s} \ldots \left(\frac{a}{q} + n_h\right)^{-s}$$

$$= \sum_{r=1}^{h} \sum_{(n_1,\ldots,n_h) \in N(r)} \sum_{a=1}^{q} \left(\frac{a}{q} + n_1\right)^{-s} \ldots \left(\frac{a}{q} + n_h\right)^{-s},$$

where $N(r)$ is the set of all h-tuples $(n_1, \ldots, n_h)$ of non-negative integers whose components take just r different values. We denote those r values by m_1, $m_1 + m_2, \ldots, m_1 + m_2 + \ldots + m_r$ ($m_1 \geqslant 0$, $m_2 \geqslant 1, \ldots, m_r \geqslant 1$). Let $h_1, \ldots, h_r$ be positive integers satisfying $h_1 + \ldots + h_r = h$, and denote by $N(r; h_1, \ldots, h_r)$ the set of all $(n_1,\dots,n_h)\in N(r)$ whose h_1 components take the value m_1, h_2 components take the value m_1+m_2, and so on. Then we have

$$U_h(s,q)=\sum_{r=1}^{h}\sum_{\substack{h_1+\dots+h_r=h\\ h_j\geqslant 1\,(1\leqslant j\leqslant r)}}\sum_{m_1=0}^{\infty}\sum_{m_2=1}^{\infty}\cdots\sum_{m_r=1}^{\infty}$$
$$\times\sum_{(n_1,\dots,n_h)\in N(r;h_1,\dots,h_r)}\sum_{a=1}^{q}\left(\frac{a}{q}+m_1\right)^{-h_1 s}\left(\frac{a}{q}+m_1+m_2\right)^{-h_2 s}$$
$$\times\dots\times\left(\frac{a}{q}+m_1+m_2+\dots+m_r\right)^{-h_r s}. \tag{8.3}$$

Since the number of elements of the set $N(r;h_1,\dots,h_r)$ is

$$\binom{h}{h_1}\binom{h-h_1}{h_2}\cdots\binom{h-h_1-\dots-h_{r-2}}{h_{r-1}}=\frac{h!}{h_1!\dots h_r!},$$

from (8.3) we obtain (8.1). The proof of (8.2) is similar. □

9. *Proof of Theorem* 3

From (1.9) with $h=3$, we obtain

$$\begin{aligned}U_3(s,q)&=\zeta(3s)q^{3s}\\&+3\left\{X_2(s,2s)q+\sum_{n=0}^{N-1}\frac{(-1)^n}{n!}\zeta(s-n)Y_{2,n}(2s)q^{s-n}+Z_{2,N}(s,2s;\varepsilon,q)\right\}\\&+3\left\{X_2(2s,s)q+\sum_{n=0}^{N-1}\frac{(-1)^n}{n!}\zeta(2s-n)Y_{2,n}(s)q^{2s-n}+Z_{2,N}(2s,s;\varepsilon,q)\right\}\\&+6\left\{X_3(s,s,s)q+\sum_{n=0}^{N-1}\frac{(-1)^n}{n!}\zeta(s-n)Y_{3,n}(s,s)q^{s-n}+Z_{3,N}(s,s,s;\varepsilon,q)\right\}.\end{aligned} \tag{9.1}$$

The explicit form of $Y_{2,n}(s)$ is given as (3.9). Also, from (1.4), we have

$$Y_{3,n}(s,s)=\sum_{k=0}^{n}\binom{n}{k}\frac{\Gamma(s+k)\Gamma(s+n-k)}{\Gamma(s)^2}\zeta_2(s+k,\,s+n-k;1). \tag{9.2}$$

The possible singularities of $\zeta(s_1,s_2;1)$ are located only on $s_1+s_2=2-\ell$ ($\ell=0,1,2,\dots$) or $s_2=1$ (see [**13**, (4.5)], or [**15**]). Hence the right-hand side of (9.2) for $n\geqslant 2$ is holomorphic at $s=1/2$. Moreover, for $n=1$, we have

$$\begin{aligned}Y_{3,1}(s,s)&=s\{\zeta_2(s,\,s+1;1)+\zeta_2(s+1,\,s;1)\}\\&=s\{\zeta(s)\zeta(s+1)-\zeta(2s+1)\},\end{aligned}$$

which is actually holomorphic at $s=1/2$. Therefore, there are just four factors on the right-hand side of (9.1) that are singular at $s=1/2$. They are (i) $Y_{2,0}(2s)$, (ii) $X_2(2s,s)$, (iii) $\zeta(2s)$ and (iv) $Y_{3,0}(s,s)$.

The contributions of (i) and (iv) are

$$\begin{aligned} &= 3\zeta(s)Y_{2,0}(2s)q^s + 6\zeta(s)Y_{3,0}(s,s)q^s \\ &= 3\zeta(s)\{\zeta(2s) + 2\zeta_2(s,s;1)\}q^s \\ &= 3\zeta(s)\{\zeta(2s) + \zeta(s)^2 - \zeta(2s)\}q^s \\ &= 3\zeta(s)^3 q^s, \end{aligned} \tag{9.3}$$

and hence holomorphic at $s = 1/2$. The contributions of (ii) and (iii) are

$$\begin{aligned} &= 3X_2(2s,s)q + 3\zeta(2s)Y_{2,0}(s)q^{2s} \\ &= 3\frac{\Gamma(1-2s)}{\Gamma(s)}\Gamma(-1+3s)\zeta(-1+3s)q + 3\zeta(2s)\zeta(s)q^{2s}, \end{aligned}$$

by (3.8). Putting $s = 1/2 + \delta$ with a small δ, we find that the above is

$$\begin{aligned} &= -\frac{3}{2\delta}\zeta\left(\tfrac12\right)q - \frac{3q}{2\sqrt{\pi}}\left\{3\sqrt{\pi}\zeta'\left(\tfrac12\right) + 2\Gamma'\left(\tfrac12\right)\zeta\left(\tfrac12\right) - 2\Gamma'(1)\Gamma\left(\tfrac12\right)\zeta\left(\tfrac12\right)\right\} \\ &\quad + \frac{3}{2\delta}\zeta\left(\tfrac12\right)q + 3q\left\{\gamma\zeta\left(\tfrac12\right) + \tfrac12\zeta'\left(\tfrac12\right) + \zeta\left(\tfrac12\right)\log q\right\} + O(\delta). \end{aligned}$$

Hence, by using $\Gamma'(1) = -\gamma$ and $\Gamma'(1/2) = -(\gamma + \log 4)\sqrt{\pi}$, we find that the above tends to

$$3\zeta\left(\tfrac12\right)q\log q + 3\left\{(\gamma + \log 4)\zeta\left(\tfrac12\right) - \zeta'\left(\tfrac12\right)\right\}q \tag{9.4}$$

when $\delta \to 0$. Now the assertion of Theorem 3 follows when we take the limit $s \to 1/2$ on the right-hand side of (9.1), and note (9.3), (9.4) and the fact that $X_2(1/2, 1) = \pi\zeta(1/2)$. □

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