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Renormalized expansions in the theory of turbulence with the use of the Lagrangian position function

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A method of renormalized expansions in theory of turbulence is developed with the use of the Lagrangian position function. The introduction of this function makes it easy to express the Lagrangian development of the velocity field. A simple truncation of a set of renormalized expansions is shown to lead to an approximation which is compatible with Kolmogorov's inertial range energy spectrum.

1. Introduction

Among attempts to construct a statistical theory of turbulence, there are approaches based on systematic renormalized expansions; see, for example, Kraichnan (1977), Kaneda (1977) and references cited there. Direct-interaction approximation (DIA) (Kraichnan 1964*a*) is one of the best-known approximations among those which can be obtained by simple truncations of such expansions. It is known that DIA has several desirable properties (see, for example, Leslie 1973) and that it is in good numerical agreement with numerical simulations of isotropic turbulence at moderate Reynolds numbers (Orszag & Patterson 1972*a*, *b*; Herring & Kraichnan 1972).

However it is also known that DIA has a fundamental defect in that it cannot represent the inertial range properly. This may be because the expansions (and consequently DIA also) are constructed in terms of Eulerian multiple-time correlation functions which are greatly affected by the behaviour of big eddies. This failure of DIA suggests the importance of a suitable choice of the quantities in terms of which the renormalized expansions are constructed (see Kraichnan 1964*c*, 1977). Hereafter we call such quantities representatives, in the hope that if they are chosen suitably then approximations accurately representing the physics may be obtained. It is desirable to develop a simple method of renormalized expansions based on a suitable choice of the representatives.

One of the natural choices of such representatives is that of Lagrangian correlation functions, as used by Kraichnan in deriving the Lagrangian-history direct-interaction (LHDI) or the abridged Lagrangian-history direct-interaction (ALHDI) approximation (Kraichnan 1965*a*, 1977). To avoid the complexities accompanied with the use of fully Lagrangian equations of motion, he has introduced a generalized velocity field $u_i(\mathbf{x}, t|s)$ defined as the velocity measured at time s in the fluid element whose space–time trajectory passes through $(\mathbf{x}, t)$. He called t and s in $u_i(\mathbf{x}, t|s)$ the labelling and measuring time, respectively. The Eulerian velocity $u_i(\mathbf{x}, t)$ is

$u_i(\mathbf{x}, t|t)$ and the usual Lagrangian velocity is $u_i(\mathbf{x}, 0|t)$ if $t = 0$ is the initial time. The generalized field obeys

$$\frac{\partial}{\partial t} u_i(\mathbf{x}, t|s) = -u_j(\mathbf{x}, t) \frac{\partial}{\partial x_j} u_i(\mathbf{x}, t|s), \tag{1.1}$$

which we call here labelling-time transformation (LTT). In his approach the field $u_i(\mathbf{x}, t|s)$ is a fundamental quantity and LTT (1.1) plays basic roles.

We call here a time derivative with respect to labelling time keeping measuring time fixed a 'labelling-time derivative' (LTD), and that with respect to measuring time, keeping labelling time fixed, a 'measuring-time derivative' (MTD). The time derivative in (1.1) is a LTD. In the procedures of the LHDI/ALHDI formalism, the use of equations describing MTD's is avoided. The resulting LHDI/ALHDI approximation does not contain MTD.

In this paper a method of renormalized expansions is proposed with the use of the Lagrangian position function ψ defined by

$$\psi(\mathbf{x}, t; \mathbf{x}', t') \equiv \delta^3(\mathbf{x} - \mathbf{y}(\mathbf{x}', t'|t)), \tag{1.2}$$

where $\delta^3(\ldots)$ is the three-dimensional Dirac function and $y(\mathbf{x}', t'|t)$ is the position at time t of the fluid element whose space–time trajectory passes through $(\mathbf{x}', t')$. It is well known that in an incompressible fluid ψ obeys

$$\frac{\partial}{\partial t} \psi(\mathbf{x}, t; \mathbf{x}', t') = -u_j(\mathbf{x}, t) \frac{\partial}{\partial x_j} \psi(\mathbf{x}, t; \mathbf{x}', t'). \tag{1.3}$$

Clearly the field $u_i(\mathbf{x}, t|s)$ is given by

$$u_i(\mathbf{x}, t|s) = \int d^3\mathbf{x}'' u_i(\mathbf{x}'', s)\, \psi(\mathbf{x}'', s; \mathbf{x}, t). \tag{1.4}$$

The relation (1.4) is known to be useful for studies of Lagrangian correlation functions.

In distinction to Kraichnan's approach, in the present one the field $u_i(\mathbf{x}, t|s)$ is not regarded as a fundamental one. It is separated into $u_i(\mathbf{x}, t)$ and ψ fields as in (1.4). The introduction of ψ makes it easy to express various quantities, for example MTD's (e.g. $\partial u_i(\mathbf{x}, t|s)/\partial s$, cf. (2.16)) the use of which has been avoided in the LHDI/ALHDI formalism. In § 2, with the use of ψ we construct renormalized expansions by procedures similar to those of Kraichnan (1977).

We must note here that the time derivative which appears in the usual Lagrangian equation of motion is taken with respect to measuring time (i.e. $\partial u_i(\mathbf{x}, t|s)/\partial s$). Lagrangian mechanics corresponds to integrating with respect to measuring time keeping labelling time fixed, and MTD's may be regarded as containing essential information on Lagrangian mechanics. Hence, from the viewpoint of studying Lagrangian properties, it is interesting to construct an approximation based on equations describing such derivatives. The approximation in § 2 is in fact constructed on the basis of such equations (cf. (2.19) and (2.20)). In this point and with the difference of the choice of representatives, it is fundamentally different from the LHDI/ALHDI approximation.

Moreover, throughout the whole procedure of the construction, we need not use LTT (1.1) and there appear naturally only correlation and response functions (U, G, U^0, G^0, etc.) which have only two, not four, time arguments. Hence there is no need to invoke changes of time arguments like relabellings as done by Kraichnan. Furthermore, there is no need to introduce a fictitious compressible field. These facts simplify the construction of the expansions.

A simple truncation of expansions thus obtained yields a closed set of approximate equations which resembles (but is not identical with) Kraichnan's test-field model (TFM) approximation (see Kraichnan 1971, § 4; or Leslie 1973, cha. 11)). In the case of homogeneous and isotropic turbulence, it reduces to a fairly simple form. In § 4, we discuss about a model representation in a statistically stationary case concerned with this form. In § 3, we estimate the Kolmogorov constant based on the present approximation by assuming, following Kraichnan (1965*b*), Gaussian time dependence of the velocity correlation and averaged response functions. In § 5, we discuss the alternative choices of representatives other than that in § 2.

2. Renormalized expansions and truncated approximation

In this paper we assume the fluid to be incompressible. After the elimination of the pressure term by the use of the incompressibility condition, we may write the Navier–Stokes equation in the form

$$(\partial/\partial t-\nu\nabla_{\mathbf{x}}^2)\,u_i(\mathbf{x},t) = -\tfrac{1}{2}\lambda P_{ijm}(\nabla_{\mathbf{x}})\,[u_j(\mathbf{x},t)\,u_m(\mathbf{x},t)], \tag{2.1}$$

and rewrite (1.3) as

$$\frac{\partial}{\partial t}\psi(\mathbf{x},t;\,\mathbf{x}',t') = -\lambda u_j(\mathbf{x},t)\frac{\partial}{\partial x_j}\psi(\mathbf{x},t;\,\mathbf{x}',t'), \tag{2.2}$$

where

$$\psi(\mathbf{x},t';\,\mathbf{x}',t') = \delta^3(\mathbf{x}-\mathbf{x}'). \tag{2.3}$$

Here the parameter $\lambda = 1$ is introduced for later convenience, ν is the kinematic viscosity and

$$\left.\begin{aligned} P_{ijm}(\nabla_{\mathbf{x}}) &= P_{ij}(\nabla_{\mathbf{x}})\,\partial/\partial x_m + P_{im}(\nabla_{\mathbf{x}})\,\partial/\partial x_j,\\ P_{ij}(\nabla_{\mathbf{x}}) &= \delta_{ij}-\Pi_{ij}(\nabla_{\mathbf{x}}),\end{aligned}\right\} \tag{2.4}$$

and, for any $g(\mathbf{x})$,

$$\Pi_{ij}(\nabla_{\mathbf{x}})\,g(\mathbf{x}) = \frac{\partial^2}{\partial x_i\,\partial x_j}\int D(\mathbf{x},\mathbf{y})\,g(\mathbf{y})\,d^3\mathbf{y}. \tag{2.5}$$

The integration in (2.5) extends over the whole volume occupied by the fluid. $D(\mathbf{x},\mathbf{y})$ has zero normal derivative on the boundaries and satisfies

$$\nabla_{\mathbf{x}}^2 D(\mathbf{x},\mathbf{y}) = \delta^3(\mathbf{x}-\mathbf{y}). \tag{2.6}$$

In (2.1) the contributions from the surface boundary integrals are assumed to be negligible. As for the derivation of (2.1) connected with the boundary conditions, see Kraichnan (1964*b*, 1965*a*).

We introduce here two quantities as candidates for the representatives discussed in § 1. One of them is the two-time two-point velocity correlation function defined by

$$U_{ij}(\mathbf{x},t;\,\mathbf{x}',t') \equiv \langle\{\textstyle\int d^3\mathbf{x}''\,u_i(\mathbf{x}'',t)\,\psi(\mathbf{x}'',t;\,\mathbf{x},t')\}\,u_j(\mathbf{x}',t')\rangle \quad (t \geqslant t'), \tag{2.7a}$$

which may be written as

$$U_{ij}(\mathbf{x},t;\,\mathbf{x}',t') \equiv \langle u_i(\mathbf{x},t'|t)\,u_j(\mathbf{x}',t')\rangle \quad (t \geqslant t'), \tag{2.7b}$$

by virtue of (1.4). The other is the averaged Lagrangian infinitesimal response function defined by

$$G_{ij}(\mathbf{x},t;\,\mathbf{x}',t') \equiv \langle \hat{G}_{ij}(\mathbf{x},t;\,\mathbf{x}',t')\rangle, \tag{2.8}$$

where

$$\hat{G}_{ij}(\mathbf{x},t;\,\mathbf{x}',t') \equiv \delta\{\textstyle\int d^3\mathbf{x}''\, u_i(\mathbf{x}'',t)\,\psi(\mathbf{x}'',t;\,\mathbf{x},t')\}/\delta f_j(\mathbf{x}',t') \tag{2.9a}$$

$$= \delta u_i(\mathbf{x},t'|t)/\delta f_j(\mathbf{x}',t') \tag{2.9b}$$

$$= \textstyle\int d^3\mathbf{x}''\{\hat{G}^E_{ij}(\mathbf{x}'',t;\,\mathbf{x}',t')\,\psi(\mathbf{x}'',t;\mathbf{x},t') + u_i(\mathbf{x}'',t)\,\Psi_j(\mathbf{x}'',t;\,\mathbf{x},\mathbf{x}',t')\} \quad (t \geqslant t'), \tag{2.9c}$$

$$\hat{G}_{ij}(\mathbf{x},t;\,\mathbf{x}',t') \equiv 0 \quad (t < t'), \tag{2.9d}$$

in which $f_i(\mathbf{x},t)$ is an arbitrary source term added to the right-hand side of (2.1) and (δ/δ) denotes functional differentiation, $\hat{G}^E_{ij}$ is the Eulerian infinitesimal response function defined by (cf. Kraichnan 1964*b*)

$$\hat{G}^E_{ij}(\mathbf{x}'',t;\,\mathbf{x}',t') \equiv \delta u_i(\mathbf{x}'',t)/\delta f_j(\mathbf{x}',t') \quad (t \geqslant t') \tag{2.10}$$
$$\equiv 0 \quad (t < t'),$$

and Ψ_j is defined by

$$\Psi_j(\mathbf{x}'',t;\,\mathbf{x},\mathbf{x}',t') \equiv \delta\psi(\mathbf{x}'',t;\,\mathbf{x},t')/\delta f_j(\mathbf{x}',t') \quad (t \geqslant t') \tag{2.11}$$
$$\equiv 0 \quad (t < t').$$

The function $\hat{G}^E_{ij}$ obeys

$$[\partial/\partial t - \nu\nabla^2_{\mathbf{x}}]\,\hat{G}^E_{ij}(\mathbf{x},t;\,\mathbf{x}',t') = -\lambda P_{imn}(\nabla_{\mathbf{x}})\,[u_m(\mathbf{x},t)\,\hat{G}^E_{nj}(\mathbf{x},t;\,\mathbf{x}',t')] \quad (t > t'), \tag{2.12}$$

and

$$\hat{G}^E_{ij}(\mathbf{x},t+0;\,\mathbf{x}',t) = \delta^3(\mathbf{x}-\mathbf{x}')\,\delta_{ij}. \tag{2.13}$$

While Ψ_j obeys

$$\frac{\partial}{\partial t}\Psi_j(\mathbf{x}'',t;\,\mathbf{x},\mathbf{x}',t') = -\lambda\Big\{u_m(\mathbf{x}'',t)\frac{\partial}{\partial x''_m}\Psi_j(\mathbf{x}'',t;\,\mathbf{x},\mathbf{x}',t') + \hat{G}^E_{mj}(\mathbf{x}'',t;\,\mathbf{x}',t')\frac{\partial}{\partial x''_m}\psi(\mathbf{x}'',t;\,\mathbf{x},t')\Big\} \quad (t \geqslant t'), \tag{2.14}$$

and

$$\Psi_j(\mathbf{x}'',t;\,\mathbf{x},\mathbf{x}',t') = 0 \quad (t \leqslant t'). \tag{2.15}$$

By using (2.1), (2.2) and (1.4), we can write the MTD of $u_i(\mathbf{x},t|s)$ as

$$\frac{\partial}{\partial s}u_i(\mathbf{x},t|s) = L_i(\mathbf{x},t|s), \tag{2.16a}$$

where

$$L_i(\mathbf{x},t|s) \equiv \nu\textstyle\int d^3\mathbf{x}''\{[\nabla^2_{\mathbf{x}''}u_i(\mathbf{x}'',s)]\,\psi(\mathbf{x}'',s;\,\mathbf{x},t)\}$$
$$-\lambda\textstyle\int d^3\mathbf{x}''\Big\{[\tfrac{1}{2}P_{imn}(\nabla_{\mathbf{x}''})\,[u_m(\mathbf{x}'',s)\,u_n(\mathbf{x}'',s)]]\,\psi(\mathbf{x}'',s;\,\mathbf{x},t) + u_i(\mathbf{x}'',s)\,u_m(\mathbf{x}'',s)\frac{\partial}{\partial x''_m}\psi(\mathbf{x}'',s;\,\mathbf{x},t)\Big\}. \tag{2.16b}$$

Similarly, by using (2.9c) and (2.1), (2.2), (2.12), (2.14), or from (2.16), we obtain

$$\begin{aligned}
\frac{\partial}{\partial t}\hat{G}_{ij}(\mathbf{x},t;\mathbf{x}',t') = \nu\int d^3\mathbf{x}''\{[\nabla^2_{\mathbf{x}''}\hat{G}^E_{ij}(\mathbf{x}'',t;\mathbf{x}',t')]\,\psi(\mathbf{x}'',t;\mathbf{x},t')\} \\
-\lambda\int d^3\mathbf{x}''\Big\{(P_{imn}(\nabla_{\mathbf{x}''})[u_m(\mathbf{x}'',t)\,\hat{G}^E_{nj}(\mathbf{x}'',t;\mathbf{x}',t')])\,\psi(\mathbf{x}'',t;\mathbf{x},t') \\
+\hat{G}^E_{ij}(\mathbf{x}'',t;\mathbf{x}',t')\,u_m(\mathbf{x}'',t)\frac{\partial}{\partial x''_m}\psi(\mathbf{x}'',t;\mathbf{x},t')\Big\} \\
+\nu\int d^3\mathbf{x}''\{[\nabla^2_{\mathbf{x}''}u_i(\mathbf{x}'',t)]\,\Psi_j(\mathbf{x}'',t;\mathbf{x},\mathbf{x}',t')\} \\
-\lambda\int d^3\mathbf{x}''\Big\{(\tfrac{1}{2}P_{imn}(\nabla_{\mathbf{x}''})[u_m(\mathbf{x}'',t)\,u_n(\mathbf{x}'',t)])\,\Psi_j(\mathbf{x}'',t;\mathbf{x},\mathbf{x}',t') \\
+u_i(\mathbf{x}'',t)\Big[u_m(\mathbf{x}'',t)\frac{\partial}{\partial x''_m}\Psi_j(\mathbf{x}'',t;\mathbf{x},\mathbf{x}',t') \\
+\hat{G}^E_{mj}(\mathbf{x}'',t;\mathbf{x}',t')\frac{\partial}{\partial x''_m}\psi(\mathbf{x}'',t;\mathbf{x},t')\Big]\Big\} \qquad (2.17a) \\
\equiv \hat{C}_{ij}(\mathbf{x},t;\mathbf{x}',t') \quad (t>t'). \qquad (2.17b)
\end{aligned}$$

From the above relations, we have

$$\begin{aligned}
\left(\frac{\partial}{\partial t}-\nu\nabla^2_{\mathbf{x}}-\nu\nabla^2_{\mathbf{x}'}\right)U_{ij}(\mathbf{x},t;\mathbf{x}',t) = -\tfrac{1}{2}\lambda\langle\{P_{imn}(\nabla_{\mathbf{x}})[u_m(\mathbf{x},t)\,u_n(\mathbf{x},t)]\}\,u_j(\mathbf{x}',t) \\
+u_i(\mathbf{x},t)\{P_{jmn}(\nabla_{\mathbf{x}'})[u_m(\mathbf{x}',t)\,u_n(\mathbf{x}',t)]\}\rangle \\
\equiv A_{ij}(\mathbf{x},t;\mathbf{x}',t), \qquad (2.18)
\end{aligned}$$

$$\frac{\partial}{\partial t}U_{ij}(\mathbf{x},t;\mathbf{x}',t') = \langle L_i(\mathbf{x},t'|t)\,u_j(\mathbf{x}',t')\rangle \equiv B_{ij}(\mathbf{x},t;\mathbf{x}',t') \quad (t>t'), \qquad (2.19)$$

and

$$\frac{\partial}{\partial t}G_{ij}(\mathbf{x},t;\mathbf{x}',t') = \langle\hat{C}_{ij}(\mathbf{x},t;\mathbf{x}',t')\rangle \equiv C_{ij}(\mathbf{x},t;\mathbf{x}',t') \quad (t>t'). \qquad (2.20)$$

It is to be noted here that the time derivatives in (2.19) and (2.20) are MTD's; see (2.7b) and (2.9b). Our $U_{ij}(\mathbf{x},t;\mathbf{x}',t')$ and $G_{ij}(\mathbf{x},t;\mathbf{x}',t')$ are respectively identical with

$$U_{ij}(\mathbf{x},t'|t;\mathbf{x}',t'|t') \quad \text{and} \quad G_{ij}(\mathbf{x},t'|t;\mathbf{x}',t'|t'), \qquad (2.21)$$

in Kraichnan's notation (cf. Kraichnan 1965a). Here $t \geqslant t'$ and U and G in (2.21) are respectively different from

$$U_{ij}(\mathbf{x},t|t;\mathbf{x}',t|t') \quad \text{and} \quad G_{ij}(\mathbf{x},t|t;\mathbf{x}',t|t') \quad \text{with} \quad t \geqslant t', \qquad (2.22)$$

which may be regarded as the representatives of the ALHDI approximation. The time derivatives in (2.19), (2.20) and those in the time-displaced correlation and response equations of the ALHDI approximation are all taken with respect to t (not t') in the notation of (2.21) and (2.22). As noted above those in the former are MTD's, while those in the latter are not.

For simplicity's sake, we suppose here that the distribution over the ensemble of the initial Eulerian velocity $u_i(\mathbf{x},t_0)$ is homogeneous, isotropic and Gaussian with zero mean. More general distributions can be handled but they would add complications in which we are not interested here.

By the use of (2.2) and the Navier–Stokes equation (2.1), our renormalized expansions can be constructed by steps similar to those of Kraichnan (1977). Let $u_i^0(\mathbf{x},t)$ be the solution of the linearized Navier–Stokes equation (nonlinear terms removed), $\hat{G}_{ij}^{E0}(\mathbf{x},t;\,\mathbf{x}',t')$ be the Eulerian infinitesimal response function of the linearized equation and define the correlation function

$$U_{ij}^{E0}(\mathbf{x},t;\,\mathbf{x}',t') \equiv \langle u_i^0(\mathbf{x},t)\,u_j^0(\mathbf{x}',t')\rangle \quad (t \geqslant t'). \tag{2.23}$$

Since $u_i^0(\mathbf{x},t)$ decays linearly from $u_i(\mathbf{x},t_0)$, any moment of $u^0(\equiv u_i^0(\mathbf{x},t))$ can be expressed in terms of the correlation U^{E0}. The linearization of (2.2), with the right-hand side set to zero, gives simply

$$\psi^0(\mathbf{x},t;\,\mathbf{x}',t') = \delta^3(\mathbf{x}-\mathbf{x}'), \tag{2.24}$$

and consequently

$$\hat{G}_{ij}^0 = \hat{G}_{ij}^{E0} = G_{ij}^0 = G_{ij}^{E0}, \tag{2.25}$$

and

$$\begin{aligned} U_{ij}^{E0}(\mathbf{x},t;\,\mathbf{x}',t') &= U_{ij}^0(\mathbf{x},t;\,\mathbf{x}',t') \quad (t \geqslant t') \\ &= U_{ji}^0(\mathbf{x}',t';\,\mathbf{x},t) \quad (t \leqslant t'), \end{aligned} \tag{2.26}$$

where the superscript zero refers to linearized solutions. If the nonlinear terms in the Navier–Stokes equation (2.1) and the right-hand side terms in (2.2), (2.12) and (2.14) are re-introduced as perturbations, an iteration process gives $u_i(\mathbf{x},t)$, $\psi(\mathbf{x},t;\mathbf{x}',t')$, $\hat{G}_{ij}^E(\mathbf{x},t;\,\mathbf{x}',t')$ and $\Psi_j(\mathbf{x}'',t;\,\mathbf{x},\mathbf{x}',t')$ as functional power series of $\mathbf{u}^0$, $\hat{G}^{E0}$ and ψ^0. Here, from (2.24) ψ^0 is known to be equal to a δ-function, and from (2.25) $\hat{G}^{E0} = G^0$. Hence we can expand $\mathbf{u}$, ψ, $\hat{G}^E$ and Ψ in terms of $\mathbf{u}^0$ and G^0, and consequently can obtain the expansions of $U_{ij}(\mathbf{x},t;\,\mathbf{x}',t')$ and $G_{ij}(\mathbf{x},t;\,\mathbf{x}',t')$, see (2.7) $\sim$ (2.9), in functional powers of U_{ij}^0 and G_{ij}^0.

Because of the homogeneity, the correlation function U and the averaged response function G depend on the arguments $\mathbf{x}$ and $\mathbf{x}'$ only in the combination $\mathbf{x}-\mathbf{x}'$. Hence we introduce the wave-vector functions

$$U_{ij}(\mathbf{k},t,t') \equiv (2\pi)^{-3}\int U_{ij}(\mathbf{x},t;\,\mathbf{x}',t')\,e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}\,d^3(\mathbf{x}-\mathbf{x}'), \tag{2.27}$$

$$G_{ij}(\mathbf{k},t,t') \equiv \int G_{ij}(\mathbf{x},t;\,\mathbf{x}',t')\,e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}\,d^3(\mathbf{x}-\mathbf{x}'). \tag{2.28}$$

The wave-vector functions corresponding to A_{ij}, B_{ij}, U_{ij}^{E0}, U_{ij}^0 and C_{ij}, G_{ij}^{E0}, G_{ij}^0 are defined similarly to (2.27) and (2.28) respectively.

Now it is convenient to introduce, as representatives instead of U_{ij} and G_{ij}, the following projected quantities:

$$Q_{ij}(\mathbf{k},t,t') \equiv P_{in}(\mathbf{k})\,U_{nj}(\mathbf{k},t,t'), \tag{2.29}$$

$$F_{ij}(\mathbf{k},t,t') \equiv P_{im}(\mathbf{k})\,G_{mn}(\mathbf{k},t,t')\,P_{nj}(\mathbf{k}), \tag{2.30}$$

where the projection operator $P_{in}(\mathbf{k})$ is defined by $P_{in}(\mathbf{k}) \equiv \delta_{in} - k_i k_n/k^2$. By the application of this operator, the solenoidal conditions of Q, F, Q^0 and F^0, e.g. $k_i Q_{ij}(\mathbf{k}) = k_j Q_{ij}(\mathbf{k}) = 0$, are satisfied. It is clear that

$$Q_{ij}^0(\mathbf{k}) = U_{ij}^0(\mathbf{k}) \quad \text{and} \quad F_{ij}^0(\mathbf{k}) = P_{in}(\mathbf{k})\,G_{nj}^0(\mathbf{k}) = G_{in}^0(\mathbf{k})\,P_{nj}(\mathbf{k}). \tag{2.31}$$

In order to compute $Q_{ij}(\mathbf{k},t,t')$ and $F_{ij}(\mathbf{k},t,t')$ from (2.18) $\sim$ (2.20), it is necessary to know

$$P_{in}(\mathbf{k})\,A_{nj}(\mathbf{k},t,t) \quad (=A_{ij}(\mathbf{k},t,t)), \tag{2.32}$$

$$P_{in}(\mathbf{k})\,B_{nj}(\mathbf{k},t,t') \quad (\equiv \bar{B}_{ij}(\mathbf{k},t,t')) \tag{2.33}$$

and $$P_{im}(\mathbf{k})\, C_{mn}(\mathbf{k}, t, t')\, P_{nj}(\mathbf{k}) \quad (\equiv \bar{C}_{ij}(\mathbf{k}, t, t')). \tag{2.34}$$

In the same way that we can expand $U_{ij}(\mathbf{x}, t;\ \mathbf{x}', t')$ and $G_{ij}(\mathbf{x}, t;\ \mathbf{x}', t')$ in functional powers of $U^0_{ij}(\mathbf{x}, t;\ \mathbf{x}', t')$ and $G^0_{ij}(\mathbf{x}, t;\ \mathbf{x}', t')$, we can expand $Q_{ij}(\mathbf{k})$, $F_{ij}(\mathbf{k})$, $A_{ij}(k)$, $\bar{B}_{ij}(\mathbf{k})$ and $\bar{C}_{ij}(\mathbf{k})$ in functional powers of $U^0_{ij}(\mathbf{k})$ and $G^0_{ij}(\mathbf{k})$. We note that, in these expansions, $G^0_{in}(\mathbf{k})$ is always accompanied with the projection operator $P_{nj}(\mathbf{k})$, i.e. it always appears in the form $G^0_{in}(\mathbf{k})\, P_{nj}(\mathbf{k})$. Hence, by virtue of (2.31), they can be expanded in functional powers of $Q^0_{ij}(\mathbf{k})$ and $F^0_{ij}(\mathbf{k})$.

A basic idea of our renormalized expansions lies in expanding A, $\bar{B}$ and $\bar{C}$ in terms of Q and F (not in terms of Q^0 and F^0). The steps outlined as (i), (ii) and (iii) in Kraichnan's (1977) paper are applicable also to the present case with suitable changes of a few words as follows.

(i) Invert the developments for Q and F by iteration to yield expansions for Q^0 and F^0.

(ii) Substitute the latter expansions for each Q^0 and F^0 factor in the primitive expansions for A, $\bar{B}$ and $\bar{C}$.

(iii) Multiply out and collect terms.

It is worthwhile to note that our Q^0, F^0, Q and F have respectively only two, not four, time arguments and that we need not invoke changing of time arguments like relabelling as done by Kraichnan.

By these steps, we obtain, after straightforward calculations,†

$$(\partial/\partial t + 2\nu k^2)\, Q_{ij}(\mathbf{k}, t, t) = \lambda^2 D_{ij}(\mathbf{k}, t, t), \tag{2.35}$$

$$(\partial/\partial t)\, Q_{ij}(\mathbf{k}, t, s) = -\nu X_{ij}(\mathbf{k}, t, s) + \lambda^2 I_{ij}(\mathbf{k}, t, s) \quad (t > s), \tag{2.36}$$

$$(\partial/\partial t)\, F_{ij}(\mathbf{k}, t, s) = -\nu Y_{ij}(\mathbf{k}, t, s) + \lambda^2 J_{ij}(\mathbf{k}, t, s) \quad (t > s), \tag{2.37}$$

where

$$D_{ij}(\mathbf{k}, t, t) = H_{ij}(\mathbf{k}, t, t) + H_{ji}(-\mathbf{k}, t, t) + O(\lambda), \tag{2.38}$$

$$X_{ij}(\mathbf{k}, t, s) = k^2 Q_{ij}(\mathbf{k}, t, s) + O(\lambda), \tag{2.39}$$

$$Y_{ij}(\mathbf{k}, t, s) = k^2 F_{ij}(\mathbf{k}, t, s) + O(\lambda), \tag{2.40}$$

$$H_{ij}(\mathbf{k}, t, t) = \sum_{\mathbf{p}, \mathbf{r}}^{\Delta} \int_{t_0}^{t} ds' \{-P_{imn}(\mathbf{k})\, P_{bca}(\mathbf{p})\, F_{mb}(\mathbf{p}, t, s')\, Q_{nc}(\mathbf{r}, t, s')\, Q_{ja}(-\mathbf{k}, t, s') \\ + \tfrac{1}{2} P_{imn}(\mathbf{k})\, P_{abc}(\mathbf{k})\, F_{ja}(-\mathbf{k}, t, s')\, Q_{mb}(\mathbf{p}, t, s')\, Q_{nc}(\mathbf{r}, t, s')\}, \tag{2.41}$$

$$I_{ij}(\mathbf{k}, t, s) = \left\{-\sum_{\mathbf{p}, \mathbf{r}}^{\Delta} P_{ib}(\mathbf{k})\, k_a\, S_{bmn}(\mathbf{p}) \left[\int_s^t ds'\, Q_{ma}(-\mathbf{r}, t, s')\right] Q_{nj}(\mathbf{k}, t, s)\right\} + O(\lambda), \tag{2.42}$$

$$J_{ij}(\mathbf{k}, t, s) = \left\{-\sum_{\mathbf{p}, \mathbf{r}}^{\Delta} P_{ib}(\mathbf{k})\, k_a\, S_{bmn}(\mathbf{p}) \left[\int_s^t ds'\, Q_{ma}(-\mathbf{r}, t, s')\right] F_{nj}(\mathbf{k}, t, s)\right\} + O(\lambda), \tag{2.43}$$

in which

$$P_{imn}(\mathbf{k}) = k_m P_{in}(\mathbf{k}) + k_n P_{im}(\mathbf{k}), \tag{2.44}$$

$$S_{imn}(\mathbf{k}) = 2 k_i k_m k_n / k^2, \tag{2.45}$$

† Copies of pages giving the algebraic details are available on request from either the author or the Editor.

and the operator notation

$$\sum_{\mathbf{p},\mathbf{r}}^{\Delta} \equiv \int d^3\mathbf{p}\, d^3\mathbf{r}\, \delta(\mathbf{k}-\mathbf{p}-\mathbf{r}), \tag{2.46}$$

is used.

We have assumed homogeneity in deducing (2.35) ~ (2.37). If the turbulence is also isotropic, we may write

$$Q_{ij}(\mathbf{k},t,s) = P_{ij}(\mathbf{k})\, Q(k,t,s)/2 \quad \text{and} \quad F_{ij}(\mathbf{k},t,s) = P_{ij}(\mathbf{k})\, F(k,t,s), \tag{2.47}$$

where Q and F on the right-hand sides are scalar functions of scalar variables. Now we substitute (2.47) into (2.35) ~ (2.37) and contract them. Then by discarding terms of $O(\lambda)$ in (2.38) ~ (2.43), we obtain the following energy, time-displaced correlation and response equations (see the footnote to page 137);

$$(\partial/\partial t + 2\nu k^2)\, Q(k,t,t) = D(k,t,t), \tag{2.48a}$$

$$(\partial/\partial t + \nu k^2 + \eta(k,t,s))\, Q(k,t,s) = 0 \quad (t > s), \tag{2.48b}$$

$$(\partial/\partial t + \nu k^2 + \eta(k,t,s))\, F(k,t,s) = 0 \quad (t > s), \tag{2.48c}$$

where

$$D(k,t,t) = 2\pi \iint_\Delta kpr\, dp\, dr \int_{t_0}^{t} ds'\, b(k,p,r)\, Q(r,t,s') \times \{F(k,t,s')\, Q(p,t,s') - F(p,t,s')\, Q(k,t,s')\}, \tag{2.49}$$

and

$$\eta(k,t,s) = \pi \iint_\Delta kpr\, dp\, dr \int_{s}^{t} ds'\, d(k,p,r)\, Q(r,t,s'). \tag{2.50}$$

Here $\iint_\Delta$ denotes the integration over all regions of the (p,r) plane such that k,p,r can be the sides of a triangle, and the geometrical factors are

$$b(k,p,r) = (p/k)(xy + z^3), \tag{2.51}$$

$$d(k,p,r) = (1-y^2)(1-z^2), \tag{2.52}$$

where x, y, z are cosines of the interior angles opposite the triangle sides k, p, r, respectively.

Equations (2.48b) and (2.48c) with $F(k, t+0, t) = 1$ give that

$$Q(k,t,s) = F(k,t,s)\, Q(k,s,s) \quad (t > s), \tag{2.53}$$

and that $Q(k,t,s)$ and $F(k,t,s)$ is always positive (of course, we are here assuming $Q(k,s,s)$ to be positive). Also, (2.52) gives $d(k,p,r) \geqslant 0$. Hence $Q(K,t,s)$ and $F(k,t,s)$ decrease monotonically with t.

It is interesting to note the resemblance of our approximation (2.48) to Kraichnan's test-field model (TFM) approximation. The energy equations of both the approximations are similar to that of DIA. Half of the geometrical factor $d(k,p,r)$ given by (2.52) is equal to b^G_{kpr} which appears in the TFM approximation. We note here a comment due to D. C. Leslie (private communication); the time structure

$$\eta(k,t,s)\, Q(k,t,s) \tag{2.54}$$

in (2.48) is intermediate between that of DIA

$$\int^{t} \eta(k,t,s')\, Q(k,s',s)\, ds' \tag{2.55}$$

and that of the TFM approximation

$$\eta(k,t)\,Q(k,t,s). \tag{2.56}$$

We conclude this section with the discussion about the renormalization of the ψ field. As is clear from the above discussions, we have not introduced the renormalization corresponding to the ψ field. However, it is natural to ask why not. Although the main reason for not doing so is for the sake of simplicity, there is another reason. To see this point, let us consider the renormalization corresponding to ψ.

At first, it is to be noted that $\langle\psi\rangle$ is not a suitable representative because it is essentially affected by big eddies. As we have introduced U and G given by (2.7) ~ (2.9) instead of the Eulerian correlation function $\langle uu\rangle$ and response function $\langle\hat{G}^E\rangle$, a natural candidate for the representative corresponding to the ψ field would be, for example,

$$\Phi(\mathbf{x},t;\mathbf{x}',t') \equiv \langle\int d^3\mathbf{x}''\,\psi(\mathbf{x}'',t,\mathbf{x},t')\,\psi(\mathbf{x}'',t;\mathbf{x}',t')\rangle. \tag{2.57}$$

If we introduce Φ as well as Q and F as representatives, the renormalization procedure will be as follows.

(i) Expand Q, F and Φ in terms of Q^0, F^0 and $\Phi^0 = \psi^0$.

(ii) Invert the above expansions to obtain Q^0, F^0 and Φ^0 in terms of Q, F and Φ.

(iii) Substitute these expansions for Q^0, F^0 and Φ^0 into the primitive expansions for the 'triple moments', where 'triple moments' means terms like A, $\bar{B}$ and $\bar{C}$ in (2.32) ~ (2.34). However, it is clear that

$$\Phi(\mathbf{x},t;\mathbf{x}',t') = \delta^3(\mathbf{x}-\mathbf{x}') = \psi^0(\mathbf{x},t;\mathbf{x}',t'). \tag{2.58}$$

Consequently the resulting approximation would be the same one as that without introducing Φ. This is another reason why the renormalization corresponding to the ψ field has not been introduced above.

3. Estimation of Kolmogorov constant

In this section we consider the statistically stationary inertial range. If, according to Kolmogorov hypotheses, the inertial range forms of $Q(k,t,s)$ and $F(k,t,s)$ are required to depend only on k, ϵ and $t-s$, then they may be written in the forms

$$\begin{aligned} Q(k,t,s) &= \{K/(2\pi)\}\epsilon^{\frac{2}{3}}k^{-\frac{11}{3}}q(\tau) \quad (\tau \geqslant 0), \\ \left.\begin{aligned} F(k,t,s) &= f(\tau) \quad (\tau > 0) \\ &= 0 \quad (\tau < 0), \end{aligned}\right\}&, \end{aligned} \tag{3.1}$$

where $\tau \equiv \epsilon^{\frac{1}{3}}k^{\frac{2}{3}}(t-s)$, ϵ is the rate of energy dissipation per unit mass and K is a universal constant called the Kolmogorov constant.

The substitution of (3.1) into (2.48*b*, *c*) yields equations for $q(\tau)$ and $f(\tau)$. By noting (2.53), we may put $q(\tau) = f(\tau)$ $(\tau > 0)$. However it is still not an easy task to solve them. Hence, following Kraichnan (1965*b*), we assume here Gaussian time dependence of $q(\tau)$ and $f(\tau)$, i.e.

$$q(\tau) = \exp\{-\tfrac{1}{4}\pi\alpha^2\tau^2\}, \quad f(\tau) = \exp\{-\tfrac{1}{4}\pi\beta^2\tau^2\} \quad (\tau > 0), \tag{3.2}$$

and assume

$$\epsilon = \int_{k_1}^{\infty} dk \iint_{\Delta}^{(p\text{ or }r<k_1)} T(k;p,r)\,dp\,dr, \tag{3.3}$$

where

$$(2\pi k^2)^{-1}\iint_{\Delta} T(k;p,r)\,dp\,dr$$

is the right-hand side of (2.48*a*) and k_1 is an inertial range wavenumber. As noted above it is clear that $q(\tau)=f(\tau)$, i.e. $\alpha=\beta$. By integrating (2.48*b*) or (2.48*c*) with respect to t from s to ∞ with the substitution of (3.1) and (3.2), we can find the value of α^2/K, whereas we can find the value of α/K^2 from (3, 3) by setting the initial time t_0 at $-\infty$. From these values, the constants $\alpha=\beta$ and K can be calculated. It is worthwhile to note that, with the substitutions (3.1) and (3.2) into (2.48*b*, *c*) and (3.3), the integrations over p and r in them can be shown to converge properly at zero and infinite wavenumbers. Edwards (see Leslie 1973, cha. 6) has shown the importance of such a kind of convergence in connection with the inertial range forms of DIA.

After transforming (3.3) to a form convenient for the numerical calculation (cf. equation (5.10) in Kraichnan 1966), the above values were calculated numerically. Some details of the numerical calculations are shown in §5. The calculations gave

$$K = 1{\cdot}61, \quad \alpha=\beta=0{\cdot}81. \tag{3.4}$$

Experiments are known that suggest that K is nearly 1·5 (see Leslie 1973, cha. 11).

4. An almost-Markovian model representation in a statistically stationary turbulence

Up to now we have not introduced an external driving force for simplicity's sake. However, if we want to permit statistically stationary turbulence, the introduction is necessary. We assume here that this force is represented by an additional divergence-free term $E_i(\mathbf{x},t)$ to the Navier–Stokes equation (2.1), and we restrict E_i to be a homogeneous isotropic and Gaussian white noise process with zero mean and correlation proportional, in the wavevector space, to $Z(k,t)$ (cf. (4.9)). It is not difficult to generalize the discussions of §2 to include such a case. As for the energy equation (2.48*a*), we have only to add to it one term proportional to $Z(k,t)$, cf. (4.10). The time-displaced correlation and response equations (2.48*b*), (2.48*c*) are unaltered. We denote these equations by (2.48*a*′), (2.48*b*′) and (2.48*c*′), or simply as (2.48′).

It is easy to construct a model representation of (2.48′) in a statistically stationary state, in which the velocity amplitude obeys a generalized Langevin equation similar to the almost-Markovian model amplitude equation proposed by Kraichnan (1971). Generalized Langevin models were also discussed by Leith (1971) and Herring & Kraichnan (1972).

Let us consider the following model equations in wavevector space:

$$[\partial/\partial t+\nu k^2+\zeta(k,t)]\,u_i(\mathbf{k},t) = q_i(\mathbf{k},t)+E_i(\mathbf{k},t), \tag{4.1}$$

$$[\partial/\partial t+\nu k^2+\eta(k,t,t')]\,u_i(\mathbf{k},t'|t) = 0 \quad (t>t'), \tag{4.2}$$

$$u_i(\mathbf{k},t|t) = u_i(k,t), \tag{4.3}$$

where

$$\zeta(k,t) = \pi\iint_{\Delta} b(k,p,r)\,\theta(k,p,r,t)\,Q(r,t)\,kpr\,dp\,dr, \tag{4.4}$$

$$q_i(\mathbf{k},t) = -iP_{ijm}(\mathbf{k}) \sum_{\mathbf{p}+\mathbf{r}=\mathbf{k}} [\theta(k,p,r,t)]^{\frac{1}{2}} w(t)\, \xi_j(\mathbf{p},t)\, \xi'_m(\mathbf{r},t), \tag{4.5}$$

$$\langle \xi_i(\mathbf{k},t)\, \xi_j(-\mathbf{p},t)\rangle = \langle \xi'_i(\mathbf{k},t)\, \xi'_j(-\mathbf{p},t)\rangle = \langle u_i(\mathbf{k},t)\, u_j(-\mathbf{p},t)\rangle, \tag{4.6}$$

and the form of $\eta(k,t,t')$ is given by (2.50). Here $w(t)$ is a white noise process,

$$\langle w(t)\, w(t')\rangle = 2\delta(t-t'), \tag{4.7}$$

and the random fields ξ_i and ξ'_j are statistically independent of each other and of the initial velocity field $u_i(\mathbf{k}, t=t_0)$. The correlations in (4.6) are zero if $\mathbf{k} \neq \mathbf{p}$. The field $u_i(\mathbf{k},t)$ is assumed to be isotropic and homogeneous. The velocity correlation function $Q(k,t,t')$ is defined by

$$(L/2\pi)^3 \langle u_i(\mathbf{k},t'|t)\, u_j(-\mathbf{k},t')\rangle = \tfrac{1}{2} P_{ij}(\mathbf{k})\, Q(k,t,t') \quad (t \geqslant t'), \tag{4.8}$$

and $Q(k,t) \equiv Q(k,t,t)$, where L is the cyclic box size ($L \to \infty$, eventually). The external random force $E_i(\mathbf{k},t)$ is assumed to be a white noise process with zero mean and correlation

$$(L/2\pi)^3 \langle E_i(\mathbf{k},t)\, E_j(\mathbf{p},t)\rangle = 2P_{ij}(\mathbf{k})\, Z(k,t)\, \delta_{\mathbf{k}+\mathbf{p}} \delta(t-t'), \tag{4.9}$$

where $\delta_{\mathbf{k}+\mathbf{p}} = 1$ if $\mathbf{k}+\mathbf{p} = 0$ and $\delta_{\mathbf{k}+\mathbf{p}} = 0$ if $\mathbf{k}+\mathbf{p} \neq 0$. The coefficient $\theta(k,p,r,t)$ will be determined below.

Equations (4.1) and (4.2) give the energy equation

$$[\partial/\partial t + 2\nu k^2 + 2\zeta(k,t)]\, Q(k,t)$$
$$= 2Z(k,t) + 2\pi \iint_\Delta a(k,p,r)\, \theta(k,p,r,t)\, Q(p,t)\, Q(r,t)\, kpr\, dp\, dr, \tag{4.10}$$

where $a(k,p,r) = \frac{1}{2}\{b(k,p,r) + b(k,r,p)\}$, and the time-displaced correlation equation which is identical in form to (2.48b'). If we define the averaged response function F as

$$P_{im}(\mathbf{k}) \langle \delta u_m(\mathbf{k},t'|t)/\delta f_j(\mathbf{k},t')\rangle = P_{ij}(\mathbf{k})\, F(k,t,t') \quad (t \geqslant t'), \tag{4.11}$$

where $f_i(\mathbf{k},t)$ is an arbitrary source term added to the right-hand side of (4.2), then it is clear that $F(k,t,t')$ also satisfies an equation identical to (2.48c'). In a statistically stationary case, we have from (2.48b') and (2.48c')

$$Q(k,t,t') = F(k,t,t')\, Q(k,t') = F(k,t,t')\, Q(k,t) \quad (t > t'), \tag{4.12}$$

where we have used $Q(k,t) = Q(k,t')$. Hence, if we put

$$\theta(k,p,r,t) = \int_{t_0}^{t} F(k,t,s)\, F(p,t,s)\, F(r,t,s)\, ds, \tag{4.13}$$

then (4.10) takes precisely the same form as (2.48a'). Thus in a statistically stationary case we can construct a model representation of (2.48′).

Now let us define the Eulerian velocity correlation function Q^E as

$$Q^E(k,t,s) \equiv \langle u_i(\mathbf{k},t)\, u_i(-\mathbf{k},s)\rangle\, (L/2\pi)^3, \tag{4.14}$$

and consider the characteristic decorrelation times τ^E of $Q^E(k,t,s)$ and τ^L of $Q(k,t,s)$ for k in the inertial subrange. By applying similar arguments to those of § 3, we can show that the integration in (4.4) over p and r diverges at the low-r limit with the substitution of the inertial range form of Q and F, cf. (3.1), (3.2), while that in η (see (2.50)) converges properly. This suggests that ζ and η are of order $(V_0 k)$ and $(\epsilon^{\frac{1}{3}} k^{\frac{2}{3}})$, respectively, where V_0 is the characteristic velocity of energy-containing eddies (cf. Leslie 1973, cha. 6). Hence our model (4.1) and (4.2) give $\tau^E = O((V_0 k)^{-1})$ and $\tau^L = O((\epsilon^{\frac{1}{3}} k^{\frac{2}{3}})^{-1})$. These properties are in agreement with our expectation about the Eulerian and the Lagrangian velocity correlation functions in a real turbulence, cf. e.g. Kraichnan (1964*c*, 1965*a*).

5. Alternative choices of representatives

In § 2, we chose Q and F given by (2.29) and (2.30) as the representatives and in terms of them constructed the renormalized expansions and truncated approximation (2.48). However, it is also possible to choose other representatives and to construct renormalized expansions in terms of them.

For example we may choose, instead of F given by (2.30), the following F:

$$F_{ij}(\mathbf{k}) \equiv P_{im}(\mathbf{k})\, \tilde{G}_{mn}(\mathbf{k})\, P_{nj}(\mathbf{k}), \tag{5.1}$$

where $\tilde{G}_{mn}$ is the Fourier transform (see (2.28)) of

$$\tilde{G}_{mn}(\mathbf{x},t;\,\mathbf{x}',t') \equiv \langle \textstyle\int d^3\mathbf{x}''\, \hat{G}^E_{mn}(\mathbf{x}'',t;\,\mathbf{x}',t')\,\psi(\mathbf{x}'',t;\,\mathbf{x},t')\rangle, \tag{5.2a}$$

or

$$\tilde{G}_{mn}(\mathbf{x},t;\,\mathbf{x}',t') \equiv \langle \textstyle\int d^3\mathbf{x}''\, \hat{G}^E_{ma}(\mathbf{x},t;\,\mathbf{x}'',t')\, P_{an}(\nabla_{\mathbf{x}'})\,\psi(\mathbf{x}',t;\,\mathbf{x}'',t')\rangle. \tag{5.2b}$$

Here we are assuming the homogeneity of the turbulence as in (2.27) and (2.28). Similarly we may choose instead of Q given by (2.29) the following Q:

$$Q_{ij}(\mathbf{k}) \equiv \tilde{U}_{im}(\mathbf{k})\, P_{mj}(\mathbf{k}), \tag{5.3}$$

where $\tilde{U}_{im}(\mathbf{k})$ is the Fourier transform (see (2.27)) of

$$\tilde{U}_{im}(\mathbf{x},t;\,\mathbf{x}',t') \equiv \langle u_i(\mathbf{x},t)\{\textstyle\int d^3\mathbf{x}''\, u_m(\mathbf{x}'',t')\,\psi(\mathbf{x}',t;\,\mathbf{x}'',t')\}\rangle \quad (t \geqslant t'). \tag{5.4}$$

In the same way as in § 2, we can construct renormalized expansions by using these representatives. By defining scalar functions $Q(k)$ and $F(k)$ as in (2.47) for isotropic turbulence, we obtain three approximate equations – energy equation (E), time-displaced correlation equation (TD) and response equation (R), which correspond to (2.48*a*), (2.48*b*) and (2.48*c*) respectively. For each of the following choices (*a*) ~ (*e*) of the representatives, (E) is identical in form to (2.48*a*). While the forms of (TD) and (R) depend on the choice as follows, in which the brackets [; ...] denote the choice of the representatives.

Choice (*a*). [; Q_j given by (2.29) and F given by (5.1), (5.2*a*)]. The form of (TD) is identical to (2.48*b*). That of (R) is given by

$$(\partial/\partial t + \nu k^2)\, F(k,t,s) = \tilde{C}(k,t,s) \quad (t > s), \tag{5.5}$$

where

$$\begin{aligned}\tilde{C}(k,t,s) \equiv -\pi \iint_\Delta kpr\,dp\,dr \int_s^t ds' \{ & d(k,p,r)\, Q(r,t,s')\, F(k,t,s) \\ & + [b(k,p,r) - (p^2/k^2)\, a_1(p,k,r)]\, Q(r,t,s')\, F(p,t,s')\, F(k,s',s)\},\end{aligned} \tag{5.6}$$

in which $a_1(k,p,r) = a(k,p,r) + \frac{1}{2}(z^2 - y^2)$. The latter is in accordance with the notation of Kraichnan (1965*a*). If the second term on the right-hand side of (2.9*c*) is omitted, $\tilde{G}_{ij}$ in (2.9) becomes identical to $\tilde{G}_{ij}$ in (5.2*a*). With this choice of (5.2*a*) we need not be concerned with the above-noted second term of (2.9 c) or the Ψ term (e.g. the last four terms of (2.17 a) do not appear).

Choice (b): [; Q given by (2.29) and F given by (5.1), (5.2 b)]. The form of (TD) is identical to (2.48 b). That of (R) is given by

$$(\partial/\partial t+\nu k^2)\,F(k,t,s) = \tilde{\tilde{C}}(k,t,s) \quad (t > s), \tag{5.7}$$

where

$$\tilde{\tilde{C}}(k,t,s) \equiv \pi \iint_\Delta kpr\,dp\,dr \int_s^t ds'\,Q(r,t,s')\{a_1(k,p,r)\,F(p,t,s)-c(k,p,r)\,F(k,t,s) + a_1(k,p,r)\,F(k,t,s')\,F(p,s',s)-b(k,p,r)\,F(p,t,s')\,F(k,s',s)\}, \tag{5.8}$$

in which $c(k,p,r) = (1-y^2)$.

Choice (c): [; Q given by (5.3) and F given by (2.30)]. The form of (TD) is identical to the corresponding one in the ALHDI approximation (see eq. (10.5) in Kraichnan (1965 a), if U and G in the latter are replaced by the present Q and F, respectively. (It is not difficult to show that for an incompressible fluid $\tilde{U}_{ij}(\mathbf{x},t;\mathbf{x}',t')$ in (5.4) is equal to Kraichnan's $U_{ij}(\mathbf{x},t|t;\mathbf{x}',t|t')$, cf. (2.22), which may be regarded as one of the representatives of the ALHDI approximation.) We call this equation TD of ALHDI and do not reproduce here the explicit form of it. The form of (R) is identical to (2.48 c).

Choice (d): [; Q given by (5.3) and F given by (5.1), (5.2 a)]. The forms of (TD) and (R) are identical to TD of ALHDI and (5·5), respectively.

Choice (e): [; Q given by (5.3) and F given by (5.1), (5.2 b)]. The forms of (TD) and (R) are identical to TD of ALHDI and (5.7), respectively.

In the same way as in § 3, by assuming (3.3) and the Gaussian-time dependence of Q and F as in (3.1) and (3.2), we can estimate the constants α, β and K based on these approximations. It can be checked that the integrations over p and r discussed in § 3 converge properly for all the above approximations. The results of the numerical computations are as follows.

Choice (a):

$$K = 1{\cdot}60, \quad \alpha = 0{\cdot}81, \quad \beta = 0{\cdot}79, \tag{5.9a}$$

Choice (b):

$$K = 1{\cdot}51, \quad \alpha = 0{\cdot}78, \quad \beta = 0{\cdot}60, \tag{5.9b}$$

Choice (c):

$$K = 1{\cdot}47, \quad \alpha = 0{\cdot}41, \quad \beta = 0{\cdot}87, \tag{5.9c}$$

Choice (d):

$$K = 1{\cdot}46, \quad \alpha = 0{\cdot}41, \quad \beta = 0{\cdot}86, \tag{5.9d}$$

Choice (e):

$$K = 1{\cdot}30, \quad \alpha = 0{\cdot}36, \quad \beta = 0{\cdot}67. \tag{5.9e}$$

(The results given by (3.4) are $K = 1{\cdot}61$, $\alpha = \beta = 0{\cdot}81$.)

It is interesting to note that in spite of the differences between the (TD) and/or (R) equations of the above choices the resulting K values are all not very different from the experimental value ($\sim 1{\cdot}5$). The results (5.9) and (3.4) show that the α values given by (3.4), (5.9 a, b) are nearly twice those given by (5.9 c, d, e). This suggests that the characteristic decorrelation time of Q given by (2.29) is nearly half that of Q given by (5.3).

As a check on the above numerical calculations and those of § 3, some calculations were done concerning the ALHDI approximation, and the results were compared with those of Kraichnan (1965*b*). Kraichnan reported that the ALHDI approximation yields $\alpha = 0{\cdot}50$, $\beta = 0{\cdot}99$ and $K = 1{\cdot}60$ in the present notation (α, β and K for the ALHDI approximation are defined in the same way as in the above one). These results mean that the time-displaced correlation equation (TD), the response equation (R) and (3.3) give

$$\alpha^2/K \simeq 0{\cdot}16, \quad \alpha^2/K \simeq 0{\cdot}16 \quad \text{and} \quad \alpha/K^2 \simeq 0{\cdot}20, \tag{5.10}$$

respectively, for $\beta/\alpha = 1{\cdot}98$. While the author's calculations gave for $\beta/\alpha = 1{\cdot}98$ the values $0{\cdot}11$, $0{\cdot}16$ and $0{\cdot}20$ instead of (5.10) in order of appearance. Thus his calculations of (R) and (3.3) agree well with those of Kraichnan, but that of (TD) does not.

Hence it seems to be worthwhile to show some details of the author's calculation of (TD). That of (R) was carried out similarly. By setting the initial time t_0 at $-\infty$, the integration with respect to time can be carried out analytically without difficulty. Then the total integral region $S \equiv \{|k-p| \leqslant r \leqslant k+p\}$ over the (p, r) plane was divided into several subregions:

$$A_n \equiv \{k-p \leqslant r \leqslant k+p, \tfrac{1}{2}\times 4^{-n-1} \leqslant p/k < \tfrac{1}{2}\times 4^{-n}\},$$

$$B_n \equiv \{k-r \leqslant p \leqslant k+r, \tfrac{1}{2}\times 4^{-n-1} \leqslant r/k < \tfrac{1}{2}\times 4^{-n}\},$$

$$C \equiv \{p/k \geqslant \tfrac{1}{2}, r/k \geqslant \tfrac{1}{2}, (p+r)/k \leqslant 2\}$$

and

$$D_m \equiv \{k-p \leqslant r \leqslant k+p, d_m < (p+r)/k \leqslant d'_m$$

$$\text{with } d_0 \equiv 2, (d_j-1)/\sqrt{2} \equiv 4^{j-1} \text{ for } j \geqslant 1, \text{ and } (d'_m-1)/\sqrt{2} \equiv 4^m\},$$

where $n, m = 0, 1, 2, \ldots$. The contributions from A_n, B_n and D_m with $n \geqslant 7$, $m \geqslant 8$ were neglected. For example, the absolute values of the contributions to the above-quoted value $0{\cdot}11$ from A_6, B_6 and D_7 were all checked to be less than $0{\cdot}2\times 10^{-3}$. The numerical integration of each subregion was carried out by using a library program at the Nagoya University Computer Center written by Ninomiya and Hatano. The program is based on the automatic quadrature by the Clenshaw–Curtis method. Up to now the author has not found out the reason for the discrepancy between Kraichnan's numerical results and those obtained by the author.

It is clear from the discussions of this section that different choices of representatives yield different approximate equations. Similar cases are known in other branches of physics. For example, the method of trial function in finding eigenvalues of an operator yields different results for different choices of trial functions. Even if different choices yield different results, this itself does not detract from the value of the method. The success depends on the choice.

Although at present we cannot judge definitely which choice is best, the choice of § 2 seems to be better than choices (*a*)–(*e*) for the following reasons:

(1) the time derivatives of (TD) and (R) of this choice seem to be the most natural ones (they are MTD's);

(2) the resulting forms of (TD) and (R) are most simple;

(3) the resulting approximation has a simple model representation as discussed in § 4.

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