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The drag on a sparse random array of fixed spheres in flow at small but finite Reynolds number

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The drag on a sphere in a random array of fixed spheres of volume concentration c ($\ll 1$) at Reynolds number R ($\ll 1$) is discussed. In the case when c and R^2 are of the same order of magnitude, the drag is determined up to terms of $O(R)$.

1. Introduction

This paper treats a steady flow of an incompressible Newtonian fluid through a sparse random array of fixed solid particles. For simplicity the particles are supposed to be spheres of equal radius a and distributed in a statistically homogeneous manner with mean number density n per unit volume.

Brinkman (1947) considered the mean drag F on a sphere in such a flow and gave

$$F = 6\pi a\mu U[1+\alpha a+\ldots], \tag{1.1}$$

where
$$\alpha^2 = 9c/2a^2,$$

$c = \frac{4}{3}\pi a^3 n$ is the volume concentration of the particles, U the mean flow velocity, and μ the fluid viscosity. His result (1.1) has been confirmed (up to terms of order $c^{\frac{1}{2}}$ in the brackets of (1.1)) by Childress (1972), Lundgren (1972), Howells (1974) and Hinch (1977).

These studies have been based on the use of the Stokes equation for the fluid surrounding the spheres. However, in real flows the particle Reynolds number R is not exactly zero, even though it may be very small. Is (1.1) still valid in such flows?

Let the particle Reynolds number $R \equiv \rho Ua/\mu$ (ρ is the fluid density) be small but finite. As is well known in the study of low-Reynolds-number flow, the disturbance due to an isolated particle is not correctly described at the outer region of the particle (i.e. the region where the distance from the particle is of order not smaller than a/R) by the use of the Stokes equation. Hence it is not surprising that the hydrodynamic interaction of particles is not properly taken into account by an analysis based on the Stokes equation when the fixed random array is so sparse that the nearest-neighbour particle of a particle (say A) lies mostly in the outer region of the particle A. This suggests that even if $R \ll 1$, such an analysis does not always yield the correct asymptotic dependence of F on c for small c.

We define here a non-dimensional drag $\hat{F}$ by

$$\hat{F}(R, c) = \frac{F}{6\pi a\mu U} - 1, \tag{1.2}$$

as a measure of the strength of the effects of R and c. When both R and c are infinitely small, the drag F is given by Stokes' formula $F = 6\pi a\mu U$, and $\hat{F} = 0$. When R is finite and c is infinitely small, F is given by Oseen's formula $F \sim 6\pi a\mu U(1+\frac{3}{8}R)$, so that

$$\hat{F}(R, +0) \sim \tfrac{3}{8}R; \tag{1.3}$$

while when c is finite and R is infinitely small, Brinkman's result (1.1) implies

$$\hat{F}(+0, c) \sim (\tfrac{9}{2}c)^{\frac{1}{2}}. \tag{1.4}$$

The purpose of this paper is to demonstrate that the drag F or $\hat{F}$ may have a dependence on c basically different from Brinkman's result (1.1) or (1.4) for very small c if R is not zero, no matter how small R may be. For the sake of simplicity, we confine ourselves to the case when c and R^2 are of the same order of magnitude ($\ll 1$). It will be shown that in this case $\hat{F}(R, c)/R$ depends on R and c only through c/R^2 to leading order in $R \sim c^{\frac{1}{2}}$, and the small effects of R and c are not simply additive.

2. Basic equations and assumptions

The fluid velocity v and the pressure p are assumed to satisfy the steady Navier–Stokes equation and the continuity equation

$$\rho(\boldsymbol{v}\cdot\nabla)\,\boldsymbol{v}+\nabla p-\mu\nabla^2\boldsymbol{v} = 0, \quad \nabla\cdot\boldsymbol{v} = 0, \tag{2.1 a, b}$$

and the boundary condition

$$\boldsymbol{v} = 0 \text{ on the surfaces of the spheres.} \tag{2.2}$$

It is convenient to extend the domain of definition of the flow quantities throughout the whole space (cf. Lundgren 1972; Howells 1974). We make the velocity and pressure gradient zero in the interiors of the spheres. The extended velocity field is then continuous everywhere and the extended velocity and pressure fields satisfy (2.1) inside and outside (not on the surfaces) of the spheres. We set the pressure in any sphere equal to the mean value at its centre.

We shall denote the statistical ensemble average of a quantity, say $q(\boldsymbol{x})$, by $\langle q\rangle_0(\boldsymbol{x})$. We shall also denote the conditional ensemble average of $q(\boldsymbol{x})$, given a sphere centred at $\boldsymbol{x}_1$, by $\langle q\rangle_1(\boldsymbol{x}|\boldsymbol{x}_1)$. Similarly the conditional average of $q(\boldsymbol{x})$, given two spheres at $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, shall be denoted by $\langle q\rangle_2(\boldsymbol{x}|\boldsymbol{x}_1, \boldsymbol{x}_2)$. Where the arguments of the averaged fields are clear we shall sometimes omit them.

Taking the conditional averages of (2.1) and (2.2) given a sphere centred at $\boldsymbol{x}_1$, we obtain

$$\rho\langle(\boldsymbol{v}\cdot\nabla)\,\boldsymbol{v}\rangle_1+\nabla\langle p\rangle_1-\mu\nabla^2\langle \boldsymbol{v}\rangle_1 = -\boldsymbol{d}, \quad \nabla\cdot\langle\boldsymbol{v}\rangle_1 = 0, \tag{2.3 a, b}$$

and

$$\langle\boldsymbol{v}\rangle_1 = 0 \quad \text{on } |\boldsymbol{x}-\boldsymbol{x}_1| = a, \qquad \langle\boldsymbol{v}\rangle_1 \to U\boldsymbol{i} \quad \text{as } |\boldsymbol{x}-\boldsymbol{x}_1|\to\infty, \tag{2.4}$$

where $\boldsymbol{U} \equiv \langle\boldsymbol{v}\rangle_0$ and $\boldsymbol{i} \equiv \boldsymbol{U}/U$. The distributed resistance $\boldsymbol{d}$ is given by (see, for example, Hinch 1977)

$$\boldsymbol{d}(\boldsymbol{x}|\boldsymbol{x}_1) = \int_{|\boldsymbol{x}-\boldsymbol{x}_2|=a} \langle\boldsymbol{\sigma}\rangle_2(\boldsymbol{x}|\boldsymbol{x}_1, \boldsymbol{x}_2)\cdot\boldsymbol{n}_2\, P(\boldsymbol{x}_2|\boldsymbol{x}_1)\,\mathrm{d}A_2,$$

in which $\boldsymbol{\sigma}$ is the stress tensor, $\boldsymbol{n}_2$ the unit normal vector on A_2 directing into the fluid, and $P(\boldsymbol{x}_2|\boldsymbol{x}_1)$ is the probability of finding a second sphere centred at $\boldsymbol{x}_2$ given that there is one centred at $\boldsymbol{x}_1$. We take here the simplest case of a uniform probability $P(\boldsymbol{x}_2|\boldsymbol{x}_1) = 3c/4\pi a^3$ $(= n)$ for $|\boldsymbol{x}_1-\boldsymbol{x}_2| \geqslant 2a$ and $P(\boldsymbol{x}_2|\boldsymbol{x}_1) = 0$ for $|\boldsymbol{x}_1-\boldsymbol{x}_2| < 2a$.

In terms of dimensionless quantities defined by

$$\bar{\boldsymbol{x}} = \frac{\boldsymbol{x}}{a}, \quad \bar{\boldsymbol{v}} = \frac{\boldsymbol{v}}{U}, \quad \bar{p} = \frac{p}{\mu U/a}, \quad \bar{\boldsymbol{d}} = \frac{\boldsymbol{d}}{\mu U/a^2}, \quad \bar{\boldsymbol{N}} = \langle (\bar{\boldsymbol{v}}\cdot\bar{\boldsymbol{\nabla}})\,\bar{\boldsymbol{v}}\rangle_1, \quad R = \frac{\rho U a}{\mu},$$

(2.3) and (2.4) may be written as

$$\bar{\nabla}^2\langle\bar{\boldsymbol{v}}\rangle_1 - \bar{\boldsymbol{\nabla}}\langle\bar{p}\rangle_1 = R\bar{\boldsymbol{N}} + \bar{\boldsymbol{d}}, \quad \bar{\boldsymbol{\nabla}}\cdot\langle\bar{\boldsymbol{v}}\rangle_1 = 0, \tag{2.5a, b}$$

and

$$\langle\bar{\boldsymbol{v}}\rangle_1 = 0 \quad \text{on} \quad r = 1, \qquad \langle\bar{\boldsymbol{v}}\rangle_1 \to \boldsymbol{i} \quad \text{as } r\to\infty, \tag{2.6}$$

where $\boldsymbol{r} = \bar{\boldsymbol{x}} - \bar{\boldsymbol{x}}_1$, $r = |\boldsymbol{r}|$.

In order to solve $(\langle\bar{\boldsymbol{v}}\rangle_1, \langle\bar{p}\rangle_1)$ and obtain the mean drag $\boldsymbol{F}$, we introduce here assumptions on the resistance $\bar{\boldsymbol{d}}$ and the average $\bar{\boldsymbol{N}}$. It is known that when $R = 0$, $\bar{\boldsymbol{d}}$ may be approximated by $\bar{\boldsymbol{d}} \sim \bar{\alpha}^2\langle\bar{\boldsymbol{v}}\rangle_1$ to a first approximation for small c, where

$$\bar{\alpha}^2 = \tfrac{9}{2}c + o(c),$$

cf. for example Hinch (1977). We therefore assume

$$\bar{\boldsymbol{d}} \sim \bar{\alpha}^2\langle\bar{\boldsymbol{v}}\rangle_1. \tag{2.7}$$

Also, we note that in the dilute limit, i.e. $c\to 0$, $\bar{\boldsymbol{N}} \sim (\langle\bar{\boldsymbol{v}}\rangle_1\cdot\bar{\boldsymbol{\nabla}})\,\langle\bar{\boldsymbol{v}}\rangle_1$. Hence we assume

$$\bar{\boldsymbol{N}} \sim (\langle\bar{\boldsymbol{v}}\rangle_1\cdot\bar{\boldsymbol{\nabla}})\,\langle\bar{\boldsymbol{v}}\rangle_1. \tag{2.8}$$

In the next section we shall consider the case $c = O(R^2)$ and find the non-dimensional drag $\boldsymbol{f} \equiv \boldsymbol{F}/(\mu U a)$ or $\hat{F} = f/(6\pi) - 1$ up to terms of $O(R)$ by assuming (2.7) and (2.8). The validity of using these assumptions will be discussed at the end of the next section.

3. The dependence of F on c when c is $O(R^2)$

If c is $O(R^2)$, then $\bar{\alpha}^2$ is $O(R^2)$. When $\bar{\alpha}^2 = SR^2$, with S being a constant of order unity, we try an inner expansion of the form

$$\left.\begin{aligned} \langle\bar{\boldsymbol{v}}\rangle_1 &= \boldsymbol{v}_0(\boldsymbol{r}) + R\boldsymbol{v}_1(\boldsymbol{r}) + \dots, \\ \langle\bar{p}\rangle_1 &= p_0(\boldsymbol{r}) + Rp_1(\boldsymbol{r}) + \dots. \end{aligned}\right\} \tag{3.1}$$

From (2.5)–(2.8) and (3.1), we obtain

(i) *in* $O(R^0)$:

$$\nabla^2\boldsymbol{v}_0 - \boldsymbol{\nabla}p_0 = 0, \quad \boldsymbol{\nabla}\cdot\boldsymbol{v}_0 = 0, \tag{3.2}$$

$$\boldsymbol{v}_0 = 0 \quad \text{on } r = 1, \quad \boldsymbol{v}_0 \to \boldsymbol{i} \quad \text{as } r\to\infty, \tag{3.3}$$

(ii) *in* $O(R^1)$:

$$\nabla^2\boldsymbol{v}_1 - \boldsymbol{\nabla}p_1 = (\boldsymbol{v}_0\cdot\boldsymbol{\nabla})\,\boldsymbol{v}_0, \quad \boldsymbol{\nabla}\cdot\boldsymbol{v}_1 = 0, \tag{3.4}$$

$$\boldsymbol{v}_1 = 0 \quad \text{on } r = 1. \tag{3.5}$$

Here and hereafter in this section we sometimes omit the bar for ease of writing. Correspondingly to (3.1) the non-dimensional drag $\boldsymbol{f}$ is expanded as $\boldsymbol{f} = \boldsymbol{f}_0 + R\boldsymbol{f}_1 + \dots$.

In the outer region, assuming

$$\left.\begin{aligned} \langle\bar{\boldsymbol{v}}\rangle_1 &= \boldsymbol{i} + R\boldsymbol{V}_1(\tilde{\boldsymbol{r}}) + \dots, \\ \boldsymbol{\nabla}\langle\bar{p}\rangle_1 + \bar{\alpha}^2\boldsymbol{i} &= R^3\tilde{\boldsymbol{\nabla}}P_1(\tilde{\boldsymbol{r}}) + \dots \quad (\tilde{\boldsymbol{r}} \equiv R\boldsymbol{r}), \end{aligned}\right\} \tag{3.6}$$

we obtain

(iii) *in* $O(R^3)$:
$$(\tilde{\nabla}^2 - \boldsymbol{i}\cdot\tilde{\nabla} - S)\,\boldsymbol{V}_1 - \tilde{\nabla}P_1 = 0, \quad \tilde{\nabla}\cdot\boldsymbol{V}_1 = 0, \tag{3.7a, b}$$

$$\boldsymbol{V}_1 \to 0 \quad \text{as } r\to\infty. \tag{3.8}$$

The zeroth-order inner solution $(\boldsymbol{v}_0, p_0)$ is just the well-known Stokes solution for the flow past a single sphere and is expanded for large r as

$$\boldsymbol{v}_0 = \boldsymbol{i} - \boldsymbol{s}(\boldsymbol{r}) + O(r^{-3}), \quad p_0 = -t(\boldsymbol{r}),$$

which yields the matching condition for the first-order outer solution $(\boldsymbol{V}_1, P_1)$

$$\boldsymbol{V}_1 \sim -\boldsymbol{s}(\tilde{\boldsymbol{r}}) \quad \text{and } P_1 \sim -t(\tilde{\boldsymbol{r}}) \quad \text{as } \tilde{r}\to 0, \tag{3.9}$$

where
$$\boldsymbol{s}(\boldsymbol{r}) = \frac{3}{4}\frac{\boldsymbol{i} + (\boldsymbol{i}\cdot\boldsymbol{r})\,\boldsymbol{r}/r^2}{r}, \quad t(\boldsymbol{r}) = \frac{3}{2}\frac{\boldsymbol{i}\cdot\boldsymbol{r}}{r^3}.$$

The field $(\boldsymbol{s}(\boldsymbol{r}), t(\boldsymbol{r}))$ is the so-called Stokeslet due to a force $6\pi\delta(\boldsymbol{r})\,\boldsymbol{i}$. The solution $(\boldsymbol{V}_1, P_1)$ may therefore be obtained by solving (3.7) with $6\pi\delta(\tilde{\boldsymbol{r}})\,\boldsymbol{i}$ on the right-hand side of (3.7*a*). By using three-dimensional Fourier transforms, we obtain

$$\boldsymbol{V}_1(\tilde{\boldsymbol{r}}) = \frac{-3}{4\pi^2}\int \boldsymbol{\Gamma}(\boldsymbol{k})\, e^{i\boldsymbol{k}\cdot\tilde{\boldsymbol{r}}}\, d^3\boldsymbol{k} \tag{3.10a}$$

$$= \frac{-3}{4\pi^2}\int [\boldsymbol{\Gamma}(\boldsymbol{k}) - \boldsymbol{\Gamma}_s(\boldsymbol{k})]\, e^{i\boldsymbol{k}\cdot\tilde{\boldsymbol{r}}}\, d^3\boldsymbol{k} - \boldsymbol{s}(\tilde{\boldsymbol{r}}), \tag{3.10b}$$

$$P_1(\tilde{\boldsymbol{r}}) = -t(\tilde{\boldsymbol{r}}), \tag{3.10c}$$

where
$$\boldsymbol{\Gamma}(\boldsymbol{k}) = \frac{\boldsymbol{i} - \boldsymbol{k}(\boldsymbol{i}\cdot\boldsymbol{k})/k^2}{k^2 + i(\boldsymbol{i}\cdot\boldsymbol{k}) + S}, \quad \boldsymbol{\Gamma}_s(\boldsymbol{k}) = \frac{\boldsymbol{i} - \boldsymbol{k}(\boldsymbol{i}\cdot\boldsymbol{k})/k^2}{k^2}.$$

In order to evaluate $\boldsymbol{V}_1$ for small $\tilde{r}$, we divide the region of integration in (3.10*b*) into two parts, $0 \leqslant k \leqslant \tilde{r}^{-\gamma}$ and $k > \tilde{r}^{-\gamma}$, where $0 < \gamma < 1$ (cf. Childress 1964). We then obtain

$$\int [\boldsymbol{\Gamma}(\boldsymbol{k}) - \boldsymbol{\Gamma}_s(\boldsymbol{k})]\, e^{i\boldsymbol{k}\cdot\tilde{\boldsymbol{r}}}\, d^3\boldsymbol{k} = \int_{k\leqslant \tilde{r}^{-\gamma}} [\boldsymbol{\Gamma}(\boldsymbol{k}) - \boldsymbol{\Gamma}_s(\boldsymbol{k})]\, d^3\boldsymbol{k}$$
$$- \int_{k>\tilde{r}^{-\gamma}} \frac{i(\boldsymbol{i}\cdot\boldsymbol{k})\,[\boldsymbol{i} - \boldsymbol{k}(\boldsymbol{i}\cdot\boldsymbol{k})/k^2]}{k^4}\, e^{i\boldsymbol{k}\cdot\tilde{\boldsymbol{r}}}\, d^3\boldsymbol{k} + O(\tilde{r}^{1-\gamma}) + O(\tilde{r}^{\gamma}), \tag{3.11}$$

so that
$$\boldsymbol{V}_1(\tilde{\boldsymbol{r}}) = -\boldsymbol{s}(\tilde{\boldsymbol{r}}) + \boldsymbol{A} + \boldsymbol{B} + o(\tilde{r}^0) \quad \text{as } \tilde{r}\to 0, \tag{3.12}$$

where
$$\boldsymbol{A} = -\frac{3}{4\pi^2}\int [\boldsymbol{\Gamma}(\boldsymbol{k}) - \boldsymbol{\Gamma}_s(\boldsymbol{k})]\, d^3\boldsymbol{k} = K\boldsymbol{i},$$

with
$$K = -\frac{3}{2\pi}\int\!\!\int (1-\cos^2\theta)\left[\frac{1}{k^2 + ik\cos\theta + S} - \frac{1}{k^2}\right] k^2 \sin\theta\, d\theta\, dk,$$

and $\boldsymbol{B}$ is the contribution from the second term on the right-hand side of (3.11) which is $O(\tilde{r}^0)$ and odd in $\tilde{\boldsymbol{r}}$. Putting $\cos\theta = t$, we obtain

$$K = -\frac{3}{2\pi}\int_0^\infty \mathrm{d}k \int_{-1}^{1} \mathrm{d}t(1-t^2)\left[\frac{1}{k^2+\mathrm{i}kt+S}-\frac{1}{k^2}\right]k^2$$
$$= -\frac{3}{4\pi}\int_0^1 \mathrm{d}t(1-t^2)\int_{-\infty}^{\infty} \mathrm{d}k\left[\frac{k^2}{k^2+\mathrm{i}kt+S}+\frac{k^2}{k^2-\mathrm{i}kt+S}-2\right].$$

The integration with respect to k can be evaluated using a contour integral. After some algebra we obtain

$$K = K(S) = \frac{3}{8}\left[(2S+1)(4S+1)^{\frac{1}{2}} - 4S^2 \ln\frac{(4S+1)^{\frac{1}{2}}+1}{(4S+1)^{\frac{1}{2}}-1}\right]. \qquad (3.13)$$

Equations (3.12) and (3.10*c*) yield the matching condition for the first-order inner solution $(\boldsymbol{v}_1, p_1)$:

$$\boldsymbol{v}_1 \sim K\boldsymbol{i}+\boldsymbol{B}, \quad p_1 = o(r^{-1}) \quad \text{as } r\rightarrow\infty.$$

This condition and (3.4), (3.5) suffice to determine $(\boldsymbol{v}_1, p_1)$. It is known (see, for example, Brenner and Cox 1963) that the first-order non-dimensional force $\boldsymbol{f}_1$ due to such a field $(\boldsymbol{v}_1, p_1)$ is $6\pi K\boldsymbol{i}$. While, as is well known, the zeroth-order non-dimensional force $\boldsymbol{f}_0$ due to $(\boldsymbol{v}_0, p_0)$ is $6\pi\boldsymbol{i}$, so that the dimensional mean drag $\boldsymbol{F}$ is given by

$$\boldsymbol{F} = a\mu U[\boldsymbol{f}_0+R\boldsymbol{f}_1+\ldots] = 6\pi a\mu U[1+RK(S)+\ldots]. \qquad (3.14)$$

We are now in position to discuss the consistency of the approximations. In order to see the validity of assuming (2.7) and (2.8), it is convenient to write (2.5*a*) as

$$\bar{\nabla}^2\langle\bar{\boldsymbol{v}}\rangle_1 - \bar{\nabla}\langle\bar{p}\rangle_1 - R(\langle\bar{\boldsymbol{v}}\rangle_1\cdot\bar{\nabla})\langle\bar{\boldsymbol{v}}\rangle_1 - \bar{\alpha}^2\langle\bar{\boldsymbol{v}}\rangle_1 = R\Delta\bar{\boldsymbol{N}}+\Delta\bar{\boldsymbol{d}}, \qquad (3.15)$$

where $$\Delta\bar{\boldsymbol{N}} = \bar{\boldsymbol{N}} - (\langle\bar{\boldsymbol{v}}\rangle_1\cdot\bar{\nabla})\langle\bar{\boldsymbol{v}}\rangle_1, \quad \Delta\bar{\boldsymbol{d}} = \bar{\boldsymbol{d}} - \bar{\alpha}^2\langle\bar{\boldsymbol{v}}\rangle_1.$$

Making the assumptions (2.7) and (2.8) is equivalent to neglecting $R\Delta\bar{\boldsymbol{N}}$ and $\Delta\bar{\boldsymbol{d}}$. These quantities $R\Delta\bar{\boldsymbol{N}}$ and $\Delta\bar{\boldsymbol{d}}$ may be estimated in a way fundamentally the same as in Acrivos, Hinch & Jeffrey (1980, §§3.3 and 4.2) as follows.

(i) *Estimation of* $\Delta\boldsymbol{d}$.

We consider here the hydrodynamic interaction of two spheres centred at $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ on the basis of the 'effective-medium' equation (3.15) with the right-hand side set at zero and the average $\langle\,\rangle_1$ replaced by the conditional average $\langle\,\rangle_2$, given two spheres centred at $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$. (cf. Vasseur & Cox 1977; Kaneda & Ishii 1982 for the hydrodynamic interaction between two particles in a pure solvent when $R \neq 0$.)

To leading order for large $|\boldsymbol{x}_1-\boldsymbol{x}_2|$, the $\boldsymbol{x}_2$ particle exerts a non-dimensional force $-\boldsymbol{f}$, where from (3.14) $-\boldsymbol{f} = -\lambda\boldsymbol{i}$ with $\lambda \doteq 6\pi\,(1+KR+\ldots)$. The first correction to the effect of the $\boldsymbol{x}_2$ particle is a modification of this force due to the velocity disturbance $\langle\boldsymbol{u}\rangle_1(\boldsymbol{x}_2|\boldsymbol{x}_1) \equiv \langle\boldsymbol{v}\rangle_1(\boldsymbol{x}_2|\boldsymbol{x}_1)-\boldsymbol{i}$ from the $\boldsymbol{x}_1$ particle, which changes this force $-\boldsymbol{f}$ to, say, $-\boldsymbol{f}^1$. Here the correction $\delta\boldsymbol{f} \equiv \boldsymbol{f}^1-\boldsymbol{f}$ is given by $\delta\boldsymbol{f} = \tilde{\lambda}\langle\boldsymbol{u}\rangle_1(\boldsymbol{x}_2|\boldsymbol{x}_1)$ in which $\tilde{\lambda} = \lambda+O(R)$. When the $\boldsymbol{x}_2$ particle is in the inner region of the $\boldsymbol{x}_1$ particle, the next correction is a change in the force exerted by the $\boldsymbol{x}_2$ particle in response to a similar change to $-\boldsymbol{f}-\delta\boldsymbol{f}$ in the force exerted by the $\boldsymbol{x}_1$ particle – the so-called second reflection. While if the $\boldsymbol{x}_2$ particle is in the outer region of the $\boldsymbol{x}_1$ particle, the next correction is of order higher (in R) than $\delta\boldsymbol{f}$.

We assumed earlier that $P(\boldsymbol{x}_2|\boldsymbol{x}_1) = 3c/4\pi$ for $|\boldsymbol{x}_1-\boldsymbol{x}_2| > 2$ and $P(\boldsymbol{x}_2|\boldsymbol{x}_1) = 0$ for $|\boldsymbol{x}_1-\boldsymbol{x}_2| < 2$ (in the non-dimensional form). We now choose $\bar{\alpha}$ as

$$\bar{\alpha}^2 = \frac{3c\lambda}{4\pi},$$

which implies $\bar{\alpha}^2 = \frac{9}{2}c+o(c)$. Then we may write $\Delta \boldsymbol{d}$ as

$$\Delta \boldsymbol{d} = \int_{|\boldsymbol{x}_1-\boldsymbol{x}_2|\geqslant 1} \mathrm{d}V_2 \Big\{P(\boldsymbol{x}_2|\boldsymbol{x}_1)\int_{|\boldsymbol{x}'-\boldsymbol{x}_2|=1} \langle\sigma\rangle_2(\boldsymbol{x}'|\boldsymbol{x}_1,\boldsymbol{x}_2)\cdot\boldsymbol{n}'\delta(\boldsymbol{x}'-\boldsymbol{x})\,\mathrm{d}A' - \lambda\langle\boldsymbol{v}\rangle_1(\boldsymbol{x}_2|\boldsymbol{x}_1)\,\delta(\boldsymbol{x}_2-\boldsymbol{x})\Big\}.$$

It can be shown from (3.10a) that V_1 decays like $\tilde{r}^{-3}$ for large $\tilde{r}$, so that the disturbance velocity $\boldsymbol{u}\equiv\boldsymbol{v}-\boldsymbol{i}$ decays like $R\tilde{r}^{-3}$. Hence, the correction δf is $O(R(R|\boldsymbol{x}_2-\boldsymbol{x}_1|)^{-3})$ when $R|\boldsymbol{x}_2-\boldsymbol{x}_1|\gtrsim 1$. Noting that when $R|\boldsymbol{x}_2-\boldsymbol{x}_1|\sim\tilde{r}\equiv Rr\gtrsim 1$, $(f+\delta f)-\lambda\langle\boldsymbol{v}\rangle_1(\boldsymbol{x}_2|\boldsymbol{x}_1) = \delta f-\lambda\langle\boldsymbol{u}\rangle_1(\boldsymbol{x}_2|\boldsymbol{x}_1) = (\tilde{\lambda}-\lambda)\langle\boldsymbol{u}\rangle_1(\boldsymbol{x}_2|\boldsymbol{x}_1) = O(R)\times O(R\tilde{r}^{-3})$ and the correction next to δf is of order smaller than δf, we find that the force density $\Delta\boldsymbol{d}$ is of order smaller than $c\delta f = O(cR\tilde{r}^{-3}) = O(R^3\tilde{r}^{-3})$ in the outer region. It is therefore consistent to neglect $\Delta\boldsymbol{d}$ in the outer equation (3.7a), while in the inner region the disturbance velocity decays like r^{-1}, and the density $\Delta\boldsymbol{d}$ coming from the second reflection is $O(cr^{-2}) = O(R^2r^{-2})$. It is therefore consistent to neglect $\Delta\boldsymbol{d}$ in the inner equations (3.2) and (3.4).

(ii) *Estimation of* ΔN.

If we use $\boldsymbol{v}'$ to denote the fluctuation of $\boldsymbol{v}$ about its conditional average $\langle\boldsymbol{v}\rangle_1$, then

$$\Delta N \equiv \langle(\boldsymbol{v}\cdot\nabla)\,\boldsymbol{v}\rangle_1-(\langle\boldsymbol{v}\rangle_1\cdot\nabla)\,\langle\boldsymbol{v}\rangle_1 = \langle(\boldsymbol{v}'\cdot\nabla)\,\boldsymbol{v}'\rangle_1.$$

As $c\rightarrow 0$, the fluctuation $\boldsymbol{v}'$ is primarily due to the infrequent occurrence of a nearby particle, i.e.

$$\Delta N \sim \int(\langle\boldsymbol{v}-\langle\boldsymbol{v}\rangle_1\rangle_2\cdot\nabla)\,\langle\boldsymbol{v}-\langle\boldsymbol{v}\rangle_1\rangle_2\,P(\boldsymbol{x}_2|\boldsymbol{x}_1)\,\mathrm{d}V_2. \tag{3.16}$$

For large $|\boldsymbol{x}-\boldsymbol{x}_1| = r$, the main contribution to this integral comes from $\boldsymbol{x}_2$ near $\boldsymbol{x}$. If we estimate (3.16) using the fluctuation $\boldsymbol{v}'$ for an isolated $\boldsymbol{x}_2$ particle with no $\boldsymbol{x}_1$ particle, then the integral vanishes because this fluctuation essentially gives the term in the bulk average $\langle(\boldsymbol{u}'\cdot\nabla)\,\boldsymbol{u}'\rangle_0 = \langle(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\rangle_0-(\langle\boldsymbol{v}\rangle_0\cdot\nabla)\,\langle\boldsymbol{v}\rangle_0$, and with the assumption of homogeneity $\langle(\boldsymbol{v}\cdot\nabla)\,\boldsymbol{v}\rangle_0 = \Sigma_i\,\partial_i\,\langle(v_i\,\boldsymbol{v})\rangle_0 = 0$ and $\nabla\langle\boldsymbol{v}\rangle_0 = 0$ so that $\langle(\boldsymbol{u}'\cdot\nabla)\,\boldsymbol{u}'\rangle_0 = 0$, where $\boldsymbol{u}'$ denotes the fluctuation of $\boldsymbol{v}$ about $\langle\boldsymbol{v}\rangle_0$. Hence ΔN decays as $r\rightarrow\infty$.

The $\boldsymbol{x}_1$ particle induces in the neighbourhood of the $\boldsymbol{x}_2$ particle a disturbance velocity of $O(|\boldsymbol{x}_2-\boldsymbol{x}_1|^{-1})$ when $|\boldsymbol{x}_2-\boldsymbol{x}_1|\lesssim R^{-1}$, and $O(R^{-2}|\boldsymbol{x}_2-\boldsymbol{x}_1|^{-3})$ when $R|\boldsymbol{x}_2-\boldsymbol{x}_1|\gtrsim 1$. The leading effect of the hydrodynamic interaction of two particles on (3.16) is a modification of the velocity disturbance around the $\boldsymbol{x}_2$ particle caused by this velocity disturbance of $O(|\boldsymbol{x}_2-\boldsymbol{x}_1|^{-1}, R^{-2}|\boldsymbol{x}_2-\boldsymbol{x}_1|^{-3})$ induced by the $\boldsymbol{x}_1$ particle. Noting that the velocity gradient outside the $\boldsymbol{x}_2$ particle is $O(|\boldsymbol{x}_2-\boldsymbol{x}_1|^{-2}, R^{-2}|\boldsymbol{x}_2-\boldsymbol{x}_1|^{-3})$, we find after the integration that $R\Delta N$ is $Ro(c\tilde{r}^{-3}) = o(R^3\tilde{r}^{-3})$ in the outer region, and $RO(c\ln R) = O(R^3\ln R)$ in the inner region. It is therefore consistent to neglect ΔN in (3.2), (3.4) and (3.7).

4. Results and Discussion

There are two lengthscales in the present problem: a/R (say l_1) and the shielding length $a/c^{\frac{1}{2}}$ (say l_2). The lengthscales l_1 and l_2 respectively become infinite as R and c tend to zero. If R is finite and c is small enough, i.e. l_1 is finite and l_2 is large enough, then the Oseen interactions, which cannot be described by the Stokes equation, become more dominant than the Stokes interactions, where we call the hydrodynamic interaction of two particles as Oseen (Stokes) interaction if each of them lies within the outer (inner) region of the other. If $c = O(R^2)$, i.e. $l_1 = O(l_2)$, then the occurrence of a nearby particle in the inner region of a particle is so unlikely that the effect of the $\boldsymbol{d}$-term ($\sim \alpha^2\langle v\rangle_1$, see (2.3) and (2.7)) – the so-called shielding effect – does not appear in the equations of motions (3.2) and (3.4) for the zeroth- and first-order inner fields. The effect does appear in (3.7) for the first-order outer fields, but it is to be noted that (3.7) also contains the term $(\boldsymbol{i}\cdot\tilde{\nabla})\,\boldsymbol{V}_1$ which comes from the nonlinear convective term in the Navier–Stokes equation; we need take into account both the shielding and convective effects at the same time.

According to (3.14), the non-dimensional drag $\hat{F}$ defined by (1.2) is given by

$$\hat{F}(R, c) \sim RK(S), \tag{4.1}$$

where $S = \bar{\alpha}^2/R^2 \sim \frac{9}{2}c/R^2$ and K is given by (3.13), which is expanded as

$$K(S) = \begin{cases} \frac{3}{8}+\frac{3}{2}S+\ldots, & \text{for small } S, \\ S^{\frac{1}{2}}+\frac{3}{40}/S^{\frac{1}{2}}+\ldots, & \text{for large } S. \end{cases} \tag{4.2}$$

Thus the leading term of (4.1) for small (large) S agrees with Oseen's formula (1.3) (Brinkman's result (1.4)). The values of K are plotted in figure 1 for intermediate values of S.

We may define a quantity Δ as a measure indicating the strength of the effect of the hydrodynamic interactions between particles by

$$\Delta(R, c) \equiv [F(R, c)-F(R, 0)]/(6\pi a\mu U) = \hat{F}(R, c)-\hat{F}(R, 0);$$

if such interactions are negligible, then $\Delta = 0$. Brinkman's result (1.1) or (1.4) implies

$$\Delta(+0, c) \sim (\tfrac{9}{2}c)^{\frac{1}{2}} \sim \bar{\alpha}, \tag{4.3}$$

while (1.3), (3.14) or (4.1), and (4.2) give

$$\Delta(R, c) \sim R[K(S)-\tfrac{3}{8}] \tag{4.4a}$$

$$\sim \begin{cases} \dfrac{3}{2}\dfrac{\bar{\alpha}^2}{R}, & \text{for small } S, \quad (4.4b) \\[2ex] \bar{\alpha}-\tfrac{3}{8}R+\dfrac{3}{40}\dfrac{R^2}{\bar{\alpha}}, & \text{for large } S. \quad (4.4c) \end{cases}$$

The dependence of Δ and also $\hat{F}$ on R and c (in the (R^2, c)-plane) is sketched in figure 2. If we increase $S \sim 9c/(2R^2)$ (but within the limit $\bar{\alpha}^2 = O(R^2)$) then, as seen in (4.4c), Δ does approach $\bar{\alpha}$ in agreement with Brinkman's result (4.3). However, the dependence of Δ on c shown by (4.4a) or (4.4b) is clearly different from (4.3), although both Δ's in (4.3) and (4.4) are $O(c^{\frac{1}{2}})$ because $c = O(R^2)$. Figure 2 suggests a transition from $c^{\frac{1}{2}}$-dependence to c/R-dependence of Δ as c is sufficiently decreased with R fixed.

Corresponding to the fact that the shielding and convective effects cannot be treated separately, the small effects of c and R are not simply additive, as seen in (4.1) and (4.4).

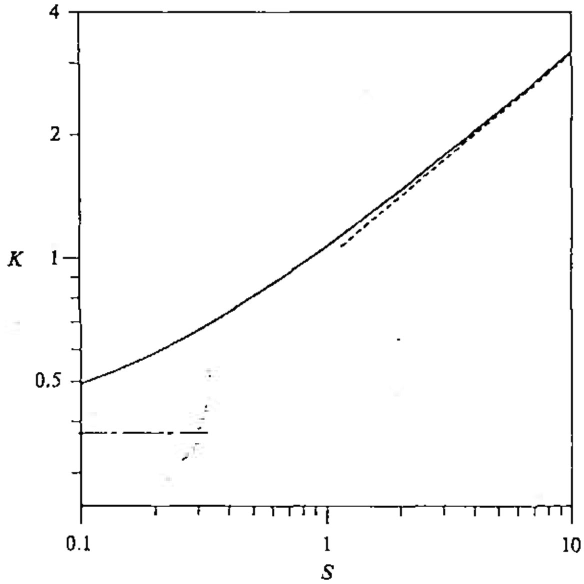

FIGURE 1. Log–log plot of K given by (3.13) vs. $S \sim \frac{3}{2}c/R^2$ (——). $K = S^{\frac{1}{2}}$ (– – –) and $K = \frac{3}{8}$ (–·–·–) are also plotted.

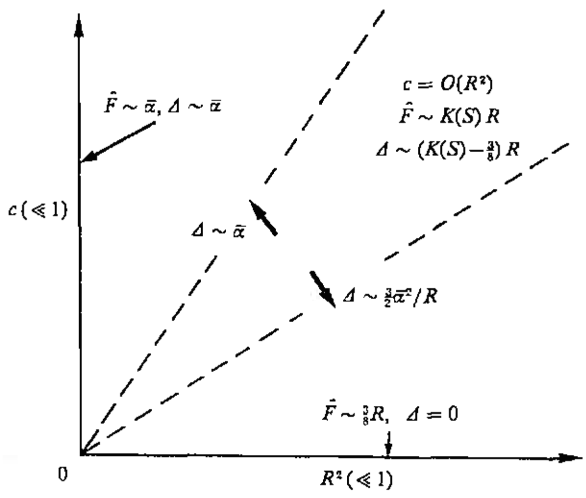

FIGURE 2. Dependence of Δ and $\hat{F}$ on R and c. Here $S = \bar{\alpha}^2/R^2$ and $\bar{\alpha}^2 \sim \frac{3}{2}c$.

Finally, we mention a similarity of the present problem with the others. As mentioned above, the present problem has two lengthscales $l_1 = a/R$ and $l_2 = a/c^{\frac{1}{2}}$, and the small effects of c and R are not simply additive in a certain range of the ratio l_1/l_2. This is also true for the problem of heat transfer from a dilute fixed bed of heated spheres studied by Acrivos *et al.* (1980), in which there are two lengthscales $l_1 = a/\epsilon$ and $l_2 = a/c^{\frac{1}{2}}$, where ϵ is the Péclet number. They showed that when $c = O(\epsilon^2)$, i.e. $l_1/l_2 = O(1)$, the small effects of c and ϵ are not simply additive. Another example of such a problem is provided by the unsteady flow of viscous fluid past an object at finite $R(\ll 1)$. This flow has two lengthscales $l_1 = a/R$ and $l_2 = a/\gamma^{\frac{1}{2}}$ (the so-called skin depth), where $\gamma = \omega a^2\rho/\mu$ and ω represents the frequency or $\omega \boldsymbol{v} \sim (\mathrm{d}/\mathrm{d}t)\,\boldsymbol{v}$. If the term $(\mathrm{d}/\mathrm{d}t)\,\boldsymbol{v}$ is replaced by $\omega\boldsymbol{v}$, then the usual unsteady Navier–Stokes equation for an incompressible fluid becomes of the same form as the 'effective-medium' equation (3.15) with $\Delta\bar{\boldsymbol{d}}$ and $\Delta\bar{N}$ neglected. When $\gamma = O(R^2)$, i.e. $l_1/l_2 = O(1)$, this problem yields an equation similar to (3.7), cf. Bentwich & Miloh (1978) and also Ockendon (1968). In fact, (15) in Bentwich & Miloh is mathematically equivalent to (3.7) for axisymmetric flows; (3.12) and (3.13) could be obtained also from the stream function (19) in their paper.

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