Approximation Algorithms for Weighted Independent Set Problem

Hirata Laboratory (Dept. of Comput. Sci. and Math. Info.)

Unweighted Independent Set Problem (IS)

Find a **maximum size independent set** in the graph $G = (V, E)$ (Fig. 1).

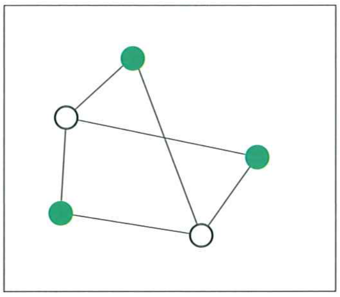

Figure 1: Green vertices form a maximum independent set (ave. deg. $\bar{d} = 2.4$)

IS is one of the most famous NP-hard problems. Approximation ratio

$$\text{app. ratio} = \max_{\text{instances}} \frac{\text{opt. size}}{\text{app. size}}$$

is analyzed for many algorithms in terms of the **average degree**

$$\bar{d} = \frac{\sum_{v \in V} d(v)}{|V|},$$

where

$$\text{degree of } v\text{: } d(v) = (\#\text{ of neighbors of } v).$$

For example:

$$\frac{\bar{d}+1}{2} : \text{Hochbaum[83] (LP)} \quad (1)$$

$$\frac{2\bar{d}+3}{5} : \text{Halldórsson, Radhakrishnan[97] (LP), detailed}$$

$$O\left(\frac{\bar{d}\log\log\bar{d}}{\log\bar{d}}\right) : \text{Halldórsson[00] (SDP)} \quad (2)$$

Weighted Independent Set Problem (WIS)

Find a **maximum weight independent set** in the graph $G = (V, E)$ with vertex weight $w\colon V \to \mathbb{R}^+$ (Fig. 2).

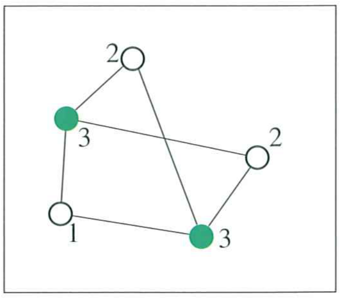

Figure 2: Green vertices form a max. weight ind. set with weight 6 (w. av. deg. $\bar{d}_w \sim 2.54$).

There are **no previous results** in terms of average degree $\bar{d}$.

In fact, $\bar{d}$ is **inappropriate** parameter for WIS: The graphs in Fig. 3 are essentially same (for sufficiently small ϵ) for WIS but they have very different value of $\bar{d}$ (2.8, 6.8, resp.).

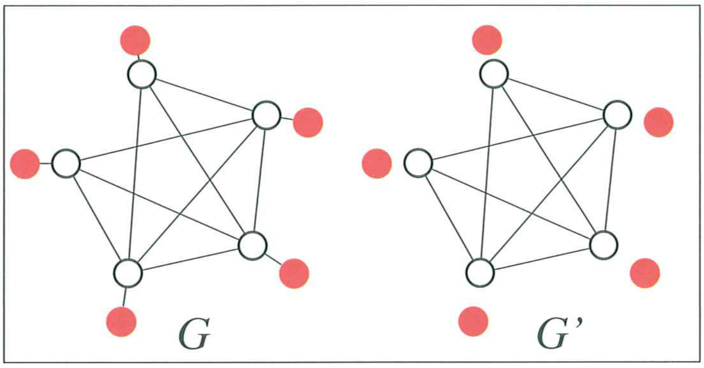

Figure 3: Graphs with almost same weighted ind. set (Black vertices have weight 1 and red vertices have weight ϵ. Every red vertex in G' is connected to any other vertices).

New Parameter: Weighted Average Degree

For WIS, we introduced **weighted degree**:

$$d_w(v) = \frac{\sum_{u\text{: neighbor of } v} w(u)}{w(v)}$$

and the **weighted average degree**:

$$\bar{d}_w = \frac{\sum_{v \in V} w(v) \cdot d_w(v)}{\sum_{v \in V} w(v)}.$$

The weighted average degree is seemingly simple extension of the unweighted average degree, but more suitable: Two graphs in Fig. 3 have almost the same average weighted degree $4.6 + O(\epsilon)$.

Our Results: Analysis With Weighted Average Degree

- A Weighted Greedy algorithm (select one vertex with minimum weighted degree in turn to form an independent set) has approximation ratio $\bar{d}_w + 1$. The result is similar to the unweighted case but proof is more complicated.
- We can extend the results (1) and (2) to

$$\text{LP + W.Greedy} : \frac{\bar{d}_w + 1}{2},$$

$$\text{SDP + W.Greedy} : O\left(\frac{\bar{d}_w \log\log\bar{d}_w}{\log\bar{d}_w}\right).$$