

# Class of viable modified $f(R)$ gravities describing inflation and the onset of accelerated expansion

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A general approach to viable modified  $f(R)$  gravity is developed in both the Jordan and the Einstein frames. A class of exponential, realistic modified gravities is introduced and investigated with care. Special focus is made on step-class models, most promising from the phenomenological viewpoint and which provide a natural way to classify all viable modified gravities. One- and two-step models are explicitly considered, but the analysis is extensible to  $N$ -step models. Both inflation in the early universe and the onset of recent accelerated expansion arise in these models in a natural, unified way. Moreover, it is demonstrated that models in this category easily pass all local tests, including stability of spherical body solution, nonviolation of Newton's law, and generation of a very heavy positive mass for the additional scalar degree of freedom.

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## I. INTRODUCTION

Modified gravity models constitute an interesting dynamical alternative to the  $\Lambda$ CDM cosmology in that they are able to describe with success the current acceleration in the expansion of our Universe, the so-called dark energy epoch. Moreover, modified  $F(R)$  gravity (for a review, see e.g. [1]) has undergone many studies which conclude that this gravitational alternative to dark energy [2,3] is able to pass the solar system tests. The investigation of cosmic acceleration as well as the study of the cosmological properties of  $F(R)$  models has been carried out in Refs. [1–7].

Recently the importance of those models was reassessed, namely, with the appearance of the so-called “viable”  $F(R)$  models [8–11]. Those are models which do satisfy the cosmological as well as the local gravity constraints, which had caused a number of problems to some of the first-generation theories of that kind. The final aim of all these phenomenological models is to describe a segment as large as possible of the whole history of our universe, as well as to recover all local predictions of Einstein's gravity, which have been verified experimentally to very good accuracy, at the solar system scale.

Let us recall that, in general (see e.g. [1], for a review), the total action for the modified gravitational models reads

$$S = \frac{1}{\kappa^2} \int d^4x \sqrt{-g} [R + f(R)] + S_{(m)}. \quad (1)$$

Here  $f(R)$  is a suitable function, which defines the modified gravitational part of the model. The general equation of motion in  $F(R) \equiv R + f(R)$  gravity with matter is given by

$$\begin{aligned} \frac{1}{2} g_{\mu\nu} F(R) - R_{\mu\nu} F'(R) - g_{\mu\nu} \square F'(R) + \nabla_\mu \nabla_\nu F'(R) \\ = -\frac{\kappa^2}{2} T_{(m)\mu\nu}, \end{aligned} \quad (2)$$

where  $T_{(m)\mu\nu}$  is the matter energy-momentum tensor.

In this paper we investigate two classes of “viable” modified gravitational models which means, roughly speaking, they have to incorporate the vanishing (or fast decrease) of the cosmological constant in the flat ( $R \rightarrow 0$ ) limit, and must exhibit a suitable constant asymptotic behavior for large values of  $R$ . A huge family of these models, which we will term first class—and to which almost all of the models proposed in the literature belong—can be viewed as containing all possible smooth versions of the following sharp step-function model. To discuss this toy model at the distribution level will prove to be very useful in order to grasp the essential features that *all* models in this large family are bound to satisfy. In other words, to extract the general properties of the whole family in a rather simple fashion (which will involve, of course, precise distribution calculus).

This simple model reads

$$f(R) = -2\Lambda_{\text{eff}}\theta(R - R_0), \quad (3)$$

where  $\theta(R - R_0)$  is Heaviside's step distribution. Models in this class are characterized by the existence of one or more transition scalar curvatures, an example being  $R_0$  in the above toy model (but there can be more, as we will later see).

The other class of modified gravitational models that has been considered contains a sort of “switching on” of the cosmological constant as a function of the scalar curvature  $R$ . A simplest version of this kind reads

$$f(R) = 2\Lambda_{\text{eff}}(e^{-bR} - 1). \quad (4)$$

Here the transition is smooth. The above two models may be combined in a natural way if one is also interested in the phenomenological description of the inflationary epoch. For example, a two-step model may be the smooth version of

$$f(R) = -2\Lambda_0\theta(R - R_0) - 2\Lambda_I\theta(R - R_I), \quad (5)$$

with  $R_0 \ll R_I$ , the latter being the inflation scale curvature.

The typical, smooth behavior of  $f(R)$  associated with the one- and two-step models is given, in the smooth case, in Figs. 1 and 2, respectively. The main problem associated with these sharp models is the appearance of possible antigravity regime in a region around the transition point and antigravity in a past epoch, which is not phenomenologically acceptable. On the other hand, an analytical study of these models can be easily carried out, as discussed in the appendix.

The existence of viable (or ‘‘chameleon’’)  $f(R)$  theories with a phase of early-time inflation is not known to us from the literature. The fact that we are able to provide several classes of models of this kind that are consistent also with the late-time accelerated expansion is thus a novelty worth noting.

The paper is organized as follows. The next section reviews  $f(R)$  gravity in the physical, Jordan frame. Equations of motion are presented and the solar system tests (absence of a tachyon, stability of the spherical body solution, and nonviolation of Newton law) are discussed. In Sec. III we give a number of viable modified gravities which may lead to the unification of early-time inflation with late-time acceleration and satisfy the solar system tests. Some properties of such viable modified gravity are discussed in detail. Section IV is devoted to the presentation of viable models via conformal transformation, as a kind of scalar-tensor theory that is mathematically equivalent to the original theory. In some examples, the explicit form of the scalar potential is derived. Corrections to Newton’s law are also obtained for some of the realistic theories here considered. It turns out that these models do

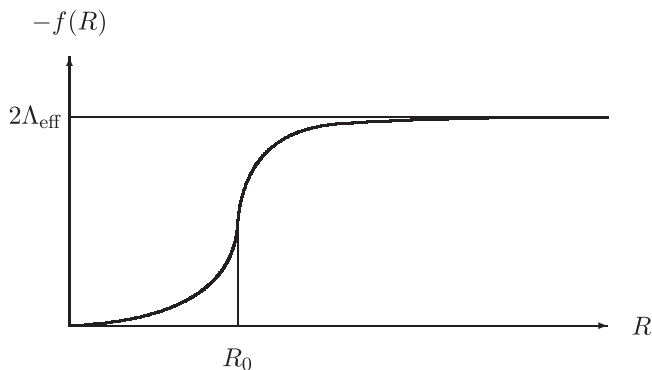


FIG. 1. Typical behavior of  $f(R)$  in the one-step model.

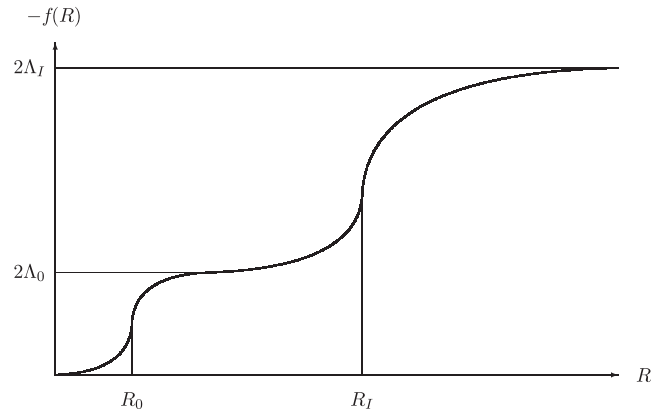


FIG. 2. Typical behavior of  $f(R)$  in the two-step model.

pass the stringent Newton law bounds, since the corrections to Newton’s law in these cases turn out to be negligible. Some summary and outlook is given in the discussions section. Finally, in an appendix we show how to evaluate the positions of the corresponding de Sitter critical points by considering the sharp version of the one- and two-step models, expressed in terms of the Heaviside and Dirac distributions.

## II. MODIFIED GRAVITY IN THE JORDAN FRAME

Regarding the precise determination of the modifying term  $f(R)$ , we here revisit this issue in the Jordan frame (instead of the Einstein one). Let us recall the two sufficient conditions which often lead to realistic models (see, for example [8])

$$f(0) = 0, \quad \lim_{R \rightarrow R_1} f(R) = -\alpha, \quad (6)$$

where  $\alpha$  is a suitable curvature scale which represents an effective cosmological constant, being  $R_1 \gg R_0$ , with  $R_0 > 0$ , the transition point. The condition  $f(0) = 0$  ensures the disappearance of the cosmological constant in the limit of flat space-time.

By using these conditions, some models in this class are seen to be able to pass the local tests (with some extra bounds on the theory parameters) and are also capable of explaining the observed recent acceleration of the universe expansion, provided that  $\alpha = \Lambda_0 = 2H_0^2$ ,  $H_0$  being the Hubble constant at the epoch of reference. However, they do not incorporate early-time inflation, which comes into play at higher values of  $R$ . Thus, one might also reasonably require that [10]

$$f(0) = 0, \quad \lim_{R \rightarrow R_2} f(R) = -(\alpha + \alpha_I), \quad (7)$$

where  $\alpha_I \gg \alpha$  is associated with the inflation cosmological constant  $\Lambda_I$ , and where  $R_2 \gg R_I \gg R_0$ ,  $R_I$  being the corresponding transition large scalar curvature.

Further restrictions, like small corrections to Newton’s law and the stability of planetlike gravitational solutions

need to be fulfilled too [3]. All those can be also formulated in the mathematically equivalent Einstein frame. Here we present a short review of them in the Jordan frame which we consider as the physical one (see Ref. [6] for a discussion of the physical (non-)equivalence of the Einstein and Jordan frames).

The starting point is the trace of the equations of motion, which is trivial in the Einstein theory but gives precious dynamical information in the modified gravitational models. It reads

$$3\nabla^2 f'(R) = R + 2f(R) - Rf'(R) - \kappa^2 T. \quad (8)$$

The above trace equation can be interpreted as an equation of motion for the nontrivial ‘‘scalon’’  $f'(R)$  (since it is indeed associated with the corresponding scalar field in the other frame). For solutions with constant scalar curvature  $R_*$ , the scalon field is constant and one obtains the following vacuum solution:

$$R_* + 2f(R_*) - R_*f'(R_*) = 0. \quad (9)$$

Furthermore, according to [10], we can describe the degree of freedom associated with the scalon by means of a scalar field  $\chi$ , defined by  $F'(R) = 1 + f'(R) = e^{-\chi}$ . If we consider a perturbation around the vacuum solution of constant curvature  $R_*$ , given by  $R = R_* + \delta R$ , where

$$\delta R = -\frac{1 + f'(R_*)}{f''(R_*)} \delta\chi, \quad (10)$$

then the equation of motion for the scalon field is

$$\square \delta\chi - \frac{1}{3} \left( \frac{1 + f'(R_*)}{f''(R_*)} - R_* \right) \delta\chi = -\frac{\kappa^2}{6(1 + f'(R_*))} T. \quad (11)$$

As a result, in connection with the local and with the planetary tests, the following effective mass plays a very crucial role:

$$M^2 \equiv \frac{1}{3} \left( \frac{1 + f'(R_*)}{f''(R_*)} - R_* \right). \quad (12)$$

If  $M^2 < 0$ , a tachyon appears and this leads to an instability. Even if  $M^2 > 0$ , when  $M^2$  is small, it is  $\delta R \neq 0$  at long ranges, which generates a large correction to Newton’s law. As a result,  $M^2$  has to be positive and very large in order to pass both the local and the astronomical tests. This stability condition can be also derived within QFT in de Sitter space-time (see, for instance, [12]).

Concerning the matter instability [3,13,14], this might occur when the curvature is rather large, as on a planet, as compared with the average curvature of the universe  $R \sim (10^{-33} \text{ eV})^2$ . In order to arrive to a stability condition, we can start by noting that the scalon equation can be rewritten in the form [3,13]

$$\square R + \frac{f'''(R)}{f''(R)} \nabla_\rho R \nabla^\rho R + \frac{(1 + f'(R))R}{3f''(R)} - \frac{2(R + f(R))}{3f''(R)} = \frac{\kappa^2}{6f''(R)} T. \quad (13)$$

If we now consider a perturbation,  $\delta R$ , of the Einstein gravity solution  $R = R_e = -\frac{\kappa^2 T}{2} > 0$ , we obtain

$$0 \simeq (-\partial_t^2 + U(R_e))\delta R + C, \quad (14)$$

with the effective potential

$$U(R_e) \equiv \left( \frac{F''''(R_e)}{F''(R_e)} - \frac{F''''(R_e)^2}{F''(R_e)^2} \right) \nabla_\rho R_e \nabla^\rho R_e + \frac{R_e}{3} - \frac{F'(R_e)F''''(R_e)R_e}{3F''(R_e)^2} - \frac{F'(R_e)}{3F''(R_e)} + \frac{2F(R_e)F''''(R_e)}{3F''(R_e)^2} - \frac{F''''(R_e)R_e}{3F''(R_e)^2}. \quad (15)$$

If  $U(R_e)$  is positive, then the perturbation  $\delta R$  becomes exponentially large and the whole system becomes unstable. Thus, the matter stability condition is, in this case,

$$U(R_e) < 0. \quad (16)$$

Coming back to the vacuum condition (9), we recall that, within the cosmological framework, it may be rederived making use of the dynamical system approach. This consists in rewriting the generalized Friedmann equations of the modified gravitational model in terms of a first-order differential system and looking for its critical points. For a modified gravity model, the associated dynamical system can be written as [7,15] [ $a(t)$  being the expansion factor in a FRW flat space-time]:

$$\begin{aligned} \frac{d}{d \ln a} \Omega_R &= 2\Omega_R(2 - \Omega_R)\Omega_R - \beta(1 - \Omega_F - \Omega_\rho), \\ \frac{d}{d \ln a} \Omega_F &= 2\Omega_F(2 - \Omega_R) + (\Omega_F - \Omega_R)(1 - \Omega_F - \Omega_\rho), \\ \frac{d}{d \ln a} \Omega_\rho &= [2(2 - \Omega_R) - 3(w + 1) \\ &\quad + 1 - \Omega_F - \Omega_\rho]\Omega_\rho, \end{aligned} \quad (17)$$

where  $w = \frac{p}{\rho}$  is the usual barotropic constant, being

$$\begin{aligned} \Omega_R &= \frac{R}{6H^2}, & \Omega_F &= -\frac{f(R) - Rf'(R)}{6H^2(1 + f'(R))}, \\ \Omega_\rho &= \frac{\chi\rho}{3H^2(1 + f'(R))}, \end{aligned} \quad (18)$$

with

$$\beta = \frac{1 + f'(R)}{Rf''(R)}. \quad (19)$$

There exists yet another quantity

$$\Omega_{\dot{F}} = -\frac{f'(R)}{H(1+f'(R))}, \quad (20)$$

which satisfies the constraint

$$\Omega_{\dot{F}} + \Omega_F + \Omega_\rho = 1. \quad (21)$$

The critical points are solutions of the (algebraic) system

$$\begin{aligned} 0 &= 2\Omega_R(2 - \Omega_R)\Omega_R - \beta(1 - \Omega_F - \Omega_\rho), \\ 0 &= 2\Omega_F(2 - \Omega_R) + (\Omega_F - \Omega_R)(1 - \Omega_F - \Omega_\rho), \\ 0 &= [2(2 - \Omega_R) - 3(w + 1) + 1 - \Omega_F - \Omega_\rho]\Omega_\rho. \end{aligned} \quad (22)$$

As an example of those, the de Sitter critical points are the ones in the invariant vacuum submanifold  $\Omega_\rho = 0$ . These solutions read

$$\Omega_R = 2, \quad R_* = 12H_*^2, \quad (23)$$

and

$$\Omega_F = 1, \quad R_* = R_* f'(R_*) - 2f(R_*). \quad (24)$$

The last equation coincides with Eq. (9). It is a transcendental equation, for all our models, and can be solved only by iteration (see the appendix) or either numerically by other methods.

The stability condition associated with the de Sitter critical point in this dynamical system framework can be investigated too, and reads [15]

$$1 < \beta(R_*) = \frac{1 + f'(R_*)}{R_* f''(R_*)}. \quad (25)$$

It coincides with the requirement that the effective mass (12) be positive. In the matter-radiation sector, where  $\Omega_\rho$  is nonvanishing, other critical points may also exist. In the next section, we give explicit examples of exponential, viable modified gravity.

### III. EXAMPLES OF REALISTIC EXPONENTIAL MODIFIED GRAVITY

After the general discussion above, we will here present some new viable  $f(R)$  models. We start with a most simple one

$$f(R) = \alpha(e^{-bR} - 1). \quad (26)$$

Since  $f(0) = 0$  and  $f(R) \rightarrow -\alpha$  for large  $R$ , conditions (6) are satisfied. Moreover,

$$f'(R) = -b\alpha e^{-bR}, \quad f''(R) = b^2\alpha e^{-bR}. \quad (27)$$

We have seen that in the discussion of the viability of modified gravitational models, the existence of vacuum constant curvature solutions plays a very crucial role, namely, the existence of solutions of Eq. (9). With regard to the trivial fixed point  $R_* = 0$ , this model has the properties

$$1 + f'(0) = 1 - \alpha b, \quad f''(0) = \alpha b^2. \quad (28)$$

Thus, the effective mass for  $R_* = 0$  is

$$M^2(0) = \frac{1 - \alpha b}{3\alpha b^2}, \quad (29)$$

and then Minkowski space-time is stable as soon as  $\alpha b < 1$ . Such condition is equivalent to  $1 + f'(0) > 0$ .

In order to investigate the existence of other fixed points, we first have to find the existence conditions and then make use of Newton's method or some of its variants (see the appendix). It is easy to see that for the model (26), one has the critical point only if  $\alpha b > 1$ , namely  $K'(0) > 1$ , where the function  $K(R)$  defined in the appendix. Then, one can construct an approximation procedure to solve Eq. (9) in terms of an iteration process, namely

$$R_{n+1} = R_n - \frac{R_n - R_n f'(R_n) + 2f(R_n)}{1 + f'(R_n) - R_n f''(R_n)}. \quad (30)$$

For a starting point,  $R = R_1$ , large enough,  $f(R)$  is approximately constant and a few iterations give

$$R_{*,1} \simeq 2\alpha. \quad (31)$$

This critical point is stable and corresponds to the current acceleration in the universe expansion. This follows from that fact that the effective mass is

$$M^2 \simeq \frac{1}{3\alpha b^2} e^{2b\alpha}, \quad (32)$$

namely it is positive and large. However, since  $\alpha b > 1$ , for very small  $R$ , one has antigravity effects, namely  $1 + f'(R) < 0$ , as we shall see in Sec. IV.

Matter instability can also be investigated. Equation (15) gives

$$U(R_e) \simeq -\frac{1}{3\alpha b^2} e^{2bR_e}, \quad (33)$$

which is negative, thus the matter stability condition is fulfilled.

The model we have discussed so far does not exhibit a sharp transition curvature. But there are many models where one or more transitions of this kind appear. The Hu-Sawicki (HS) model [8] belongs to this one-step class family of models. A simple choice for a one-step model is in our case

$$f(R) = \alpha \left[ \frac{1 + e^{-bR_0}}{1 + e^{b(R-R_0)}} - 1 \right] = -\alpha \frac{e^{bR} - 1}{e^{bR} + e^{bR_0}}. \quad (34)$$

When  $R$  is very small, we have

$$f(R) \simeq -\frac{\alpha b}{1 + e^{bR_0}} R + O(R^2), \quad (35)$$

while for suitable values of  $b$ , one has the same behavior as in the HS model, where the continuous parameter  $b$  plays the role of the integer  $n$  in the above-mentioned model.

Higher values of  $b$  give rise to a sharper transition, occurring at  $R_0$  from very small values of  $f(R)$  towards a constant value  $-\alpha$ . This model has an effective mass, evaluated at  $R_* = 0$ , which turns out to be negative; thus Minkowski space-time is unstable in this case, and it might happen that  $1 + f'(R) < 0$  around the transition.

A simple modification of the above model which incorporates the inflationary era, namely, the requirement (7), is a combination of the two models discussed so far; that is

$$f(R) = \alpha(e^{-bR} - 1) - \alpha_I \frac{e^{bR} - 1}{e^{bR} + e^{bR_I}}, \quad (36)$$

or, as a two-step model,

$$f(R) = -\alpha \frac{e^{bR} - 1}{e^{bR} + e^{bR_0}} - \alpha_I \frac{e^{bR} - 1}{e^{bR} + e^{bR_I}}. \quad (37)$$

Again,  $f(0) = 0$  and, at the value  $R = R_I$ , there is a transition to a higher constant value  $-(\alpha + \alpha_I)$  which can be related to inflation.

We should note that  $f(R)$  in (37) is a monotonically decreasing function. Then, when  $R$  is very large,  $f(R)$  tends to a constant value, which could correspond to the effective cosmological constant generating inflation. In order to describe the recent accelerating expansion of the universe,  $f(R)$  should remain almost constant, that is,  $f'(R) = 0$ , for sufficiently small values of  $R$  corresponding to the curvature of the present universe.

The main problem with this model is the appearance of antigravity in connection with the transition point when the function is sharp (see the discussion in the appendix). The appearance of antigravity in the past, namely, around  $R_I$ , is not acceptable, as we have already commented.

A possible modification of the previous model is the following:

$$f(R) = -\alpha(e^{-bR} - 1) + cR^N \frac{e^{bR} - 1}{e^{bR} + e^{bR_I}}, \quad (38)$$

with  $N > 2$  and  $c > 0$ . In this variant, similarly to the theory [11], during the inflationary era at  $R > R_I$ ,  $f(R)$ , the model acquires also a power dependence on the scalar curvature, which may help to exit from the inflationary stage.

As discussed in detail in the appendix, for the sharp, theta models, besides the problem of antigravity, for  $R_0 \ll \alpha$  and  $R_I \ll \alpha_I$ , they possess, generically, two de Sitter critical points: one around the transition point  $R_* \simeq \frac{5R_0}{4}$  and the other being

$$R_{*,2} \simeq 2\alpha. \quad (39)$$

We can also investigate the matter instability. For the two-step model (37), we now assume

$$R_0 \ll R \sim R_e \ll R_I. \quad (40)$$

Then  $f(R)$  in (37) can be approximated as

$$f(R) \sim -\alpha \left\{ -1 + (1 + e^{-bR_0})e^{-b(R-R_0)} \right\} - \frac{\alpha_I b R}{1 + e^{bR_I}}. \quad (41)$$

We may assume

$$\frac{\alpha_I b}{1 + e^{bR_I}} \ll 1, \quad (42)$$

since  $bR_I$  could be very large [see the argument around (78) about antigravity]. Then we find

$$U(R_e) \simeq -\frac{e^{b(R_e - R_0)}}{3\alpha b^2(1 + e^{-bR_0})}, \quad (43)$$

which is negative and there is no instability.

We conclude this section with a variant of the above model which facilitates the analytic computation and the discussion concerning antigravity. In fact, as a smoothed one-step function, we may consider

$$\begin{aligned} f(R) &= -\alpha \left( \tanh\left(\frac{b(R - R_0)}{2}\right) + \tanh\left(\frac{bR_0}{2}\right) \right) \\ &= -\alpha \left( \frac{e^{b(R-R_0)} - 1}{e^{b(R-R_0)} + 1} + \frac{e^{bR_0} - 1}{e^{bR_0} + 1} \right). \end{aligned} \quad (44)$$

When  $R \rightarrow 0$ , we find that

$$f(R) \rightarrow -\frac{\alpha b R}{2 \cosh^2\left(\frac{bR_0}{2}\right)}. \quad (45)$$

and thus  $f(0) = 0$ , as required. On the other hand, when  $R \rightarrow +\infty$ ,

$$f(R) \rightarrow -2\Lambda_{\text{eff}} \equiv -\alpha \left( 1 + \tanh\left(\frac{bR_0}{2}\right) \right). \quad (46)$$

If  $R \gg R_0$  in the present universe,  $\Lambda_{\text{eff}}$  plays the role of the effective cosmological constant. We also obtain

$$f'(R) = -\frac{\alpha b}{2 \cosh^2\left(\frac{b(R-R_0)}{2}\right)}, \quad (47)$$

which has a minimum when  $R = R_0$ :

$$f'(R_0) = -\frac{\alpha b}{2}. \quad (48)$$

Then in order to avoid antigravity, we find

$$0 < 1 + f'(R_0) < 1 - \frac{\alpha b}{2}. \quad (49)$$

The model given by Eq. (44) is able to describe late acceleration. In order to show that the de Sitter critical points exist, we can compute the function  $K(R) = Rf'(R) - 2f(R)$  of the appendix and we have  $K'(R) = Rf''(R) - f'(R)$ ,  $K''(R) = Rf'''(R)$ ,  $1 - K'(0) = 1 + f'(0)$ , where

$$K'(R) = -\left[ bR \tanh\left(\frac{b(R - R_0)}{2}\right) + 1 \right] f'(R), \quad (50)$$

Thus, enforcing the absence of antigravity, one has

$$1 + f'(0) > 0. \quad (51)$$

namely,  $K'(0) < 1$ . In this case however  $K'(R) > 0$  for  $R > R_0$ , and it is negative for  $R < R_0$ . Therefore, we cannot use the argument of the appendix. For the one-step model, however, one can actually live with antigravity in the future, thus in the sharp version, the analysis in the appendix leads again in fact to the existence of two dS critical points.

As a model that is able to describe both the inflation and the late acceleration epochs, we can consider the following two-step model:

$$f(R) = -\alpha_0 \left( \tanh\left(\frac{b_0(R-R_0)}{2}\right) + \tanh\left(\frac{b_0 R_0}{2}\right) \right) - \alpha_I \left( \tanh\left(\frac{b_I(R-R_I)}{2}\right) + \tanh\left(\frac{b_I R_I}{2}\right) \right). \quad (52)$$

We now assume

$$R_I \gg R_0, \quad \alpha_I \gg \alpha_0, \quad b_I \ll b_0, \quad (53)$$

and

$$b_I R_I \gg 1. \quad (54)$$

When  $R \rightarrow 0$  or  $R \ll R_0, R_I$ ,  $f(R)$  behaves as

$$f(R) \rightarrow -\left( \frac{\alpha_0 b_0}{2 \cosh^2\left(\frac{b_0 R_0}{2}\right)} + \frac{\alpha_I b_I}{2 \cosh^2\left(\frac{b_I R_I}{2}\right)} \right) R \quad (55)$$

and find  $f(0) = 0$  again. When  $R \gg R_I$ , we find

$$f(R) \rightarrow -2\Lambda_I \equiv -\alpha_0 \left( 1 + \tanh\left(\frac{b_0 R_0}{2}\right) \right) - \alpha_I \left( 1 + \tanh\left(\frac{b_I R_I}{2}\right) \right) \sim -\alpha_I \left( 1 + \tanh\left(\frac{b_I R_I}{2}\right) \right). \quad (56)$$

On the other hand, when  $R_0 \ll R \ll R_I$ , we find

$$f(R) \rightarrow -\alpha_0 \left[ 1 + \tanh\left(\frac{b_0 R_0}{2}\right) \right] - \frac{\alpha_I b_I R}{2 \cosh^2\left(\frac{b_I R_I}{2}\right)} \sim -2\Lambda_0 \equiv -\alpha_0 \left[ 1 + \tanh\left(\frac{b_0 R_0}{2}\right) \right]. \quad (57)$$

Here we have assumed (54). We also find

$$f'(R) = -\frac{\alpha_0 b_0}{2 \cosh^2\left(\frac{b_0(R-R_0)}{2}\right)} - \frac{\alpha_I b_I}{2 \cosh^2\left(\frac{b_I(R-R_I)}{2}\right)}, \quad (58)$$

which has two valleys when  $R \sim R_0$  or  $R \sim R_I$ . When  $R = R_0$ , we obtain

$$f'(R_0) = -\alpha_0 b_0 - \frac{\alpha_I b_I}{2 \cosh^2\left(\frac{b_I(R_0-R_I)}{2}\right)} > -\alpha_I b_I - \alpha_0 b_0. \quad (59)$$

On the other hand, when  $R = R_I$ , we get

$$f'(R_I) = -\alpha_I b_I - \frac{\alpha_0 b_0}{2 \cosh^2\left(\frac{b_0(R_0-R_I)}{2}\right)} > -\alpha_I b_I - \alpha_0 b_0. \quad (60)$$

Then, in order to avoid the antigravity period, we find

$$\alpha_I b_I + \alpha_0 b_0 < 2. \quad (61)$$

The existence of the de Sitter critical points in this two-step model is much more difficult to investigate. However, in order to get the acceleration of the Universe expansion it is sufficient that  $\omega_{\text{eff}} < -\frac{1}{3}$ .

We now investigate the correction to the Newton's law and the matter instability issue. In the solar system domain, on or inside the earth, where  $R \gg R_0$ ,  $f(R)$  in (44) can be approximated by

$$f(R) \sim -2\Lambda_{\text{eff}} + 2\alpha e^{-b(R-R_0)}. \quad (62)$$

On the other hand, since  $R_0 \ll R \ll R_I$ , by assuming Eq. (54),  $f(R)$  in (52) could be also approximated by

$$f(R) \sim -2\Lambda_0 + 2\alpha e^{-b_0(R-R_0)}, \quad (63)$$

which has the same expression, after having identified  $\Lambda_0 = \Lambda_{\text{eff}}$  and  $b_0 = b$ . Then, we may check the case of (62) only.

We find that the effective mass has the following form

$$M^2 \sim \frac{e^{b(R-R_0)}}{4\alpha b^2}, \quad (64)$$

which could be very large again, as in the last section, and the correction to Newton's law can be made negligible. We also find that  $U(R_b)$  in (15) has the form

$$U(R_e) = -\frac{1}{2\alpha b} \left( 2\Lambda + \frac{1}{b} \right) e^{-b(R_e-R_0)}, \quad (65)$$

which could be negative and would suppress any instability.

Thus, we have here presented several realistic exponential models which naturally unify the inflation with the dark energy epochs (with a radiation/matter dominance phase between, as in Refs. [10,11]). In addition, the Newton law is respected and all spherical body solutions (Earth, Sun, etc.) are stable.

#### IV. VIABLE MODELS IN THE EINSTEIN FRAME

As is well known from previous studies, it is often quite convenient to go from the Jordan (physical) frame to the mathematically equivalent Einstein frame description, where  $f(R)$  models become scalar-tensor theories with a suitable potential. In particular, corrections to Newton's law and the matter instability can be also investigated in the Einstein frame directly, where the relevant degrees of freedom are a new tensor metric and a scalar field. More specifically, concerning the inflation issue, the Einstein

frame can indeed be very useful. Following e.g. Ref. [3], we can introduce the auxiliary field  $A$  and rewrite the action (1) as

$$S = \frac{1}{\kappa^2} \int d^4x \sqrt{-g} \{ (1 + f'(A))(R - A) + A + f(A) \}. \quad (66)$$

From the equation of motion with respect to  $A$ , if  $f''(A) \neq 0$ , it follows that  $A = R$ . By using the conformal transformation  $g_{\mu\nu} \rightarrow e^\sigma g_{\mu\nu}$ , with  $\sigma = -\ln(1 + f'(A))$ , we obtain the Einstein frame action [3]:

$$\begin{aligned} S_E &= \frac{1}{\kappa^2} \int d^4x \sqrt{-g} \left\{ R - \frac{3}{2} \left( \frac{F''(A)}{F'(A)} \right)^2 g^{\rho\sigma} \partial_\rho A \partial_\sigma A \right. \\ &\quad \left. - \frac{A}{F'(A)} + \frac{F(A)}{F'(A)^2} \right\} \\ &= \frac{1}{\kappa^2} \int d^4x \sqrt{-g} \left( R - \frac{3}{2} g^{\rho\sigma} \partial_\rho \sigma \partial_\sigma \sigma - V(\sigma) \right), \quad (67) \end{aligned}$$

$$V(\sigma) = e^\sigma g(e^{-\sigma}) - e^{2\sigma} F(g(e^{-\sigma})) = \frac{A}{F'(A)} - \frac{F(A)}{F'(A)^2}. \quad (68)$$

Here  $g(e^{-\sigma})$  is given by solving  $\sigma = -\ln(1 + f'(A)) = \ln F'(A)$ , as  $A = g(e^{-\sigma})$ . After the scale transformation  $g_{\mu\nu} \rightarrow e^\sigma g_{\mu\nu}$  is done, there appears a coupling of the scalar field  $\sigma$  with matter. For example, if matter is a scalar field  $\Phi$ , with mass  $M$ , whose action is given by

$$S_\phi = \frac{1}{2} \int d^4x \sqrt{-g} (-g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - M^2 \Phi^2), \quad (69)$$

then there appears a coupling with  $\sigma$  (in this Einstein frame):

$$S_{\phi E} = \frac{1}{2} \int d^4x \sqrt{-g} (-e^\sigma g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - M^2 e^{2\sigma} \Phi^2). \quad (70)$$

The strength of the coupling is of the same order as that of the gravitational coupling,  $\kappa$ . Unless the mass corresponding to  $\sigma$ , which is defined by

$$m_\sigma^2 \equiv \frac{1}{2} \frac{d^2 V(\sigma)}{d\sigma^2} = \frac{1}{2} \left\{ \frac{A}{F'(A)} - \frac{4F(A)}{(F'(A))^2} + \frac{1}{F''(A)} \right\}, \quad (71)$$

is big, there will appear a large correction to the Newton law. Newton's law has been investigated in the solar system, as well as on earth, where the curvature is much larger than  $R_0$ . For the model (26), we find

$$m_\sigma^2 \sim \frac{e^{bR}}{2\alpha b^2}, \quad (72)$$

which is positive. As we will discuss soon [(74)–(78)], we find  $1/b \ll R_0 \ll R$ , and therefore  $bR \gg 1$ , which tell us that  $m_\sigma^2$  could be very large and the correction to the Newton law would be very small. For the model (37), we

find

$$m_\sigma^2 \sim \frac{e^{b(R-R_0)}}{2\alpha b^2(1 + e^{-bR_0})}, \quad (73)$$

which could be very large again, and the correction to the Newton law correspondingly very small.

Equation (66) also tells that if

$$1 + f'(A) < 0, \quad (74)$$

then antigravity could appear, since the effective gravitational constant is given by

$$\kappa_{\text{eff}}^2 \equiv \frac{\kappa^2}{1 + f'(A)}. \quad (75)$$

In order to avoid it, the condition (75) must be satisfied, at least until present, all the way since the beginning of the universe. Some remark is in order. In the antigravity region, there is no evolution of the universe with a flat spatial part. In the usual Einstein gravity, we have the Friedmann-Robertson-Walker (FRW) equation,  $(3/\kappa^2)H^2 = \rho$ , but in the antigravity region, since the sign of  $\kappa^2$  changes, as  $\kappa^2 \rightarrow -\kappa^2$ , we get  $-(3/\kappa^2)H^2 = \rho$ . Since the left-hand side of this equation is always negative and the right-hand side is always positive, there is no solution, which shows that there is no time-evolution of the universe in this case. We should also note that even if  $1 + f'(A)$  is negative, the conformal transformation itself can still be well defined, if we use the absolute value of  $1 + f'(A)$ ; that is,  $|1 + f'(A)|g_{\mu\nu} \rightarrow g_{\mu\nu}$ . In that case, however, in the obtained Einstein frame action, the sign of the scalar curvature  $R$  becomes negative, that is, antigravity again appears. Moreover, the initial value problem which is formulated in  $f(R)$  gravity via conformal transformation [16] is not well defined. This is the reason why we avoid the consideration of antigravity regimes.

For the simple model (4), the condition (74) reads

$$1 - \alpha b e^{-bR} > 0. \quad (76)$$

Since the scalar curvature  $R$  in the past universe could be larger than the curvature  $R_0$  in the present universe, we find  $1 - \alpha b e^{-bR} > 1 - \alpha b e^{-bR_0}$ . Therefore, if

$$1 - \alpha b e^{-bR_0} > 0 \quad (77)$$

is satisfied, the condition (74) can be satisfied too. Equation (77) tells us that

$$\alpha b < 1 \quad \text{or} \quad \frac{1}{b} \ll R_0. \quad (78)$$

In order that  $f(R)$  can play the role of an effective cosmological constant for the present universe, the second condition  $1/b \ll R_0$  should be preferred. The situation is not much changed in the two-step model (37), and the antigravity condition tells us that  $1/b \ll R_0$ , again.

We can now investigate the regions which reproduce realistic models. For the simple model (4), conditions (77)

or (78), which avoid antigravity in the history of the universe, can be expressed as

$$\frac{1}{b} \ll R_0 \sim (10^{-33} \text{ eV})^2. \quad (79)$$

On the other hand, the condition that the correction to Newton's law should be small is that  $m_\sigma$ , given by (72), must be large. Since  $\alpha$  in (72) plays the role of the effective cosmological constant in the present universe, we have

$$\alpha \sim R_0 \sim (10^{-33} \text{ eV})^2. \quad (80)$$

In the solar system, we find  $R \sim 10^{-61} \text{ eV}^2$ . Even if we choose  $1/b \sim R_0 \sim (10^{-33} \text{ eV})^2$ , we find that  $m_\sigma^2 \sim 10^{1000} \text{ eV}^2$ , which is ultimately heavy. Then, there will not be any appreciable correction to the Newton law. In the air on earth,  $R \sim 10^{-50} \text{ eV}^2$ , and even if we choose  $1/b \sim R_0 \sim (10^{-33} \text{ eV})^2$  again, we find that  $m_\sigma^2 \sim 10^{10\,000\,000\,000} \text{ eV}^2$ . Then, the correction to Newton's law is never observed in such a model.

For the model (37), since  $R_0 \ll R \ll R_I$  in the solar system or on the earth,  $f(R)$  can be approximated by (41). Then the effective gravitational constant could be given by

$$\frac{1}{\kappa_{\text{eff}}^2} \equiv \frac{1}{\kappa^2} \left( 1 - \frac{\alpha_I b}{1 + e^{bR_I}} \right). \quad (81)$$

The mass of the scalar field  $\sigma$  is given by (73), which is very large again, that is,  $m_\sigma^2 \sim 10^{1000} \text{ eV}^2$  in the solar system and  $m_\sigma^2 \sim 10^{10\,000\,000\,000} \text{ eV}^2$  in the air surrounding the earth, and therefore the correction to the Newton law is negligibly small, either.

There is a technical point which deserves more careful considerations. It is that  $A = R$  has to be expressed as a function of  $\sigma$  by solving the equation

$$f'(A) = e^{-\sigma} - 1 \quad (82)$$

and this can be explicitly done for the simplest cases only. For example, in Ref. [8] a class of models defined by means of the function

$$f(R) = -\frac{m^2 c_1 (R/m^2)^n}{1 + c_2 (R/m^2)^n}, \quad n \geq 1. \quad (83)$$

has been proposed. Here  $c_1, c_2$  are arbitrary dimensionless constants, while  $m$  has the dimension of mass. This model yields an effective cosmological constant which generates the late-time accelerated expansion. For such class of models, Eq. (82) reduces to an algebraic equation of order  $2n$ , which can be explicitly solved for  $n = 1$  and  $n = 2$ . In the simplest case,  $n = 1$ , one easily gets

$$A_\pm = \frac{m^2}{c_2} \left[ \pm e^{\sigma/2} \sqrt{\frac{c_1}{e^\sigma - 1}} - 1 \right], \quad c_2 > 0. \quad (84)$$

$$\begin{aligned} V(\sigma) &= e^\sigma (1 - e^\sigma) A - e^{2\sigma} f(A) \\ &= \frac{m^2 e^\sigma}{c_2} (\sqrt{c_1} e^{\sigma/2} - \sqrt{e^\sigma - 1})^2, \end{aligned} \quad (85)$$

where the positive solution  $A_+$  has been chosen. For  $n = 2$  the potential assumes a quite complicated form, which is practically useless.

A simple modification of the model (83) is the following [10]:

$$f(R) = -\frac{m^2 c_1 (R/m^2)^n + c_3}{1 + c_2 (R/m^2)^n}, \quad n \geq 1, \quad (86)$$

which for  $n = 1$  and  $c_2 > 0$  gives rise to the potential

$$\begin{aligned} V(\sigma) &= -\frac{m^2 e^\sigma}{c_2} [(c_1 + 1) e^\sigma \\ &\quad - 2e^{\sigma/2} \sqrt{(c_1 - c_2 c_3)(e^\sigma - 1)} - 1]. \end{aligned} \quad (87)$$

Now, we go back to the models that we have considered above. For some of them we can give an explicit form for the potential. We start with our first model (6). In such case, Eq. (82) is a simple transcendental equation which gives rise to

$$A = -\frac{1}{b} \ln \left[ \frac{1}{\alpha b} (1 - e^{-\sigma}) \right], \quad (88)$$

$$V(\sigma) = \frac{e^\sigma}{b} \left[ 1 + (\alpha b - 1) e^\sigma + (e^\sigma - 1) \ln \frac{1 - e^{-\sigma}}{\alpha b} \right]. \quad (89)$$

where  $\sigma > 0$  is understood.

Also for the one-step model (34), Eq. (82) becomes a transcendental one, but it can be solved and we can eventually write the potential in the form

$$V(\sigma) = e^\sigma (1 - e^\sigma) A - e^{2\sigma} f(A), \quad (90)$$

where  $A = A(\sigma)$  is given by

$$\begin{aligned} A_\pm &= \frac{1}{\beta} \log \left\{ \frac{\mu e^\sigma}{2\gamma(e^\sigma - 1)} \left[ 1 - \frac{2(1 - e^{-\sigma})}{\mu} \right. \right. \\ &\quad \left. \left. \pm \sqrt{1 - \frac{4(1 - e^{-\sigma})}{\mu}} \right] \right\}, \\ \mu &= \alpha\beta(1 + \gamma), \end{aligned} \quad (91)$$

where  $\gamma = e^{-bR}$ .

Also for the two-step model (37), which includes inflation, one can obtain an exact expression for the potential  $V(\sigma)$  but, since  $\alpha_I \gg \alpha$ , such expression reduces to the latter above, with the replacement  $\alpha \rightarrow \alpha_I$ . The explicit expressions of the scalar potential in the equivalent, scalar-tensor theory can be actually very useful in the study of the PPN-regime of modified gravity, in that of the stellar evolution equations, and also in some related questions.



## V. DISCUSSION AND CONCLUSIONS

In this paper, a general approach to viable modified gravity has been developed in both the Jordan and the Einstein frames. We have focussed on the so-called step-class models mainly, since they seem to be most promising from the phenomenological viewpoint and, at the same time, they provide a natural possibility to classify all viable modified gravities. We have explicitly presented the cases of one- and two-step models, but a similar analysis can be extended to the case of a  $N$ -step model, with  $N$  being finite or countably infinite. No additional problems are expected to appear and the models can be adjusted, provided one can always find smooth solutions interpolating between the de Sitter solutions (what seems at this point a reasonable possibility), to repeat at each stage the same kind of de Sitter transition. We can thus obtain multistep models which may lead to multiple inflation and multiple acceleration, in a way clearly reminiscent of braneworld inflation.

This looks quite promising, with the added bonus that the model's construction is rather simple, as we have here shown explicitly. Everywhere in the paper, as a guide for an accurate analysis, use has been made of the simple but efficient tools provided by the corresponding toy model constructed with sharp distributions, a new technique that we have here introduced too. It is to be remarked that, for the infinite-step models, one can naturally expect to construct the classical gravity analog of the stringy landscape realizations, as in the classical ideal fluid model [17].

For the model (52), both inflation in the early universe and the recent accelerated expansion could be understood in those models in a unified way. If we start with large curvature,  $f(R)$  becomes almost constant, as in (56), and plays the role of the effective cosmological constant, which would generate inflation. For a successful exit from the inflationary epoch we may need, in the end, more (say small nonlocal or small  $R^n$ ) terms. When curvature becomes smaller, matter could dominate, which would indeed lower the curvature values. Then, when the curvature  $R$  becomes small enough and  $R_0 \ll R \ll R_I$ ,  $f(R)$  becomes again an almost constant function and plays the role of the small cosmological constant which generates the accelerated expansion of the universe, which started in the recent past. Moreover, the model naturally passes all local tests and can be considered as a true viable alternative to general relativity. Some remark is however in order. On general grounds, one is dealing here with a highly non-linear system and one should investigate all possible critical points thereof (including time-dependent cosmologies), within the dynamical approach method. Of course, the existence of other critical points is possible; anyhow, for viable  $f(R)$  models, to find them is not a simple task, and we have here restricted our efforts to the investigation of the dS critical points. With regard to the stability of these points, the one associated with inflation should be unstable.

In this way, the exit from inflation could be achieved in a quite natural way. In particular, for instance, this is in fact the case for the two-step model with the  $R^3$  term discussed in the appendix.

In conclusion, a class of exponential, realistic modified gravities have been introduced and investigated with care. Some of these models ultimately lead to the unification of the inflationary epoch with the late-time accelerating epoch under quite simple and rather natural conditions. What remains to be done is to study those models in further quantitative detail, by comparing their predictions with the accurate astrophysical data coming from ongoing and proposed sky observations. It is expected that this can be done rather soon, having in mind the possibility to slightly modify the early universe features of the theories here introduced, while still preserving all of their nice, realistic current universe properties, as we have shown above.

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## APPENDIX

In this appendix, we will study how to evaluate—or at least get some information—on the number and positions of the corresponding de Sitter critical points. In order to grasp the general behavior of the larger family of models (as already discussed in Secs. I and II), let us start by considering the sharp (mathematically very clean albeit physically unrealistic) version of the one-step models, expressed in terms of Heaviside and Dirac distributions, namely

$$f(R) = -\alpha\theta(R - R_0), \quad f'(R) = -\alpha\delta(R - R_0), \quad \alpha > 0. \quad (\text{A1})$$

This simple, idealized model leads to antigravity, since  $1 + f'(R)$  is obviously always negative at the transition. In practical terms, this means that the sharper is the smoothing of the step-function, the harder one will be involved in the antigravity problem. For the one-step model, antigravity could be arranged to happen in the future. In fact, the equation whose solutions are the de Sitter critical points becomes

$$R = -\alpha R_0 \delta(R - R_0) + 2\alpha\theta(R - R_0). \quad (\text{A2})$$

This is an equation in the distribution theory sense and

requires an appropriate treatment. To start, there is the trivial solution  $R = 0$ . If  $R \neq R_0$ , the only solution is  $R = 2\alpha$ . However, eventually we have to deal with a nonideal, physical situation and we must consider not the sharp but the smoothed version of the delta and theta distribution. For the delta, we may consider its support to be contained in the interval  $-\varepsilon + R_0, R_0 + \varepsilon$ .

We can get information about the other fixed point by arguing as follows. Integrating the above equation from the value  $R = R_0 - \varepsilon$  to the value  $R_0 + \varepsilon$ :

$$\frac{1}{2}(4R_0\varepsilon) = -\alpha R_0 + 2\alpha R_0\varepsilon. \quad (\text{A3})$$

As a result, we obtain the consistency condition for  $\varepsilon$

$$\varepsilon \simeq \frac{\alpha}{2(\alpha - R_0)} R_0. \quad (\text{A4})$$

Typically, we have  $R_0 \ll \alpha$ . In this case, we obtain

$$\frac{R_0}{2} < R_{*,1} < \frac{3R_0}{2} \quad R_{*,2} \simeq 2\alpha. \quad (\text{A5})$$

Thus, with a sharp  $f(R)$  function, one has two de Sitter solutions when  $R_0 \ll \alpha$ . Antigravity effects can be confined around  $R_{*,1}$  (namely, in the future) and the current acceleration is represented by the second solution  $R_{*,2} \simeq 2\alpha$ .

For a two-step sharp model, however, this solution to the problem is not acceptable because we cannot allow for antigravity in past epochs, as discussed in Sec. IV. The way out of this is to consider, for example, a sufficiently non-sharp smoothing of the theta functions, but in this case, the above analysis is not longer valid. Another possibility—which can still make use of sharp theta functions—is the following modification of the two-step model, by a power of the curvature:

$$f(R) = -\alpha\theta(R - R_0) + \beta R^N \theta(R - R_I), \quad \alpha, \beta > 0. \quad (\text{A6})$$

In this case, near the second transition point one does not have, by construction, antigravity effects. The above analysis can now be applied again, with the result that, integrating from  $R_I$ ,  $N > 2$ ,  $N = 3$ , etc., one gets

$$R_* \simeq 2\alpha + \beta(N - 2)R_*^N. \quad (\text{A7})$$

This is an algebraic equation of  $N$ th order, whose solutions can be easily investigated, in any specific situation. For example, for  $N = 3$  and  $2\alpha$  negligible, one gets the approximate value

$$R_* \simeq \frac{1}{\sqrt{\beta}}. \quad (\text{A8})$$

In contrast, in the nonsharp case it is not so easy to find sufficient conditions for the existence of the de Sitter critical points. With regard to this issue, let us recall the following theorem, which may be useful in a direct, numerical computation of the critical points in a physically realistic setting. *Theorem:* Given a twice differentiable function,  $G(x)$ , defined in the real interval  $[a, b]$  and such that  $G'(x)$  is nonvanishing in this interval, and

$$|G''(x)G(x)| < |G'(x)^2|, \quad (\text{A9})$$

then the zeroes of  $G(x)$  are obtained by the recursive formula (Newton's tangent method)

$$x_{n+1} = x_n - \frac{G(x_n)}{G'(x_n)}. \quad (\text{A10})$$

This theorem yields a contraction mapping on the complete metric, in our case in the interval  $[a, b]$ . In fact the zeroes of  $G(x)$  are the fixed points of the function  $K(x) = x - \frac{G(x)}{G'(x)}$ , and this function is a contraction mapping, as far as  $|K'(x)| < 1$ . Since  $K'(x) = \frac{G''(x)G(x)}{G'^2(x)}$ , one gets the stated result. The recursive relation  $x_{n+1} = K(x_n)$  leads to Eq. (A10).

In the cases we consider here, this result is, however, not easy to implement. Alternatively, one can proceed as follows. Let us write

$$\begin{aligned} K(x) &= x f'(x) - 2f(x), & K(0) &= 0, \\ K(x) &\simeq -2f(x), & x &\gg 0. \end{aligned} \quad (\text{A11})$$

The fixed points  $x = K(x)$  are the de Sitter critical points. Let us suppose that  $K'(x) > 0$ , for every  $x > 0$ . Then it follows that: (i) if  $K'(0) > 1$  then there exists a fixed point; (ii) if  $K'(0) < 1$  then there are no fixed points.

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