Modified $f(R)$ gravity unifying R^m inflation with the ΛCDM epoch

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We consider modified $f(R)$ gravity which may unify R^m early-time inflation with late-time ΛCDM epoch. It is shown that such a model passes the local tests (Newton law, stability of Earth-like gravitational solution, very heavy mass for additional scalar degree of freedom) and suggests the realistic alternative for general relativity. Various scenarios for the future evolution of $f(R)$ ΛCDM era are discussed.

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I. INTRODUCTION

Modified gravity is considered a very interesting alternative proposal for dark energy. The attractive property of gravitational dark energy (for a review, see [1]) is the fact that one should not introduce some strange matter with negative pressure to describe the late-time cosmic acceleration. The (sudden) change of the decelerated expansion to the accelerated one is explained by the change of the properties of a gravitational theory in the course of the universe evolution. In other words, some subleading gravitational action terms [1–3] may become essential ones at the late universe. That is why there is much activity in the study of different versions of modified $f(R)$ gravity with applications to dark energy cosmology [1–8].

Recently, a very realistic modified $f(R)$ gravity which evades the solar system tests was proposed in Ref. [9] (for related discussion, see [10]). At the same time, such a theory leads to an effective ΛCDM epoch which complies with observational data with the same accuracy as the usual general relativity (GR) with cosmological constant. A generalization of the model [9] done in Ref. [11] suggests a quite natural unified description of early-time inflation with late-time acceleration following the earlier proposal of Ref. [3]. In the present work we propose another class of modified $f(R)$ gravity which unifies R^m inflation with the ΛCDM era. It is shown that such a theory may pass the local tests. The cosmological properties of such a model are studied. Some speculative remarks about the possibility of a (future/past) antigravity phase are made. It is also shown that such a theory may be more friendly with future observational data: unlike usual GR the current ΛCDM epoch induced by such a theory may enter the future quintessence/transient phantom era by the effective reconstruction of the action. Alternatively, one can have the eternal ΛCDM epoch as a result of such reconstruction.

II. UNIFYING R^m INFLATION WITH ΛCDM COSMOLOGY

We start from the following action of general $f(R)$ gravity

$$S = \frac{1}{\kappa^2}\int d^4x\sqrt{-g}(R + f(R)). \tag{1}$$

The equation of motion in $f(R)$ gravity with matter is given by

$$\frac{1}{2}g_{\mu\nu}F(R) - R_{\mu\nu}F'(R) - g_{\mu\nu}\Box F'(R) + \nabla_\mu\nabla_\nu F'(R) = -\frac{\kappa^2}{2}T_{(m)\mu\nu}. \tag{2}$$

Here $F(R) = R + f(R)$ and $T_{(m)\mu\nu}$ is the matter energy-momentum tensor. By introducing the auxiliary field A one may rewrite the action (1) in the following form:

$$S = \frac{1}{\kappa^2}\int d^4x\sqrt{-g}\{(1 + f'(A))(R - A) + A + f(A)\}. \tag{3}$$

As is clear from (3), if $F'(R) = 1 + f'(R) < 0$, $\kappa^2_{\mathrm{eff}} \equiv \kappa^2/F'(A)$ becomes negative and the theory enters the antigravity regime. Note that it is not the case for usual GR.

Recently a viable $f(R)$ model has been proposed in Ref. [9] (for other recent proposals/study of properties of viable $f(R)$ gravity, see [7,8,10]). In this model $f(R)$ is chosen to be

$$f_{\mathrm{HS}}(R) = -\frac{m^2 c_1 (R/m^2)^n}{c_2 (R/m^2)^n + 1}, \tag{4}$$

which satisfies the conditions

$$\lim_{R\to\infty} f_{\mathrm{HS}}(R) = \mathrm{const}, \qquad \lim_{R\to 0} f_{\mathrm{HS}}(R) = 0. \tag{5}$$

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The second condition means that there is a flat spacetime solution (vanishing cosmological constant). The estimation of Ref. [9] suggests that R/m^2 is not so small but rather large even in the present universe and $R/m^2 \sim 41$. Hence,

$$f_{\rm HS}(R) \sim -\frac{m^2 c_1}{c_2} + \frac{m^2 c_1}{c_2^2}\left(\frac{R}{m^2}\right)^{-n}, \tag{6}$$

which gives an "effective" cosmological constant $-m^2 c_1/c_2$ and generates the late-time accelerating expansion. One can show that

$$H^2 \sim \frac{m^2 c_1 \kappa^2}{c_2} \sim (70\ {\rm km/s\cdot pc})^2 \sim (10^{-33}\ {\rm eV})^2. \tag{7}$$

Hence, the above model describes an effective ΛCDM cosmology.

Although the model [9] is very successful, early-time inflation is not included there. We have suggested the modified gravity model to treat the inflation and the late-time accelerating expansion in a unified way [11]. We have considered a simple extension of the model [9] to include the inflation at the early universe. In order to generate the inflation, one may require

$$\lim_{R\to\infty} f(R) = -\Lambda_i. \tag{8}$$

Here Λ_i is an effective cosmological constant at the early universe and therefore it is natural to assume $\Lambda_i \gg (10^{-33}\ {\rm eV})^2$. For instance, it could be $\Lambda_i \sim 10^{20\sim 38}\ ({\rm eV})^2$. In order that the current cosmic acceleration could be generated, let us consider that currently $f(R)$ is a small constant, that is,

$$f(R_0) = -2\tilde{R}_0, \qquad f'(R_0) \sim 0. \tag{9}$$

Here R_0 is the current curvature $R_0 \sim (10^{-33}\ {\rm eV})^2$. Note that $R_0 > \tilde{R}$ due to the contribution from matter. In fact, if we can regard $f(R_0)$ as an effective cosmological constant, the effective Einstein equation gives

$$R_0 = \tilde{R}_0 - \kappa^2 T_{\rm matter}. \tag{10}$$

Here $T_{\rm matter}$ is the trace of the matter energy-momentum tensor. We should note that $f'(R_0)$ need not vanish exactly. Since we are considering the time scale of one-10×10^9 years, we only require $|f'(R_0)| \ll (10^{-33}\ {\rm eV})^4$. The last condition corresponding to the second one in (5) is

$$\lim_{R\to 0} f(R) = 0. \tag{11}$$

In the above class of models, the universe starts from the inflation driven by the effective cosmological constant (8) at the early stage, where the curvature is very large. As the curvature becomes smaller, the effective cosmological constant also becomes smaller. After that radiation/matter dominates. When the density of the radiation and matter becomes smaller and the curvature goes to the value R_0 (9), there appears a small effective cosmological constant (9). Hence, the current cosmic expansion could start.

Equation (3) indicates that there could appear an antigravity regime when $1 + f'(A) = 1 + f'(R) < 0$. If we assume that the antigravity does not appear through the known universe history (even in the future), the condition $f'(R) > -1$, combined with the condition (11), gives

$$f(R) > -R, \tag{12}$$

which could contradict with (9). Since 70% of the total energy density of the present universe could be dark energy, we find $\tilde{R}_0 \sim 0.7 R_0$, that is

$$f(R_0) \sim -1.4 R_0, \tag{13}$$

which seems to conflict with Eq. (12). The problem could occur even for the model [9]. Note, however, (12) does not always mean that the antigravity regime occurs at the current universe. The condition (11) might be a condition in the future and therefore the antigravity might appear in the future when the curvature becomes smaller than the present value. A more detailed discussion will be given later.

Since the model corresponding to (8) has been investigated in [11], we now propose a model which satisfies

$$\lim_{R\to\infty} f(R) = \alpha R^m, \tag{14}$$

with a positive integer m and a constant α. The condition to avoid the antigravity $f'(R) > -1$ tells $\alpha > 0$ and therefore $f(R)$ should be positive at the early universe. On the other hand, Eq. (9) or (13) shows that $f(R)$ is negative at the present universe. Therefore $f(R)$ should cross zero in the past.

At the early universe, if the scalar curvature is large, the $f(R)$-term could behave as (14) and would dominate the Einstein-Hilbert term (if m is bigger than 1). Now let us assume that there exists matter with an equation of state parameter w. For a spatially flat Friedmann-Robertson-Walker (FRW) universe,

$$ds^2 = -dt^2 + a(t)^2 \sum_{i=1,2,3} (dx^i)^2, \tag{15}$$

as shown in [4], the scale factor $a(t)$ behaves as

$$a(t) \propto t^{h_0}, \qquad h_0 \equiv \frac{2m}{3(w+1)}. \tag{16}$$

Then the effective equation of state parameter $w_{\rm eff}$, which is defined by

$$w_{\rm eff} = -1 + \frac{2}{3h_0}, \tag{17}$$

can be less than $-1/3$ and the accelerating expansion could occur if m is large enough although $w_{\rm eff} > -1$. Then the inflation could occur due to the R^m behavior of $f(R)$ in (14). The expansion of the universe is quintessencelike since $w_{\rm eff} > -1$. We need not, however, a real quintessence field as the inflaton. Note, w can vanish, which corresponds to dust, that is, cold dark matter or baryons, or w can be equal to $1/3$, which corresponds to the radiation. If m is chosen to be large enough, h_0 also becomes large and therefore $w_{\rm eff}$ goes to -1, which corresponds to the cosmological constant. Hence, for such a class of models, at the early stage, the universe starts from the inflation driven by the R^m behavior, instead of the effective cosmological constant in (8). Since $w_{\rm eff} > -1$, the curvature could become smaller as time passes and $f(R)$ could cross zero. At that stage, $f(R)$ could be neglected and the radiation/matter could dominate. After that, $f(R)$ becomes negative and its absolute value could increase as the curvature becomes smaller. When the density of the radiation and matter becomes smaller due to the expansion of the universe and the curvature goes to the value R_0 (9), there could appear a small effective cosmological constant (9). Hence, the current cosmic expansion starts. Note that another possibility to realize the above scenario is to add subleading curvature terms (dominant over R) in such a way that de Sitter-like inflation occurs. Moreover, the reconstruction of such $f(R)$ gravity at large curvature may be done so that the inflationary stage is instable and curvature-induced exit occurs. As the last possibility to achieve the exit from the inflationary era one can add (small) nonlocal gravity action of the sort recently proposed in Refs. [12,13].

A simplest example satisfying the above conditions is

$$f(R) = \alpha R^m - \beta R^n. \tag{18}$$

Here α and β are positive constant and m and n are positive integers satisfying the condition $m > n$. Since

$$f'(R) = \alpha m R^{m-1} - \beta n R^{n-1}, \tag{19}$$

we find

$$R_0 = \left(\frac{\beta n}{\alpha m}\right)^{1/(m-n)}, \qquad f(R_0) = -\beta\left(1 - \frac{n}{m}\right)\left(\frac{\beta n}{\alpha m}\right)^{n/(m-n)} < 0. \tag{20}$$

Since

$$f(R_0) = -2\tilde{R}_0 \sim R_0 \sim (10^{-33}\ \mathrm{eV})^2, \tag{21}$$

one gets $\alpha \sim R_0^{1-m}$ and $\beta \sim R_0^{1-n}$. This shows that $f(R)$ becomes larger than R, $f(R) \gg R$, even in the solar system where $R \sim 10^{-61}\ \mathrm{eV}^2$, which could be inconsistent.

Another, more realistic proposal is

$$f(R) = \frac{\alpha R^{m+l} - \beta R^n}{1 + \gamma R^l}. \tag{22}$$

Here α, β, and γ are positive constants and m, n, and l are positive integers satisfying the condition $m + l > n$. (If necessary, to achieve the exit from the inflation more curvature terms with powers less than $m + l$ may be added to Eq. (22).) For simplicity, we now choose

$$m = l = n. \tag{23}$$

Then since

$$f'(R) = \frac{nR^{n-1}(\alpha\gamma R^{2n} - 2\alpha R^n - \beta)}{(1 + \gamma R^n)^2}, \tag{24}$$

one gets R_0 satisfying (9) is given by

$$R_0 = \left\{\left(\frac{1}{\gamma}\right)\left(1 + \sqrt{1 + \frac{\beta\gamma}{\alpha}}\right)\right\}^{1/n}, \tag{25}$$

and therefore

$$f(R_0) \sim -2\tilde{R}_0 = \frac{\alpha}{\gamma^2}\left(1 + \frac{(1 - \frac{\beta\gamma}{\alpha})\sqrt{1 + \frac{\beta\gamma}{\alpha}}}{2 + \sqrt{1 + \frac{\beta\gamma}{\alpha}}}\right). \tag{26}$$

As a working hypothesis, we assume $\beta\gamma/\alpha \gg 1$, which will be justified later. Then we have

$$R_0 \sim \left(\frac{\beta}{\alpha\gamma}\right)^{1/2n}, \qquad f(R_0) = -2\tilde{R}_0 \sim -\frac{\beta}{\gamma}. \tag{27}$$

One also assumes

$$f(R_I) \sim \left(\frac{\alpha}{\gamma}\right) R_I \sim R_I, \tag{28}$$

when R is given by the scale of the inflation, $R \sim R_I$, which is $R_I \sim (10^{15}\ \mathrm{GeV})^2 = (10^{24}\ \mathrm{eV})^2$. The above conditions (27) and (28) could be solved as

$$\alpha \sim 2\tilde{R}_0 R_0^{-2n}, \qquad \beta \sim 4\tilde{R}_0^2 R_0^{-2n} R_I^{n-1}, \qquad \gamma \sim 2\tilde{R}_0 R_0^{-2n} R_I^{n-1}. \tag{29}$$

Then we find $\beta\gamma/\alpha \sim 4\tilde{R}_0^2 R_0^{-2n} R_I^{2n-2} \sim 10^{228(n-1)}$, which is surely large if $n > 1$.

The action (3) may be presented in scalar-tensor form. By using the scale transformation $g_{\mu\nu} \to e^\sigma g_{\mu\nu}$ with $\sigma = -\ln(1 + f'(A))$, the Einstein frame action follows [3]:

$$S_E = \frac{1}{\kappa^2}\int d^4x\sqrt{-g}\left(R - \frac{3}{2}g^{\rho\sigma}\partial_\rho\sigma\partial_\sigma\sigma - V(\sigma)\right),$$
$$V(\sigma) = e^\sigma g(e^{-\sigma}) - e^{2\sigma} f(g(e^{-\sigma})) = \frac{A}{F'(A)} - \frac{F(A)}{F'(A)^2}. \tag{30}$$

Here $F(R) \equiv R + f(R)$ and $g(e^{-\sigma})$ is given by solving $\sigma = -\ln(1 + f'(A)) = \ln F'(A)$ as $A = g(e^{-\sigma})$. After the scale transformation $g_{\mu\nu} \to e^\sigma g_{\mu\nu}$, there appears a coupling of the scalar field σ with the matter. The mass of σ is defined by

$$m_\sigma^2 \equiv \frac{1}{2}\frac{d^2V(\sigma)}{d\sigma^2} = \frac{1}{2}\left\{\frac{A}{F'(A)} - \frac{4F(A)}{(F'(A))^2} + \frac{1}{F''(A)}\right\}. \tag{31}$$

Unless m_σ could not be large, there appears the large correction to the Newton law.

We now investigate the above correction to the Newton law in the model (22) with (29). In the solar system, where

$R \sim (10^{-61}\ \text{eV})^2$, or in the air on the Earth, where $R \sim (10^{-50}\ \text{eV})^2$, we find

$$F(R) = R + f(R) \sim R - 2R_0 \sim R,$$
$$F'(R) = 1 + f'(R) \sim 1 + \frac{n\alpha}{\gamma} R^{n-1} \sim 1 + n\left(\frac{R}{R_I}\right)^{n-1} \sim 1,$$
$$F''(R) = f''(R) \sim \frac{n(n-1)}{R_I}\left(\frac{R}{R_I}\right)^{n-2}. \tag{32}$$

Then in the solar system, the mass m_σ of the scalar field σ is given by $m_\sigma^2 \sim 10^{-160+109n}\ \text{eV}^2$ and in the air on the Earth, $m_\sigma^2 \sim 10^{-144+98n}\ \text{eV}^2$. In both cases, the mass m_σ is very large if $n \geq 2$ and the correction to the Newton law is very small.

Thus, a viable modified gravity is proposed which unifies curvature-induced R^m inflation with effective Lambda-CDM cosmology. There is no violation of Newton law in such a theory while the known universe expansion history is reproduced.

III. ΛCDM ERA AND ITS FUTURE EVOLUTION

Let us discuss further properties of the proposed modified gravity. There may exist another type of instability (so-called matter instability) in $f(R)$ gravity [14,15]. It is known that it is absent in the model of Ref. [3]. The instability might occur when the curvature is rather large, as in the planet, compared with the average curvature at the universe $R \sim (10^{-33}\ \text{eV})^2$. By multiplying Eq. (2) with $g^{\mu\nu}$, one obtains

$$0 = \Box R + \frac{F^{(3)}(R)}{F^{(2)}(R)}\nabla_\rho R \nabla^\rho R + \frac{F'(R)R}{3F^{(2)}(R)} - \frac{2F(R)}{3F^{(2)}(R)} - \frac{\kappa^2}{6F^{(2)}(R)}T. \tag{33}$$

Here $T \equiv T_{(m)\rho}{}^{\rho}$ and $F^{(n)}(R) \equiv d^nF(R)/dR^n$. We consider a perturbation from the following solution of the Einstein gravity:

$$R = R_b \equiv -\frac{\kappa^2}{2}T > 0. \tag{34}$$

Note that T is negative since $|p| \ll \rho$ in the earth and $T = -\rho + 3p \sim -\rho$. Then we assume

$$R = R_b + R_p, \qquad (|R_p| \ll |R_b|). \tag{35}$$

Now one can get

$$0 = -\partial_t^2 R_p + U(R_b)R_p + \text{const},$$
$$U(R_b) \equiv \left(\frac{F^{(4)}(R_b)}{F^{(2)}(R_b)} - \frac{F^{(3)}(R_b)^2}{F^{(2)}(R_b)^2}\right)\nabla_\rho R_b \nabla^\rho R_b + \frac{R_b}{3} - \frac{F^{(1)}(R_b)F^{(3)}(R_b)R_b}{3F^{(2)}(R_b)^2} - \frac{F^{(1)}(R_b)}{3F^{(2)}(R_b)} + \frac{2F(R_b)F^{(3)}(R_b)}{3F^{(2)}(R_b)^2} - \frac{F^{(3)}(R_b)R_b}{3F^{(2)}(R_b)^2}. \tag{36}$$

Then if $U(R_b)$ is positive, R_p becomes exponentially large as a function of t: $R_p \sim e^{\sqrt{U(R_b)}t}$ and the system becomes unstable. In the model (22) with (29), if $n \geq 2$

$$U(R_b) \sim -\frac{R_I}{3n(n-1)}\left(\frac{R_b}{R_I}\right)^{-n+1} < 0. \tag{37}$$

Therefore there is no such instability in the model under consideration.

Let us consider what occurs when $f'(R) \to -1$. If $f'(R) < -1$, the theory enters the antigravity regime as is seen in (3). In the (effective) FRW equations with flat spatial part,

$$\frac{H^2}{3\kappa_{\text{eff}}^2} = \rho, \qquad 0 = \frac{1}{\kappa_{\text{eff}}^2}(2\dot{H} + 3H^2) + p, \tag{38}$$

antigravity means negative κ_{eff}^2. When $\kappa_{\text{eff}}^2 < 0$, there is no solution of the FRW equation (38), which means that the antigravity could not occur in the FRW universe with flat spatial part. Now we assume $f'(R) = -1$ when $R = R_A > 0$. When $f'(R) \to -1$ but $f'(R) > -1$, it follows $\kappa_{\text{eff}}^2 \to 0$. Then from (38), $H, \dot{H} \to 0$ when $\kappa_{\text{eff}}^2 \to 0$, which seems to contradict the assumption $R_A > 0$, since the scalar curvature R vanishes when $H = \dot{H} = 0$. This indicates that the scalar curvature R could not reach R_A. Then, anyway, the antigravity could not be realized for the real universe even in the future. A possibility of the transition between normal gravity and antigravity might be if ρ and p vanishes when κ_{eff} vanishes, then H and/or $\dot{H}$ might not vanish. As usual matter gives positive contribution to ρ, one needs the negative contributions to ρ. One contribution might come from the negative cosmological constant and another might come from the negative spatial curvature, which gives a contribution to ρ as $-1/a^2$. There could be one more possibility for the transition between normal gravity and antigravity, where H vanishes but $\dot{H}$ is finite, and therefore the scalar curvature does not vanish. We should note that ρ could be positive in the flat spatial geometry but p can vanish or even can be negative as for dark energy. Hence, one can speculate that the preinflationary era may result from the transition from antigravity to usual $f(R)$ gravity at the point with zero effective Newton coupling and infinite negative cosmological constant.

It is interesting to investigate if one can distinguish the ΛCDM epoch from usual GR and the same epoch which appears in the present $f(R)$ model. The analog of the first

FRW equation is

$$0 = -\frac{F(R)}{2} + 3(H^2 + \dot{H})F'(R) - 18(4H^2\dot{H} + H\ddot{H})F''(R) + \kappa^2\rho_{\rm matter}. \quad (39)$$

For the constant equation of state matter, it is known that $\rho_{\rm matter} = \rho_0 a^{-2/3(w+1)}$. Proposing (9), $f(R)$ can be expanded with respect to $R - R_0$ as

$$f(R) = -2\tilde{R}_0 + \delta f, \quad \delta f \equiv f_0(R - R_0)^2 + \mathcal{O}((R - R_0)^3). \quad (40)$$

Here f_0 is a positive constant. If we keep only the first term in (40) by putting $\delta f = 0$, we find the following solution in (39)

$$a = a_0 e^{g(t)}, \quad g(t) = g_0(t) \equiv \frac{2}{3(w+1)} \ln\left(A \sinh\left(\frac{3(1+w)t}{2l}\right)\right). \quad (41)$$

Here $A^2 \equiv \rho_0 a_0^{-3(1+w)}/\tilde{R}_0$, $l^2 \equiv 3/\tilde{R}_0$. The solution (41) is the same as for Einstein gravity with a cosmological constant and matter. One now treats δf as a perturbation, which could be justified in the near future or near past. Putting $g(t) = g_0(t) + \delta g$ and using (39) and (40) gives

$$0 = -6H_0\delta\dot{g} - \frac{2\kappa^2\rho_{\rm matter0}}{3(w+1)}\delta g - \frac{1}{2}\delta f + 3(H_0^2 + \dot{H}_0)\delta f' - 18(4H_0^2\dot{H}_0 + H_0\ddot{H}_0)\delta f'' + \mathcal{O}(\delta g^2). \quad (42)$$

Here subindex "0" expresses a quantity given when $\delta f = 0$, especially

$$H_0 = \dot{g}_0(t), \qquad \rho_{\rm matter0} = \rho_0 a_0^{-3(1+w)} e^{-3(1+w)g_0(t)}. \quad (43)$$

The solution of (42) is

$$\delta g = e^{-(\kappa^2/(9(w+1)))\int^t dt'(\rho_{\rm matter0}(t')/H_0(t'))} \int^t \frac{dt'}{6H_0(t')} \times \left(-\frac{1}{2}\delta f(t') + (3H_0(t')^2 + \dot{H}_0(t'))\delta f'(t') - 18(H_0(t)^2\dot{H}_0(t') + H_0(t')\ddot{H}_0(t'))\delta f''(t')\right) \times e^{\kappa^2/(9(w+1))\int^{t'} dt''(\rho_{\rm matter0}(t'')/H_0(t''))} + \mathcal{O}(\delta f^2). \quad (44)$$

Let the present time $t = t_0$. When $t \sim t_0$, we may assume H_0, $\dot{H}_0$, and $\ddot{H}_0$ are constants. Furthermore one may put $\delta f \propto (R - R_0)^2 \sim 0$ and $\delta f' \propto R - R_0 \sim 0$, and $\delta f = 2f_0$. Then at leading order with respect to $t - t_0$, Eq. (44) has the following form:

$$\delta g \sim -g_0(t - t_0), \qquad g_0 \equiv \frac{6(4H_0\dot{H}_0 + \ddot{H}_0)f_0}{H_0^2(1+w)}, \quad (45)$$

which gives the correction to the standard ΛCDM model from $f(R)$ gravity. Here it is assumed $\delta g = 0$ when $t = t_0$. For the model (22) with (23) and (29), one finds

$$f_0 = \frac{\alpha n^2 R_0^{2n-2}(\gamma R_0^n - 1)}{(1 + \gamma R_0^n)^2}. \quad (46)$$

From (29), we find $\gamma R_0^n \sim 2\tilde{R}_0 R_0^{-n} R_I^{n-1} \gg 1$. Then (46) could be approximated by

$$f_0 \sim \frac{\alpha n^2 R_0^{n-2}}{\gamma}. \quad (47)$$

Then, the order of g_0 may be estimated as

$$g_0 = \mathcal{O}(R_0^{n-1/2}/R_I^{n-1}) \sim 10^{-114n+81}\ {\rm eV}. \quad (48)$$

Especially for $n = 2$, we find $g_0 \sim 10^{-147}$ eV. Since 10 gigayears correspond to $(10^{-33}\ {\rm eV})^{-1}$, δg will be of the order of unity only 10^{115} gigayears later. Then the correction does not seem to be observable in the near future or past. Of course, the linear approximation used in (42) or (45) may be not enough in the model (22) with (29) and nonlinear terms account may be necessary.

In the search for footprints of nonlinear modified gravity at the current epoch one should not forget that the action of such a model may be further modified in the future so that its good properties survive. One possibility is the following term in the action:

$$\delta f(R) = -\frac{\eta R^p}{R^{p+q} + \zeta}. \quad (49)$$

Here p and q are positive integers and η and ζ are positive constants satisfying the conditions $\zeta \ll R_0^{p+q}$ and $\eta \ll R_0^{q+1}$. We should note that $\delta f(R)$ satisfies the condition (11). Hence, at the present universe

$$|\delta f(R_0)| \sim \eta R_0^{-q} \ll R_0, \quad (50)$$

and, therefore, the $\delta f(R)$ term can be neglected. When the curvature is larger than R_0, $|f(R)|$ is smaller, which shows that the $\delta f(R)$ term is irrelevant in the past universe. (Note, however, it may be used to improve the observational predictions of modified gravity when it is necessary). If the curvature becomes smaller and satisfies the condition

$$\zeta^{1/(p+q)} \ll R \ll R_0, \quad (51)$$

δR behaves as

$$\delta f(R) \sim -\eta R^{-q}. \quad (52)$$

If $\eta R^{-q} \gg R_0$, which requires

$$R_0^{q+1} \gg \eta \gg R_0 R^q \gg R_0 \zeta^{q/(p+q)}, \quad (53)$$

$\delta f(R)$ could dominate at the future universe. Using the arguments of Ref. [4], if matter is included, we find the future universe may enter to phantom era:

$$a(t) \sim (t_s - t)^{h_0}, \qquad h_0 \equiv -\frac{2q}{3(w+1)}. \tag{54}$$

In the above $a(t)$, there seems to appear the big rip singularity at $t = t_s$. Near the singularity, however, the curvature becomes large and the $\delta f(R)$ term does not dominate. Therefore, the big rip singularity does not occur and the phantom era is a transient one. After that the universe enters a quintessence or ΛCDM epoch.

Our consideration shows that even if the current universe (as predicted by modified gravity under consideration) is qualitatively/quantitatively the same as the standard ΛCDM era this may not be true in the near future. For instance, the (transient) phantom epoch may emerge without the need to introduce phantom matter.

IV. DISCUSSION

In summary, we proposed a modified $f(R)$ gravity which predicts natural unification of early-time inflation with late-time acceleration. This theory which is closely related with the models [9,11] passes the local tests (Newton law, stability of Earth-like gravitational solution, heavy mass for additional scalar degree of freedom, etc.). The speculative possibility of the past or future antigravity regime is briefly mentioned. The evolution of the $f(R)$ ΛCDM epoch is discussed. It is shown that it may get out from the cosmological constant boundary by future reconstruction of the gravitational action. As a result, the future universe may enter a quintessencelike or transient phantom era or it may continue to be an asymptotically de Sitter universe forever. At the next step, it is necessary to investigate the cosmological perturbations in the highly nonlinear gravity under discussion. However, this is a quite nontrivial task as the results should be presented in a gauge-independent formulation (the simple approximation which is analogous to the one made in GR with dark fluid does not lead to realistic predictions due to its gauge dependence). As more precise observational data for cosmological parameters are expected very soon, the further study of various cosmological predictions of our model (to distinguish it from GR) are requested. This will be done elsewhere.

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APPENDIX A: CONSISTENCY OF STELLAR SOLUTION IN $f(R)$ GRAVITY

Let us investigate if $f(R)$ gravity admits a consistent stellar solution, where vacuum solution matches onto the stellar-interior solution.

For this purpose, one may rewrite Eq. (2) in the following form:

$$\begin{aligned}\frac{1}{2}g_{\mu\nu}R - R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\Lambda + \frac{\kappa^2}{2}T_{(m)\mu\nu} \\ = -\frac{1}{2}g_{\mu\nu}(f(R)+\Lambda) + R_{\mu\nu}f'(R) + g_{\mu\nu}\Box f'(R) \\ - \nabla_\mu\nabla_\nu f'(R).\end{aligned} \tag{A1}$$

Here Λ is the value of $f(R)$ in the present universe, which corresponds to the effective cosmological constant: $\Lambda = f(R_0)$, then we have $R_0 \sim \Lambda \sim (10^{-33}\ \mathrm{eV})^2$. We now like to treat the right-hand part as a perturbation. The last two derivative terms could be dangerous when we consider the stellar configuration since there could be a jump in the value of R on the surface of the star. One may regard the order of the derivative could be the order of the inverse of the Compton length of a typical scale of the system. Since the most dangerous case corresponds to a particle, one may estimate the order of the derivative could be the Compton length of the proton: $\partial_\mu \sim m_p \sim 1\ \mathrm{GeV} \sim 10^9\ \mathrm{eV}$. Here m_p is the mass of the proton. We also assume the scalar curvature has the order of $R_e \sim 10^{-47}\ \mathrm{eV}^2$, which corresponds to the curvature inside the Earth.

First we consider the $1/R$ model:

$$f(R) = -\frac{\mu^4}{R}. \tag{A2}$$

The order of the dimensional parameter μ is 10^{-33} eV. One may estimate

$$\Box f'(R) \sim \nabla_\mu\nabla_\nu f'(R) \sim \frac{m_p^2\mu^4}{R^2} \sim 10^{-20}\ \mathrm{eV}^2, \tag{A3}$$

which is much larger than Λ or R_e and therefore the perturbative expansion breaks.

In case of the Hu-Sawicki (HS) model (4), we find

$$\begin{aligned}\Box f'(R) \sim \nabla_\mu\nabla_\nu f'(R) \sim \frac{m_p^2\Lambda}{m^2}\left(\frac{R}{m^2}\right)^{-n-1} \\ \sim 10^{-3-17n}\ \mathrm{eV}^2.\end{aligned} \tag{A4}$$

In (4), $R/m^2 \sim 41$. Equation (A4) shows that if $n > 2$, $\Box f(R)$ or $\nabla_\mu\nabla_\nu f(R)$ could be much smaller than R and therefore the perturbative expansion is consistent. For the model (22) the qualitative structure is similar to that of the HS model. This shows that for the class of models under discussion, the stellar solution is qualitatively similar to the one in Einstein gravity.

[1] S. Nojiri and S. D. Odintsov, Int. J. Geom. Methods Mod. Phys. **4**, 115 (2007).
[2] S. Capozziello, Int. J. Mod. Phys. D **11**, 483 (2002); S. Capozziello, S. Carloni, and A. Troisi, arXiv:astro-ph/0303041; S. M. Carroll, V. Duvvuri, M. Trodden, and M. Turner, Phys. Rev. D **70**, 043528 (2004).
[3] S. Nojiri and S. D. Odintsov, Phys. Rev. D **68**, 123512 (2003).
[4] M. C. B. Abdalla, S. Nojiri, S D. Odintsov, Classical Quantum Gravity **22**, L35 (2005).
[5] V. Faraoni, arXiv:gr-qc/0607016; Phys. Rev. D **72**, 124005 (2005); Phys. Rev. D **72**, 061501 (2005); Phys. Rev. D **76**, 127501 (2007); M. Ruggiero and L. Iorio, J. Cosmol. Astropart. Phys. 01 (2007), 010; A. Cruz-Dombriz and A. Dobado, Phys. Rev. D **74**, 087501 (2006); N. Poplawski, Classical Quantum Gravity **24**, 3013 (2007); A. Brookfield, C. van de Bruck, and L. Hall, Phys. Rev. D **74**, 064028 (2006); Y. Song, W. Hu, and I. Sawicki, Phys. Rev. D **75**, 044004 (2007); B. Li, K. Chan, and M. Chu, Phys. Rev. D **76**, 024002 (2007); X. Jin, D. Liu, and X. Li, arXiv:astro-ph/0610854; T. Sotiriou and S. Liberati, Ann. Phys. (N.Y.) **322**, 935 (2007); T. Sotiriou, Classical Quantum Gravity **23**, 5117 (2006); I. Navarro and K. Van Acoleyen, J. Cosmol. Astropart. Phys. 02 (2007) 022; A. Bustelo and D. Barraco, Classical Quantum Gravity **24**, 2333 (2007); G. Olmo, Phys. Rev. D **75**, 023511 (2007); F. Briscese, E. Elizalde, S. Nojiri, and S. D. Odintsov, Phys. Lett. B **646**, 105 (2007); B. Li and J. Barrow, Phys. Rev. D **75**, 084010 (2007); T. Rador, Phys. Lett. B **652**, 228 (2007); V. Faraoni, Phys. Rev. D **75**, 067302 (2007); J. C. C. de Souza and V. Faraoni, Classical Quantum Gravity **24**, 3637 (2007); S. Rahvar and Y. Sobouti, arXiv:0704.0680; O. Bertolami, C. Boehmer, T. Harko, and F. Lobo, Phys. Rev. D **75**, 104016 (2007); O. Bertolami and J. Paramos, arXiv:0709.3988; S. Carloni, A. Troisi, and P. Dunsby, arXiv:0706.0452; arXiv:0707.0106 [Phys. Rev D (to be published)]; S. Capozziello and M. Francaviglia, arXiv:0706.1146; J. Santos, J. Alcaniz, M. Reboucas, and F. Carvalho, Phys. Rev. D **76**, 083513 (2007); H. Mohseni Sadjadi, Phys. Rev. D **76**, 104024)2007); A. Iglesias, N. Kaloper, A. Padilla, and M. Park, Phys. Rev. D **76**, 104001 (2007); N. Lanahan-Tremblay and V. Faraoni, Classical Quantum Gravity **24**, 5667 (2007); N. Goheer, J. Leach and P. Dunsby, Classical Quantum Gravity **24**, 5689 (2007); O. Lecian and G. Montani, arXiv:0709.1400.
[6] S. Nojiri and S. Odintsov, Gen. Relativ. Gravit. **36**, 1765 (2004); Phys. Lett. B **576**, 5 (2003); Phys. Lett. B **599**, 137 (2004); P. Wang and X. Meng, TSPU Vestnik **44N7**, 40 (2004); Gen. Relativ. Gravit. **36**, 1947 (2004); G. Cognola, E. Elizalde, S. Nojiri, S. D. Odintsov, and S. Zerbini, J. Cosmol. Astropart. Phys. 02 (2005) 010; Phys. Rev. D **73**, 084007 (2006); D. A. Easson, Int. J. Mod. Phys. A **19**, 5343 (2004); S. Capozziello *et al.*, Phys. Rev. D **71**, 043503 (2005); Phys. Lett. B **634**, 93 (2006); G. Allemandi, A. Borowiec, M. Francaviglia, and S. D. Odintsov, Phys. Rev. D **72**, 063505 (2005); G. Allemandi, M. Francaviglia, M. Ruggiero, and A. Tartaglia, Gen. Relativ. Gravit. **37**, 1891 (2005); T. Multamaki and I. Vilja, Phys. Rev. D **76**, 064021 (2007); Phys. Rev. D **74**, 064022 (2006); arXiv:0709.3422; J. A. R. Cembranos, Phys. Rev. D **73**, 064029 (2006); T. Koivisto and H. Kurki-Suonio, Classical Quantum Gravity **23**, 2355 (2006); T. Clifton and J. Barrow, Phys. Rev. D **72**, 103005 (2005); O. Mena, J. Santiago, and J. Weller, Phys. Rev. Lett. **96**, 041103 (2006); M. Amarzguioui, O. Elgaroy, D. Mota, and T. Multamaki, Astron. Astrophys. **454**, 707 (2006); I. Brevik, Int. J. Mod. Phys. D **15**, 767 (2006); T. Koivisto, Phys. Rev. D **76**, 043527 (2007); T. Faulkner, M. Tegmark, E. Bunn, and Y. Mao, Phys. Rev. D **76**, 063505 (2007); G. Cognola, M. Castaldi, and S. Zerbini, arXiv:gr-qc/0701138; S. Capozziello *et al.*, Phys. Rev. D **76**, 104019 (2007); arXiv:0708.3038; arXiv:0709.0891; S. Nojiri, S. D. Odintsov, and P. Tretyakov, Phys. Lett. B **651**, 224 (2007); M. Movahed, S. Baghram, and S. Rahvar, Phys. Rev. D **76**, 044008 (2007); L. Amendola and S. Tsujikawa, arXiv:0705.0396; C. Boehmer, T. Harko, and F. Lobo, arXiv:0709.0046; arXiv:0710.0966; J. Miritzis, arXiv:0708.1396.
[7] S. Nojiri and S. D. Odintsov, Phys. Rev. D **74**, 086005 (2006); J. Phys. A **40**, 6725 (2007); J. Phys. Conf. Ser. **66**, 012005 (2007); S. Capozziello, S. Nojiri, S. D. Odintsov, and A. Troisi, Phys. Lett. B **639**, 135 (2006).
[8] S. Fay, S. Nesseris, and L. Perivolaropoulos, Phys. Rev. D **76**, 063504 (2007); S. Fay, R. Tavakol, and S. Tsujikawa, Phys. Rev. D **75**, 063509 (2007).
[9] W. Hu and I. Sawicki, arXiv:0705.1158.
[10] S. A. Appleby and R. A. Battye, Phys. Lett. B **654**, 7 (2007); S. Nojiri and S. D. Odintsov, Phys. Lett. B **652**, 343 (2007); L. Pogosian and A. Silvestri, arXiv:0709.0296 [Phys. Rev. D (to be published)]; S. Tsujikawa, arXiv:0709.1391 [Phys. Rev. D (to be published)].
[11] S. Nojiri and S. D. Odintsov, Phys. Lett. B **657**, 238 (2007).
[12] S. Deser and R. Woodard, Phys. Rev. Lett. **99**, 111301 (2007).
[13] S. Nojiri and S. D. Odintsov, arXiv:0708.0924.
[14] A. D. Dolgov, M. Kawasaki, Phys. Lett. B **573**, 1 (2003); M. Soussa and R. Woodard, Gen. Relativ. Gravit. **36**, 855 (2004).
[15] V. Faraoni, Phys. Rev. D **74**, 104017 (2006); T. Sotiriou, Phys. Lett. B **645**, 389 (2007); I. Sawicki and W. Hu, Phys. Rev. D **75**, 127502 (2007).