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Reconstructing the universe history, from inflation to acceleration, with phantom and canonical scalar fields

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We consider the reconstruction technique in theories with a single or multiple (phantom and/or canonical) scalar fields. With the help of several examples, it is demonstrated explicitly that the universe expansion history, unifying early-time inflation and late-time acceleration, can be realized in scalar-tensor gravity. This is generalized to the theory of a scalar field coupled nonminimally to the curvature and to a Brans-Dicke-like theory. Different examples of unification of inflation with cosmic acceleration, in which de Sitter, phantom, and quintessence type fields play the fundamental role—in different combinations—are worked out. Specifically, the frame dependence and stability properties of de Sitter space scalar field theory are studied. Finally, for two-scalar theories, the late-time acceleration and early-time inflation epochs are successfully reconstructed, in realistic situations in which the more and more stringent observational bounds are satisfied, using the freedom of choice of the scalar field potential, and of the kinetic factor.

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I. INTRODUCTION

The increasing amount and precision of observational data demand that theoretical cosmological models be as realistic as possible in their description of the evolution of our Universe. The discovery of late-time cosmic acceleration brought into this playground a good number of dark energy models (for a recent review, see [1]) that aim at describing the observed accelerated expansion, which seems to have started quite recently on the redshift scale. Keeping in mind the possibility, which is well compatible with the observational data, that the effective equation of state parameter w be less than -1, phantom cosmological models [2–5] share a place in the list of theories capable of explaining dark energy. Ideal fluid and scalar field quintessence/phantom models still remain among the easiest and most popular constructions. Nevertheless, when working with these models, one should bear in mind that such theories are at best effective descriptions of the early/late Universe, owing to a number of well-known problems.

Even in such a situation, scalar field models still remain quite popular candidates for dark energy. An additional problem with these theories–which traditionally has not been discussed in depth–is that a good mathematical theory must not be limited to the description of a single side of the cosmic evolution: it should rather provide a unified description of the whole expansion history of the Universe, from the inflationary epoch to the onset of cosmic acceleration, and beyond. Note that a similar drawback is also typical of inflationary models, most of which have problems with ending inflation and also fail to describe realistically the late-time Universe.

The purpose of this work is to show that, given a certain scale factor (or Hubble parameter) for the universe expansion history, one can in fact reconstruct it from a specific scalar field theory. Using multiple scalars, the reconstruction becomes easier due to the extra freedom brought by the arbitrariness in the scalar field potentials and kinetic factors. However, there are subtleties in these cases that can be used advantageously, and this makes the study of those models even more interesting.

Specifically, in this work we review the reconstruction technique for scalar theories with one, two, and an arbitrary number, n, of fields. After that, many explicit examples are presented in which a unified, continuous description of the inflationary era and of the late-time cosmic acceleration epoch are obtained in a rather simple and natural way.

The organization of this paper is as follows: in Sec. II we consider a universe filled with matter and show that it is possible to obtain both inflation and accelerated expansion at late times by using a single scalar field. Realistic examples are worked out in order to illustrate this fact. Section III is devoted to the theory of a scalar coupled nonminimally to gravity through the Ricci curvature. Late-time acceleration is explicitly discussed in this model, using two specific examples. In Sec. IV we discuss the issue of reconstruction for a nonminimally coupled scalar

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field theory, including Brans-Dicke theory, again clarifying this case with the help of an example. Section V deals with de Sitter space in scalar-tensor theory and studies issues relevant for the conformal transformation to the Einstein frame, arriving at stability conditions that are important in the study of the future evolution of the late-time accelerated era. In Sec. V, we discuss the case of several scalar fields, beginning with the case of two-scalar fields minimally coupled to gravity (such models have been used, e. g., in reheating scenarios after inflation). We study an explicit example and then the general case of n scalar fields. Section VII addresses our final goal, namely, the reconstruction of inflation and cosmic acceleration from a scalar field theory, by means of a two-scalar model that reproduces cosmological constraints in each epoch. Further, the cosmic acceleration is reproduced with a pair of scalar fields plus an ordinary matter term, where the observed cosmological density parameter ($\Omega_{\rm DE} \simeq 0.7$) and the equation of state (EoS) parameter ($w_{\rm DE} \simeq -1$) are actually reproduced. Finally, Sec. VIII contains the conclusions.

II. UNIFIED INFLATION AND LATE-TIME ACCELERATION IN SCALAR THEORY

Let us consider a universe filled with matter with equation of state $p_m = w_m \rho_m$ (here w_m is a constant) and a scalar field which only depends on time. We will show that it is possible to obtain both inflation and accelerated expansion at late times by using a single scalar field ϕ (see also [6,7]). In this case, the action is

$$S = \int dx^4 \sqrt{-g}\left[\frac{1}{2\kappa^2}R - \frac{1}{2}\omega(\phi)\partial_\mu\phi\partial^\mu\phi - V(\phi) + L_m\right], \tag{1}$$

with $\kappa^2 = 8\pi G$, $V(\phi)$ being the scalar potential and $\omega(\phi)$ the kinetic function, respectively, while L_m is the matter Lagrangian density. Note that for convenience the kinetic factor is introduced. At the final step of calculations, the scalar field may be always redefined so that kinetic factor is absorbed in its definition. As we work in a spatially flat Friedmann-Robertson-Walker (FRW) space-time, the metric is given by

$$ds^2 = -dt^2 + a^2(t)\sum_{i=1}^{3} dx_i^2. \tag{2}$$

The corresponding FRW equations are written as

$$H^2 = \frac{\kappa^2}{3}(\rho_m + \rho_\phi), \qquad \dot{H} = -\frac{\kappa^2}{2}(\rho_m + p_m + \rho_\phi + p_\phi), \tag{3}$$

with ρ_ϕ and p_ϕ given by

$$\rho_\phi = \frac{1}{2}\omega(\phi)\dot{\phi}^2 + V(\phi), \qquad p_\phi = \frac{1}{2}\omega(\phi)\dot{\phi}^2 - V(\phi). \tag{4}$$

Combining Eqs. (3) and (4), one obtains

$$\omega(\phi)\dot{\phi}^2 = -\frac{2}{\kappa^2}\dot{H} - (\rho_m + p_m), \qquad V(\phi) = \frac{1}{\kappa^2}(3H^2 + \dot{H}) - \frac{\rho_m - p_m}{2}. \tag{5}$$

As the matter is not coupled to the scalar field, by using energy conservation one has

$$\dot{\rho_m} + 3H(\rho_m + p_m) = 0, \qquad \dot{\rho_\phi} + 3H(\rho_\phi + p_\phi) = 0. \tag{6}$$

From the first equation, we get $\rho_m = \rho_{m0}a^{-3(1+w_m)}$. We now consider the theory in which $V(\phi)$ and $\omega(\phi)$ are

$$\omega(\phi) = -\frac{2}{\kappa^2}f'(\phi) - (w_m + 1)F_0 \mathrm{e}^{-3(1+w_m)F(\phi)}, \qquad V(\phi) = \frac{1}{\kappa^2}[3f(\phi)^2 + f'(\phi)] + \frac{w_m - 1}{2}F_0 \mathrm{e}^{-3(1+w_m)F(\phi)}, \tag{7}$$

where $f(\phi) \equiv F'(\phi)$, F is an arbitrary (but twice differentiable) function of ϕ, and F_0 is an integration constant. Then, the following solution is found (see [6–9]):

$$\phi = t, \qquad H(t) = f(t), \tag{8}$$

which leads to

$$a(t) = a_0 \mathrm{e}^{F(t)}, \qquad a_0 = \left(\frac{\rho_{m0}}{F_0}\right)^{(1/3(1+w_m))}. \tag{9}$$

We can study this system by analyzing the effective EoS parameter which, using the FRW equations, is defined as

$$w_{\rm eff} \equiv \frac{p}{\rho} = -1 - \frac{2\dot{H}}{3H^2}, \tag{10}$$

where

$$\rho = \rho_m + \rho_\phi, \qquad p = p_m + p_\phi. \tag{11}$$

Using the formulation above, one can present explicit examples of reconstruction as follows.

A. Example 1

As a first example, we consider the following model:

$$f(\phi) = h_0^2\left(\frac{1}{t_0^2 - \phi^2} + \frac{1}{\phi^2 + t_1^2}\right). \tag{12}$$

Using the solution (9), the Hubble parameter and the scale factor are given by

$$H = h_0^2\left(\frac{1}{t_0^2 - t^2} + \frac{1}{t^2 + t_1^2}\right),$$
$$a(t) = a_0\left(\frac{t+t_0}{t_0 - t}\right)^{(h_0^2/2t_0)} e^{(h_0^2/t_1)\arctan(t/t_1)}. \quad (13)$$

As one can see, the scale factor vanishes at $t = -t_0$, so we can fix that point as corresponding to the creation of the Universe. On the other hand, the kinetic function and the scalar potential are given by Eqs. (7), hence

$$\omega(\phi) = -\frac{8}{\kappa^2}\frac{h_0^2(t_1^2 + t_0^2)(\phi^2 - \frac{t_1^2+t_0^2}{2})\phi}{(t_1^2+\phi^2)^2(t_0^2-\phi^2)^2} - (w_m + 1)F_0 e^{-3(w_m+1)F(\phi)},$$
$$V(\phi) = \frac{h_0^2(t_1^2+t_0^2)}{\kappa^2(t_1^2+\phi^2)^2(t_0^2-\phi^2)^2}\left[3h_0^2(t_1^2+t_0^2) + 4\phi\left(\phi^2 - \frac{t_1^2+t_0^2}{2}\right)\right] + \frac{w_m - 1}{2}F_0 e^{-3(w_m+1)F(\phi)}, \quad (14)$$

where F_0 is an integration constant and

$$F(\phi) = \frac{h_0^2}{2t_0}\ln\left(\frac{\phi + t_0}{t_0 - \phi}\right) + \frac{h_0^2}{t_1}\arctan\frac{\phi}{t_1}. \quad (15)$$

Then, using Eq. (10), the effective EoS parameter is written as

$$w_{\text{eff}} = -1 - \frac{8}{3h_0^2}\frac{t(t-t_+)(t+t_-)}{(t_1^2+t_0^2)^2}, \quad (16)$$

where $t_\pm = \pm\sqrt{\frac{t_0^2 - t_1^2}{2}}$. There are two phantom phases that occur when $t_- < t < 0$ and $t > t_+$, and another two nonphantom phases for $-t_0 < t < t_-$ and $0 < t < t_+$, during which $w_{\text{eff}} > -1$ (matter/radiation-dominated epochs). The first phantom phase can be interpreted as an inflationary epoch, and the second one as corresponding to the current accelerated expansion, which will end in a Big Rip singularity when $t = t_0$. Note that superacceleration (i.e., $\dot{H} > 0$) is due to the negative sign of the kinetic function $\omega(\phi)$, as for "ordinary" phantom fields (to which one could reduce by redefining the scalar ϕ).

B. Example 2

As a second example, we consider the choice

$$f(\phi) = \frac{H_0}{t_s - \phi} + \frac{H_1}{\phi^2}. \quad (17)$$

We take H_0 and H_1 to be constants and t_s as the Rip time, as specified below. Using (7), we find that the kinetic function and the scalar potential are

$$\omega(\phi) = -\frac{2}{\kappa^2}\left[\frac{H_0}{(t_s-\phi)^2} - \frac{2H_1}{\phi^2}\right] - (w_m+1) \times F_0(t_s-\phi)^{3(1+w_m)H_0}\exp\left[\frac{3(1+w_m)H_1}{\phi}\right],$$
$$V(\phi) = \frac{1}{\kappa^2}\left[\frac{H_0(3H_0+1)}{(t_s-\phi)^2} + \frac{H_1}{\phi^3}\left(\frac{H_1}{\phi} - 2\right)\right] + \frac{w_m - 1}{2}F_0(t_s-\phi)^{3(1+w_m)H_0}\mathrm{e}^{(3(1+w_m)H_1/\phi)}, \quad (18)$$

respectively. Then, through the solution (9), we obtain the Hubble parameter and the scale factor

$$H(t) = \frac{H_0}{t_s - t} + \frac{H_1}{t^2}, \qquad a(t) = a_0(t_s - t)^{-H_0}\mathrm{e}^{-(H_1/t)}. \quad (19)$$

Since $a(t) \to 0^+$ for $t \to 0$, we can fix $t = 0$ as the beginning of the Universe. On the other hand, at $t = t_s$ the Universe reaches a Big Rip singularity, thus we keep $t < t_s$. In order to study the different stages that our model will pass through, we calculate the acceleration parameter and the first derivative of the Hubble parameter. They are

$$\dot{H} = \frac{H_0}{(t_s-t)^2} - \frac{2H_1}{t^3},$$
$$\frac{\ddot{a}}{a} = H^2 + \dot{H} = \frac{H_0}{(t_s-t)^2}(H_0+1) + \frac{H_1}{t^2}\left(\frac{H_1}{t^2} - \frac{2H_1}{t} + \frac{2H_0}{t_s - t}\right). \quad (20)$$

As we can observe, for t close to zero, $\ddot{a}/a > 0$, so that the Universe is accelerated during some time. Although this is not a phantom epoch, since $\dot{H} < 0$, such a stage can be interpreted as corresponding to the beginning of inflation. For $t > 1/2$ but $t \ll t_s$, the Universe is in a decelerated epoch ($\ddot{a}/a < 0$). Finally, for t close to t_s, it turns out that $\dot{H} > 0$, and then the Universe is superaccelerated, such acceleration being of phantom nature and ending in a Big Rip singularity at $t = t_s$.

C. Example 3

Our third example also exhibits unified inflation and late-time acceleration, but in this case we avoid phantom phases and, therefore, Big Rip singularities. We consider the following model:

$$f(\phi) = H_0 + \frac{H_1}{\phi^n}, \quad (21)$$

where H_0 and $H_1 > 0$ are constants and n is a positive integer (also constant). The case $n = 1$ yields an initially decelerated universe and a late-time acceleration phase. We concentrate on cases corresponding to $n > 1$ which gives, in general, three epochs: one of early acceleration (interpreted as inflation), a second decelerated phase, and, finally, accelerated expansion at late times. In this model, the scalar potential and the kinetic parameter are given, upon use of Eqs. (7) and (21), by

$$\omega(\phi) = \frac{2}{\kappa^2}\frac{nH_1}{\phi^{n+1}} - (w_m + 1) \times F_0 e^{-3(w_m+1)(H_0\phi-(H_1/(n-1)\phi^{n-1}))}, \quad (22)$$

$$V(\phi) = \frac{1}{\kappa^2}\frac{3}{\phi^{n+1}}\left[\frac{(H_0\phi^{n/2}+H_1)^2}{\phi^{n-1}} - \frac{nH_1}{3}\right] + \frac{w_m - 1}{2}F_0 e^{-3(w_m+1)(H_0\phi-(H_1/(n-1)\phi^{n-1}))}. \quad (23)$$

Then, the Hubble parameter given by the solution (9) can be written as

$$H(t) = H_0 + \frac{H_1}{t^n}, \qquad a(t) = a_0 \exp\left[H_0 t - \frac{H_1}{(n-1)t^{n-1}}\right]. \quad (24)$$

We can fix $t = 0$ as the beginning of the Universe because at this point $a \to 0$, so $t > 0$. The effective EoS parameter (10) is

$$w_{\rm eff} = -1 + \frac{2nH_1 t^{n-1}}{(H_0 t^n + H_1)^2}. \quad (25)$$

Thus, when $t \to 0$ then $w_{\rm eff} \to -1$ and we have an acceleration epoch, while for $t \to \infty$, $w_{\rm eff} \to -1$ which can be interpreted as late-time acceleration. To find the phases of acceleration and deceleration for $t > 0$, we study $\ddot{a}/a$, given by:

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 = -\frac{nH_1}{t^{n+1}} + \left(H_0 + \frac{H_1}{t^n}\right)^2. \quad (26)$$

For sufficiently large values of n we can find two positive zeros of this function, which means two corresponding phase transitions. They happen, approximately, at

$$t_\pm \approx \left[\sqrt{nH_1}\frac{(1 \pm \sqrt{1 - \frac{4H_0}{n}})}{2H_0}\right]^{2/n}, \quad (27)$$

so that, for $0 < t < t_-$, the Universe is in an accelerated phase interpreted as an inflationary epoch; for $t_- < t < t_+$ it is in a decelerated phase (matter/radiation dominated); and, finally, for $t > t_+$ one obtains late-time acceleration, which is in agreement with the current cosmic expansion.

We now consider how the exit from inflation could be realized. First, we should note that if one refines the scalar field as $\varphi = \int^\phi d\phi\sqrt{\omega(\phi)}$ in the nonphantom phase or $\varphi = \int^\phi d\phi\sqrt{-\omega(\phi)}$ in the phantom phase, then the action (1) has the following form:

$$S = \int d^4x\sqrt{-g}\left\{\frac{1}{2\kappa^2}R \mp \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \tilde{V}(\varphi)\right\} + S_m. \quad (28)$$

Here $\tilde{V}(\varphi) \equiv V(\phi(\varphi))$. The minus sign corresponds to the nonphantom phase and the plus sign to the phantom phase. For example, in the model (21) when ϕ is small (and therefore in the early Universe), we find $\varphi = -2\sqrt{2nH_1}/(n-1)\kappa\phi^{(n-1)/2}$ and $\tilde{V}(\varphi) \propto \varphi^{4+4/(n-1)}$. We should note that in the early Universe ($\phi \to 0$), φ is large. Then, the potential $\tilde{V}(\varphi)$ is of the slow-roll type. When the value of φ is large, the energy of the vacuum is large, which generates inflation. The value of φ slowly becomes small and the energy of the vacuum, and with it the curvature, decrease. When the curvature becomes small enough inflation could stop. For the successful exit from inflation we may need one more scalar field, as in the hybrid inflation scenario [10]. In the model (21), by construction, inflation ends as a purely classical theory when $\ddot{a} = a(H^2 + \dot{H}) = 0$, that is, for $t = t_e$ satisfying

$$0 = \left(H_0 + \frac{H_1}{t_e^n}\right)^2 - \frac{nH_1}{t_e^{n+1}}. \quad (29)$$

If we include, however, quantum effects the value of the scalar field φ jumps to a larger value—as a consequence of the quantum fluctuations—and the vacuum acquires higher energy that generates inflation again. In order to suppress the probability of those effects, in the hybrid inflation model at least one more scalar field and its coupling with φ had to be introduced.

D. Example 4

As our last example, we consider another model unifying early universe inflation and the accelerating expansion of the present Universe. We may choose $f(\phi)$ as

$$f(\phi) = \frac{H_i + H_l c e^{2\alpha\phi}}{1 + c e^{2\alpha\phi}}, \quad (30)$$

which gives the Hubble parameter

$$H(t) = \frac{H_i + H_l c e^{2\alpha t}}{1 + c e^{2\alpha t}}. \quad (31)$$

Here H_i, H_l, c, and α are positive constants. In the early Universe ($t \to -\infty$), we find that H becomes a constant $H \to H_i$ and at late times ($t \to +\infty$), H becomes a constant again $H \to H_l$. Then H_i could be regarded as the effective cosmological constant driving inflation, while H_l could be a small effective constant generating the late acceleration. Then, we should assume $H_i \gg H_l$. Hence, if we consider the model with action

$$S = \int d^4x\sqrt{-g}\left\{\frac{R}{2\kappa^2} - \frac{\omega(\phi)}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right\},$$
$$\omega(\phi) \equiv -\frac{f'(\phi)}{\kappa^2} = \frac{2\alpha(H_i - H_l)c\mathrm{e}^{2\alpha\phi}}{\kappa^2(1 + c\mathrm{e}^{2\alpha\phi})^2},$$
$$V(\phi) \equiv \frac{3f(\phi)^2 + f'(\phi)}{\kappa^2} = \frac{3H_i^2 + \{6H_iH_l - 2\alpha(H_i - H_l)\}c\mathrm{e}^{2\alpha\phi} + c^2H_l^2\mathrm{e}^{4\alpha\phi}}{\kappa^2(1 + c\mathrm{e}^{2\alpha\phi})^2}, \quad (32)$$

we can realize the Hubble rate given by (31) with $\phi = t$. If we redefine the scalar field as

$$\varphi \equiv \int d\phi\sqrt{\omega(\phi)} = \frac{\mathrm{e}^{\alpha\phi}}{\kappa}\sqrt{\frac{2(a-b)c}{\alpha}}, \quad (33)$$

the action S in (32) can be rewritten in the canonical form

$$S = \int d^4x\sqrt{-g}\left\{\frac{R}{2\kappa^2} - \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \tilde{V}(\varphi)\right\}, \quad (34)$$

where

$$\tilde{V}(\varphi) = V(\phi) = \frac{3H_i^2 + \frac{\kappa^2\alpha\{6H_iH_l - 2\alpha(H_i - H_l)\}}{2(H_i - H_l)}\varphi^2 + \frac{3\kappa^4H_l^2\alpha^2}{4(H_i - H_l)^2}\varphi^4}{\kappa^2(1 + \frac{\kappa^2\alpha}{2(H_i - H_l)}\varphi^2)^2}. \quad (35)$$

One should note that $\phi \to -\infty$ corresponds to $\varphi \to 0$ and $V \sim 3H_i^2/\kappa^2$, while $\phi \to +\infty$ corresponds to $\varphi \to \infty$ and $V \sim 3H_l^2/\kappa^2$, as expected. At early times ($\varphi \to 0$), $\tilde{V}(\varphi)$ behaves as

$$\tilde{V}(\varphi) \sim \frac{3H_i^2}{\kappa^2}\left\{1 - \frac{\kappa^2\alpha(3H_i + \alpha)}{3H_i^2}\varphi^2 + \mathcal{O}(\varphi^2)\right\}. \quad (36)$$

At early times, $\phi < 0$ and therefore, from Eq. (33), we find $\kappa\varphi\sqrt{\alpha/2(H_i - H_l)c} \ll 1$, from which it follows that

$$\frac{1}{3\kappa^2}\frac{\tilde{V}'(\varphi)}{\tilde{V}(\varphi)^2} \sim \frac{4\alpha^2\kappa^2(3H_i + \alpha)^2}{27H_i^4}\varphi^2 < \frac{8\alpha(3H_i + \alpha)^2(H_i - H_l)c}{27H_i^4}, \quad (37)$$
$$\frac{1}{3\kappa^2}\frac{|\tilde{V}''(\varphi)|}{\tilde{V}(\varphi)} \sim \frac{2(3H_i + \alpha)\alpha}{9H_i^2}.$$

Then, if $\alpha \ll H_i$, the slow-roll conditions can be satisfied.

We may include matter with constant EoS parameter w_m. Then $\omega(\phi)$ and $V(\phi)$ are modified as

$$\omega(\phi) \equiv -\frac{f'(\phi)}{\kappa^2} - \frac{w_m + 1}{2}g_0\mathrm{e}^{-3(1+w_m)g(\phi)},$$
$$V(\phi) \equiv \frac{3f(\phi)^2 + f'(\phi)}{\kappa^2} + \frac{w_m - 1}{2}g_0\mathrm{e}^{-3(1+w_m)g(\phi)},$$
$$g(\phi) \equiv \int d\phi f(\phi) = H_l\phi + \frac{H_i - H_l}{2}\ln(c + \mathrm{e}^{-2\alpha\phi}). \quad (38)$$

The matter-energy density is then given by

$$\rho_m = \rho_0 a^{-3(1+w_m)} = g_0\mathrm{e}^{-3(1+w_m)g(t)} = g_0(c + \mathrm{e}^{-2\alpha t})^{-3(1+w_m)(H_i - H_l)/2}\mathrm{e}^{-3(1+w_m)H_l t}. \quad (39)$$

In the early Universe $t \to -\infty$, ρ_m behaves as

$$\rho_m \sim g_0\mathrm{e}^{3(1+w_m)(2\alpha - H_l)t}. \quad (40)$$

On the other hand, the energy density of the scalar field behaves as

$$\rho_\varphi = \frac{1}{2}\dot{\varphi}^2 + \tilde{V}(\varphi) = \frac{\omega(\phi)}{2}\dot{\phi}^2 + V(\phi) \to \frac{3H_l^2}{\kappa^2}. \quad (41)$$

Then, if $2\alpha < H_l$ (and $w_m > -1$), the matter contribution could be neglected in comparison with the scalar field contribution.

Now let the present time be $t = t_0$. Then, we find that

$$\Omega_m \equiv \frac{\kappa^2\rho_m}{3H^2} = \frac{\kappa^2 g_0\mathrm{e}^{4\alpha t_0}(c + \mathrm{e}^{-2\alpha t_0})^{-3(1+w_m)(H_i - H_l)/2\alpha + 2}\mathrm{e}^{-3(1+w_m)H_l t_0}}{3(H_i + H_l c\mathrm{e}^{2\alpha t_0})^2}, \quad (42)$$

and $\Omega_\phi = 1 - \Omega_m$. If we assume $\alpha t_0 \gg 1$, we find

$$\Omega_m \sim \frac{\kappa^2 g_0}{3H_l^2}c^{-3(1+w_m)(H_i - H_l)/2\alpha}\mathrm{e}^{-3(1+w_m)H_l t_0}. \quad (43)$$

Hence, we may choose the parameters so that $\Omega_m \sim 0.27$, which could be consistent with the observed data. This model provides a quite realistic picture of the unification of the inflation with the present cosmic speed-up.

III. ACCELERATED EXPANSION IN THE NONMINIMALLY CURVATURE-COUPLED SCALAR THEORY

In the preceding section we have considered an action, (1), in which the scalar field is minimally coupled to gravity. In the present section, the scalar field couples to gravity through the Ricci scalar (see [11] for a review on cosmological applications). We begin from the action

$$S=\int d^4x\sqrt{-g}\left[(1+f(\phi))\frac{R}{\kappa^2}-\frac{1}{2}\omega(\phi)\partial_\mu\phi\partial^\mu\phi-V(\phi)\right], \quad (44)$$

where $f(\phi)$ is an arbitrary function of the scalar field ϕ. Then, the effective gravitational coupling depends on ϕ, as $\kappa_{\rm eff}=\kappa[1+f(\phi)]^{-1/2}$. One can work in the Einstein frame, by performing the scale transformation

$$g_{\mu\nu}=[1+f(\phi)]^{-1}\tilde{g}_{\mu\nu}. \quad (45)$$

The tilde over g denotes an Einstein frame quantity. Thus, the action (44) in such a frame assumes the form [12]

$$S=\int d^4x\sqrt{-\tilde{g}}\left\{\frac{\tilde{R}}{2\kappa^2}-\left[\frac{\omega(\phi)}{2(1+f(\phi))}+\frac{6}{\kappa^2(1+f(\phi))}\times\left(\frac{d(1+f(\phi)^{1/2})}{d\phi}\right)^2\right]\partial_\mu\phi\partial^\mu\phi-\frac{V(\phi)}{[1+f(\phi)]^2}\right\}. \quad (46)$$

The kinetic function can be written as $W(\phi)=\frac{\omega(\phi)}{1+f(\phi)}+\frac{3}{\kappa^2(1+f(\phi))^2}(\frac{df(\phi)}{d\phi})^2$, and the extra term in the scalar potential can be absorbed by defining the new potential $U(\phi)=\frac{V(\phi)}{[1+f(\phi)]^2}$, so that we recover the action (1) in the Einstein frame, namely

$$S=\int dx^4\sqrt{-\tilde{g}}\left(\frac{\tilde{R}}{\kappa^2}-\frac{1}{2}W(\phi)\partial_\mu\phi\partial^\mu\phi-U(\phi)\right). \quad (47)$$

We assume that the metric is FRW and spatially flat in this frame

$$d\tilde{s}^2=-d\tilde{t}^2+\tilde{a}^2(\tilde{t})\sum_i dx_i^2, \quad (48)$$

then, the equations of motion in this frame are given by

$$\tilde{H}^2=\frac{\kappa^2}{6}\rho_\phi, \quad (49)$$

$$\dot{\tilde{H}}=-\frac{\kappa^2}{4}(\rho_\phi+p_\phi), \quad (50)$$

$$\frac{d^2\phi}{d\tilde{t}^2}+3\tilde{H}\frac{d\phi}{d\tilde{t}}+\frac{1}{2W(\phi)}\left[W'(\phi)\left(\frac{d\phi}{d\tilde{t}}\right)^2+2U'(\phi)\right]=0, \quad (51)$$

where $\rho_\phi=\frac{1}{2}W(\phi)\dot{\phi}^2+U(\phi)$, $p_\phi=\frac{1}{2}W(\phi)\dot{\phi}^2-U(\phi)$, and the Hubble parameter is $\tilde{H}\equiv\frac{1}{\tilde{a}}\frac{d\tilde{a}}{d\tilde{t}}$. Then,

$$W(\phi)\dot{\phi}^2=-4\dot{\tilde{H}}, \qquad U(\phi)=6\tilde{H}^2+2\dot{\tilde{H}}. \quad (52)$$

Note that $\dot{\tilde{H}}>0$ is equivalent to $W<0$; superacceleration is due to the "wrong" (negative) sign of the kinetic energy, which is the distinctive feature of a phantom field. The scalar field could be redefined to eliminate the factor $W(\phi)$, but this would not correct the sign of the kinetic energy.

If we choose $W(\phi)$ and $U(\phi)$ as $\omega(\phi)$ and $V(\phi)$ in (7),

$$W(\phi)=-\frac{2}{\kappa^2}g'(\phi), \qquad U(\phi)=\frac{1}{\kappa^2}[3g(\phi)^2+g'(\phi)], \quad (53)$$

by using a function $g(\phi)$ instead of $f(\phi)$ in (7), we find a solution as in (8),

$$\phi=\tilde{t}, \qquad \tilde{H}(\tilde{t})=g(\tilde{t}). \quad (54)$$

In (53) and hereafter in this section, we have dropped the matter contribution for simplicity.

We consider the de Sitter solution in this frame,

$$\tilde{H}=\tilde{H}_0=\text{const.}\rightarrow\tilde{a}(\tilde{t})=\tilde{a}_0e^{\tilde{H}_0\tilde{t}}. \quad (55)$$

We will see below that accelerated expansion can be obtained in the original frame corresponding to the Einstein frame (47) with the solution (55), by choosing an appropriate function $f(\phi)$. From (55) and the definition of $W(\phi)$ and $U(\phi)$, we have

$$W(\phi)=0\rightarrow\omega(\phi)=-\frac{3}{[1+f(\phi)]\kappa^2}\left[\frac{df(\phi)}{d\phi}\right]^2,$$
$$U(\phi)=\frac{6}{\kappa^2}\tilde{H}_0^2\rightarrow V(\phi)=\frac{6}{\kappa^2}\tilde{H}_0^2[1+f(\phi)]^2. \quad (56)$$

Thus, the scalar field has a noncanonical kinetic term in the original frame, while in the Einstein frame the latter can be positive, depending on $W(\phi)$. The correspondence between conformal frames can be made explicit through the conformal transformation (45). Assuming a spatially flat FRW metric in the original frame,

$$ds^2=-dt^2+a^2(t)\sum_{i=1}^3 dx_i^2, \quad (57)$$

then, the relation between the time coordinate and the scale parameter in these frames is given by

$$t=\int\frac{d\tilde{t}}{([1+f(\tilde{t})]^{1/2}}, \qquad a(t)=[1+f(\tilde{t})]^{-1/2}\tilde{a}(\tilde{t}). \quad (58)$$

Now let us discuss the late-time acceleration in the model under discussion.

A. Example 1

As a first example, we consider the coupling function between the scalar field and the Ricci scalar

$$f(\phi)=\frac{1-\alpha\phi}{\alpha\phi}, \quad (59)$$

where α is a constant. Then, from (56), the kinetic function $\omega(\phi)$ and the potential $V(\phi)$ are

$$\omega(\phi) = -\frac{3}{\kappa^2\alpha^2}\frac{1}{\phi^3}, \qquad V(\phi) = \frac{6\tilde{H}_0}{\kappa^2\alpha^2}\frac{1}{\phi^2}, \tag{60}$$

respectively. The solution for the current example is found to be

$$\phi(t) = \tilde{t} = \frac{1}{\alpha}\left(\frac{3\alpha}{2}t\right)^{2/3}, \qquad a(t) = \tilde{a}_0\left(\frac{3\alpha}{2}t\right)^{1/3}\exp\left[\frac{\tilde{H}_0}{\alpha}\left(\frac{3\alpha}{2}t\right)^{2/3}\right]. \tag{61}$$

We now calculate the acceleration parameter to study the behavior of the scalar parameter in the original frame,

$$\frac{\ddot{a}}{a} = -\frac{2}{9}\frac{1}{t^2} + \tilde{H}_0\left(\frac{2}{3\alpha}\right)^{1/3}\left[\frac{1}{t^{4/3}} + \tilde{H}_0\left(\frac{2}{3\alpha}\right)^{1/3}\frac{1}{t^{2/3}}\right]. \tag{62}$$

We observe that for small values of t the acceleration is negative; after that we get accelerated expansion for large t; finally, the Universe ends with zero acceleration as $t \to \infty$. Thus, late-time accelerated expansion is reproduced by the action (44) with the function $f(\phi)$ given by Eq. (59).

B. Example 2

As a second example, consider the function

$$f(\phi) = \phi - t_0. \tag{63}$$

From (58), the kinetic term and the scalar potential are, in this case,

$$\omega(\phi) = -\frac{3}{\kappa^2}\frac{1}{(1+\phi-t_0)}, \qquad V(\phi) = \frac{6\tilde{H}_0^2}{\kappa^2}(1+\phi-t_0). \tag{64}$$

The solution in the original (Jordan) frame reads

$$\phi(t) = \frac{t^2}{4} + t_0 - 1, \qquad a(t) = \frac{2\tilde{a}_0}{t}\exp\left[\tilde{H}_0\left(\frac{t^2}{4} + t_0 - 1\right)\right], \tag{65}$$

and the corresponding acceleration is

$$\frac{\ddot{a}}{a} = \frac{1}{t^2}\left[\frac{\tilde{H}_0 t^2}{2} + \left(\frac{\tilde{H}_0 t^2}{2} - 1\right)^2 + 1\right]. \tag{66}$$

Notice that this solution describes acceleration at every time t and, for $t \to \infty$, the acceleration tends to be a constant value, as in de Sitter space-time, hence similar to what happens in the Einstein frame. Thus, we have proved here that it is possible to reproduce accelerated expansion in both frames, by choosing a convenient function for the coupling $f(\phi)$.

IV. RECONSTRUCTION OF NONMINIMALLY COUPLED SCALAR FIELD THEORY

We now consider the reconstruction problem in the nonminimally coupled scalar field theory, or the Brans-Dicke theory. We begin with the same scalar-tensor theory with constant parameters ϕ_0 and V_0:

$$S = \int d^4x\sqrt{-g}\left[\frac{R}{2\kappa^2} - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right], \qquad V(\phi) = V_0 e^{-2\phi/\phi_0}, \tag{67}$$

which admits the exact solution

$$\phi = \phi_0 \ln\left|\frac{t}{t_1}\right|, \qquad H = \frac{\kappa^2\phi_0^2}{2t}, \qquad t_1^2 \equiv \frac{\gamma\phi_0^2(\frac{3\gamma\kappa^2\phi_0^2}{2} - 1)}{2V_0}. \tag{68}$$

We choose $\phi_0^2\kappa^2 > 2/3$ and $V_0 > 0$ so that $t_1^2 > 0$. For this solution, the metric is given by

$$ds^2 = -dt^2 + a_0^2\left(\frac{t}{t_0}\right)^{\kappa^2\phi_0^2}\sum_{i=1}^{3}(dx^i)^2, \tag{69}$$

which can be transformed into the conformal form:

$$ds^2 = a_0^2\left(\frac{\tau}{\tau_0}\right)^{-\kappa^2\phi_0^2/((\kappa^2\phi_0^2/(\kappa^2\phi_0^2/2)-1)}\left(-d\tau^2 + \sum_{i=1}^{3}(dx^i)^2\right). \tag{70}$$

Here

$$\frac{\tau}{\tau_0} = -\left(\frac{t}{t_0}\right)^{-(\kappa^2\phi_0^2/2)+1}, \qquad \tau_0 \equiv \frac{t_0}{a_0(\frac{\kappa^2\phi_0^2}{2} - 1)}, \tag{71}$$

and therefore, by using (68), one finds

$$\frac{\tau}{\tau_0} = -\left(\frac{t_1}{t_0}\right)^{-(\kappa^2/2)+1} e^{-(1/(-(\kappa^2/2)+1))(\phi/\phi_0)}. \tag{72}$$

We now consider an arbitrary cosmology given by the metric

$$d\tilde{s}^2 = f(\tau)\left(-d\tau^2 + \sum_{i=1}^{3}(dx^i)^2\right), \tag{73}$$

where τ is the conformal time. Since

$$ds^2 = \frac{1}{f(\tau)}a_0^2\left(\frac{\tau}{\tau_0}\right)^{-\kappa^2\phi_0^2/((\kappa^2\phi_0^2/(\kappa^2\phi_0^2 2)-1)}d\tilde{s}^2 = e^{\varphi}d\tilde{s}^2, \qquad e^{\varphi} \equiv \frac{a_0^2(\frac{t_1}{t_0})^{\kappa^2\phi_0^2}e^{\kappa^2\phi_0\phi}}{f(-\tau_0(\frac{t_1}{t_0})^{-(\kappa^2/2)+1}e^{-(1/(-(\kappa^2/2)+1))(\phi/\phi_0)})}, \tag{74}$$

if we begin with the action in which $g_{\mu\nu}$ in (67) is replaced by $e^{\varphi}\tilde{g}_{\mu\nu}$,

$$S = \int d^4x\sqrt{-\tilde{g}}e^{\varphi(\phi)}\left\{\frac{R}{2\kappa^2} - \frac{1}{2}\left[1 - 3\left(\frac{d\varphi}{d\phi}\right)^2\right]\partial_\mu\phi\partial^\mu\phi - e^{\varphi(\phi)}V(\phi)\right\}, \tag{75}$$

we obtain the solution (73).

A. Example

By using the conformal time τ, the metric of de Sitter space

$$ds^2 = -dt^2 + e^{2H_0t}\sum_{i=1}^{3}(dx^i)^2, \tag{76}$$

can be rewritten as

$$ds^2 = \frac{1}{H_0^2\tau^2}\left(-d\tau^2 + \sum_{i=1}^{3}(dx^i)^2\right). \tag{77}$$

Here τ is related to t by $e^{-H_0t} = -H_0\tau$. Then $t \to -\infty$ corresponds to $\tau \to +\infty$ and $t \to +\infty$ corresponds to $\tau \to 0$.

As an example of $f(\tau)$ in (73), we may consider

$$f(\tau) = \frac{(1 + H_L^2\tau^2)}{H_L^2\tau^2(1 + H_I^2\tau^2)}, \tag{78}$$

where H_L and H_I are constants. At early times in the history of the Universe $\tau \to \infty$ (corresponding to $t \to -\infty$), $f(\tau)$ behaves as

$$f(\tau) \to \frac{1}{H_I^2\tau^2}. \tag{79}$$

Then the Hubble rate is given by a constant H_I, and therefore the Universe is asymptotically de Sitter space, corresponding to inflation. On the other hand, at late times $\tau \to 0$ (corresponding to $t \to +\infty$), $f(\tau)$ behaves as

$$f(\tau) \to \frac{1}{H_L^2\tau^2}. \tag{80}$$

Then the Hubble rate is again a constant H_L, which may correspond to the late-time acceleration of the Universe. This proves that our reconstruction program can be applied directly to the nonminimally coupled scalar theory.

V. DE SITTER SPACE IN SCALAR-TENSOR THEORY

When studying scalar field cosmology in a spatially flat FRW universe from the dynamical systems point of view, it is often convenient to redefine the scalar field ϕ used in the previous sections in such a way that its kinetic energy density has a canonical form (apart from the sign). Instead of the field ϕ appearing in Eq. (47), one can use

$$\sigma \equiv \int d\phi\sqrt{|W(\phi)|}, \tag{81}$$

in terms of which the action (47) becomes

$$S = \int d^4x\sqrt{-g}\left[\frac{R}{2\kappa^2} - \frac{\epsilon}{2}\partial^\mu\sigma\partial_\mu\sigma - \bar{U}(\sigma)\right], \tag{82}$$

where $\epsilon = \text{sign}(W)$ and $\bar{U}(\sigma) = U[\phi(\sigma)]$ (this redefinition, however, cannot change the sign of the kinetic energy of the phantom field to make it positive—for this purpose one would have to make the scalar field purely imaginary through a sort of Wick rotation).

In discussions of the phase space of spatially flat scalar field cosmology, the Hubble parameter H and the scalar field σ (or ϕ) constitute a natural choice of dynamical variables. The phase space is a two-dimensional curved surface embedded in a three-dimensional space and this suffices to guarantee the absence of chaos in the dynamics [13]. Moreover, the only fixed points of the dynamical system are de Sitter spaces with a constant scalar field (H_0, σ_0). For the solution described by Eqs. (54) and (55), the scalar field redefinition (81) yields $\sigma = \text{const.} \equiv \sigma_0$ and $(\tilde{H}_0, \sigma_0)$ is a fixed point of the system. The fixed point nature of a particular solution does depend on the specific choice of dynamical variables:, for example, the solution (54) and (55) is a fixed point with the choice (H, σ) but not with the choice (H, ϕ). While the solution is the same, it is convenient to study the dynamics using σ instead of ϕ. This is particularly important for detailed calculations of the stability of de Sitter space using gauge-invariant variables (see below), as these calculations greatly simplify in a de Sitter background.

One may wonder whether the fixed point (or even the attractor) nature of de Sitter spaces is lost when performing the conformal transformation to the Einstein frame used in the previous sections in order to find exact solutions. This is not the case, but a little care is needed because, in general, acceleration of the Universe $\ddot{a} > 0$ in the Jordan frame does not imply cosmic acceleration $\frac{d^2\tilde{a}}{d\tilde{t}^2} > 0$ in the Einstein frame, and vice versa since

$$\frac{d^2\tilde{a}}{d\tilde{t}^2} = \Omega^{-1}\left[\ddot{a} + \frac{\dot{\Omega}}{\Omega}\dot{a} + \frac{(\Omega\ddot{\Omega} - \dot{\Omega}^2)}{\Omega^2}a\right] \tag{83}$$

(here an overdot denotes differentiation with respect to the Jordan frame comoving time t). However, a de Sitter space in the Jordan frame is mapped into a de Sitter space in the Einstein frame (and vice versa). A general conformal transformation of the metric $g_{\mu\nu} \to \tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$, where the conformal factor Ω depends on the scalar field present in the theory (for example, as in Eq. (45)), yields the rescaled FRW line element

$$d\tilde{s}^2 = \Omega^2 ds^2 = -d\tilde{t}^2 + \tilde{a}^2(\tilde{t})(dx^2 + dy^2 + dz^2) \tag{84}$$

with $d\tilde{t} = \Omega dt$ and $\tilde{a} = \Omega a$. Therefore, the relation between the Hubble parameters of the two conformal frames is

$$\tilde{H} \equiv \frac{1}{\tilde{a}}\frac{d\tilde{a}}{d\tilde{t}} = \frac{1}{\Omega}\left(H + \frac{\dot{\Omega}}{\Omega}\right) = \frac{1}{\Omega}\left(H + \frac{d\Omega}{d\tilde{t}}\right). \tag{85}$$

Since $\Omega = \Omega(\sigma)$, it is clear that a Jordan frame fixed point $(\dot{H}, \dot{\sigma}) = (0, 0)$ is mapped into an Einstein frame fixed point $(\dot{\tilde{H}}, \dot{\sigma}) = (0, 0)$. Moreover, a small perturbation of this fixed point in the Jordan frame corresponds to a small perturbation in the Einstein frame, and stability of the Jordan frame fixed point corresponds to stability of the corresponding Einstein frame stationary point. In fact, assume that

$$H(t) = H_0 + \delta H(t), \tag{86}$$

$$\sigma(t) = \sigma_0 + \delta\sigma(t), \tag{87}$$

in the Jordan frame, where $|\delta H(t)/H_0|$ and $|\delta\sigma(t)/\sigma_0|$ are small perturbations which, for simplicity, are taken here to be homogeneous. Then, in the Einstein frame, $\tilde{H} = \tilde{H}_0 + \delta\tilde{H}$ with

$$\delta\tilde{H} = \frac{1}{\Omega_0}\left[\delta H + \frac{\Omega_0'}{\Omega_0}(\delta\dot{\sigma} - H_0\delta\sigma)\right] \tag{88}$$

(where a prime denotes differentiation with respect to σ, and the zero subscript denotes quantities evaluated in the background de Sitter space). The Einstein frame perturbation stays small if the Jordan frame perturbations are small, hence an attractor in the Jordan frame corresponds to an Einstein frame attractor (the converse is not always true, see [14] for a counterexample). This property is crucial when using conformal transformations to study slow-roll inflation which describes dynamics around a de Sitter attractor. It is the presence of a de Sitter attractor that justifies the use of the slow-roll approximation $H(t) = H_0 + \delta H(t)$ in inflation, and the fact that the presence and attractor nature of de Sitter space (subject to certain conditions) are guaranteed also in the Einstein frame ultimately justifies the use of conformal techniques in the study of slow-roll inflation. To summarize, in general, cosmic acceleration in one conformal frame does not correspond to acceleration in the conformally related frame (see the related discussion in [15]), however slow-roll inflation in the Jordan frame corresponds to slow-roll inflation in the Einstein frame. This is relevant also for late-time de Sitter–like expansion in a universe dominated by quintessence.

The previous discussion is restricted to homogeneous (space-independent) perturbations of de Sitter space, but it can be extended to more general (space-dependent) *inhomogeneous* perturbations. The latter are more problematic because of the notorious gauge-dependence problems associated with them, and they are best described using gauge-independent methods. A linear stability condition of de Sitter space against inhomogeneous perturbations was derived in [16]. This is a necessary condition for de Sitter space to be an attractor in the (H, σ) phase space, however it is not sufficient because it only ensures stability to first order in the gauge-invariant variables (stability to higher, or to all orders, can usually be established only numerically and with the restriction to homogeneous perturbations, see, e.g., [17]).

For a theory described by the action

$$S = \int d^4x\sqrt{-g}\left[\varphi(\phi, R) - \frac{\omega(\phi)}{2}\partial^\mu\phi\partial_\mu\phi - V(\phi)\right], \tag{89}$$

the gauge-invariant linear stability condition is [16]

$$\frac{\frac{\partial^2\varphi}{\partial\phi^2}|_0 - \frac{d^2V}{d\phi^2}|_0 + \frac{R_0\varphi_{\phi R}^2}{F_0}}{\omega_0(1 + 3\frac{\varphi_{\phi R}^2}{\omega_0 F_0})} \leq 0, \tag{90}$$

where $F \equiv \frac{\partial\varphi}{\partial R}$ and $\varphi_{\phi R} \equiv \frac{\partial^2\varphi}{\partial R\partial\phi}$. For the action (44), this condition becomes

$$\frac{R_0[f_0''(1 + f_0) + (f_0')^2] - 2\kappa^2 V_0''}{2\kappa^2\omega_0(1 + f_0) + 3(f_0')^2} \leq 0, \tag{91}$$

where $f_0 \equiv f(\phi_0)$, $f_0' \equiv f'(\phi_0)$, etc. and assuming that $1 + f(\phi_0) > 0$ in order to guarantee a positive effective gravitational coupling. For an ordinary phantom field in general relativity it is $\varphi(\phi, R) = R$, $\omega = -1$, and the condition (91) reduces to $V_0'' \leq 0$, i.e., the de Sitter space is stable if the potential has a maximum to which the phantom field can climb and settle in during the dynamical evolution. This behavior, which is opposite to that of an ordinary scalar field, is due to the negative sign of the kinetic energy of the phantom [3].

The FRW equations give two conditions for the existence of de Sitter fixed points:

$$R_0\frac{\partial\varphi}{\partial R}\bigg|_0 = 2\varphi_0 - V_0, \tag{92}$$

$$\varphi_0' = V_0'. \tag{93}$$

For the action (44) these become

$$H_0^2 = \frac{\kappa^2 V_0}{12(1 + f_0)}, \tag{94}$$

$$V_0' = \frac{1 + f_0'}{1 + f_0}V_0, \tag{95}$$

where $R_0 = 12H_0^2$ for de Sitter space. Upon use of Eq. (94), the stability condition (91) in this theory becomes

$$\frac{(V_0 f_0'' - 2V_0'')(1 + f_0) + V_0(f_0')^2}{2\kappa^2\omega_0(1 + f_0) + 3(f_0')^2} \leq 0. \tag{96}$$

These stability conditions are important in the study of the future evolution of the late-time accelerated era.

VI. LATE-TIME ACCELERATION AND INFLATION WITH SEVERAL SCALAR FIELDS

In this section we begin by considering a model with two-scalar fields minimally coupled to gravity (see [4,7,18]). Such models are used, for example, in reheating scenarios after inflation.

An additional degree of freedom appears in this case, so that for a given solution we may choose different conditions on the scalar fields, as shown below. It is possible to restrict these conditions by studying the perturbative regime for each solution. The action we consider is

$$S = \int \sqrt{-g}\left[\frac{R}{2\kappa^2} - \omega(\phi)\partial_\mu\phi\partial^\mu\phi - \sigma(\chi)\partial_\mu\chi\partial^\mu\chi - V(\phi, \chi)\right], \tag{97}$$

where $\omega(\phi)$ and $\sigma(\chi)$ are the kinetic terms, which depend on the fields ϕ and χ, respectively. We again assume a flat FRW metric. The Friedmann equations are written as

$$H^2 = \frac{\kappa^2}{3}\left[\frac{1}{2}\omega(\phi)\dot{\phi}^2 + \frac{1}{2}\sigma(\chi)\dot{\chi}^2 + V(\phi, \chi)\right], \qquad \dot{H} = -\frac{\kappa^2}{2}\left[\omega(\phi)\dot{\phi}^2 + \sigma(\chi)\dot{\chi}^2\right]. \tag{98}$$

By means of a convenient transformation, we can always redefine the scalar fields so that we can write $\phi = \chi = t$. The scalar field equations are given by

$$\omega(\phi)\ddot{\phi} + \frac{1}{2}\omega'(\phi)\dot{\phi}^2 + 3H\omega(\phi)\dot{\phi} + \frac{\partial V(\phi, \chi)}{\partial\phi} = 0, \qquad \sigma(\chi)\ddot{\chi} + \frac{1}{2}\sigma'(\chi)\dot{\chi}^2 + 3H\sigma(\chi)\dot{\chi} + \frac{\partial V(\phi, \chi)}{\partial\chi} = 0. \tag{99}$$

Then, for a given solution $H(t) = f(t)$, and combining the first Friedmann equation with each scalar field equation, respectively, we find

$$\omega(\phi) = -\frac{2}{\kappa^2}\frac{\partial f(\phi, \chi)}{\partial\phi}, \qquad \sigma(\chi) = -\frac{2}{\kappa^2}\frac{\partial f(\phi, \chi)}{\partial\chi}, \tag{100}$$

where the function $f(\phi, \chi)$ carries down to $f(t, t) \equiv f(t)$, and is defined as

$$f(\phi, \chi) = -\frac{\kappa^2}{2}\left[\int \omega(\phi)d\phi + \int \sigma(\chi)d\chi\right]. \tag{101}$$

The scalar potential can be expressed as

$$V(\phi, \chi) = \frac{1}{\kappa^2}\left[3f(\phi, \chi)^2 + \frac{\partial f(\phi, \chi)}{\partial\phi} + \frac{\partial f(\phi, \chi)}{\partial\chi}\right], \tag{102}$$

and the second Friedmann equation reads

$$-\frac{2}{\kappa^2}f'(t) = \omega(t) + \sigma(t). \tag{103}$$

Then, the kinetic functions may be chosen to be

$$\omega(\phi) = -\frac{2}{\kappa^2}[f'(\phi) + g(\phi)], \qquad \sigma(\phi) = \frac{2}{\kappa^2}g(\chi), \tag{104}$$

where g is an arbitrary function. Hence, the scalar field potential is finally obtained as

$$V(\phi, \chi) = \frac{1}{\kappa^2}[3f(\phi, \chi)^2 + f'(\phi) + g(\phi) - g(\chi)]. \tag{105}$$

A. Example 1

We can consider again the solution (17)

$$f(t) = \frac{H_0}{t_s - t} + \frac{H_1}{t^2}. \tag{106}$$

This solution, as already seen in Sec. II, reproduces unified inflation and late-time acceleration in a scalar field model with matter given by action (1). We may now understand this solution as derived from the two-scalar field model (97), where a degree of freedom is added so that we can choose various types of scalar kinetic and potential terms, as shown below for the solution (106). Then, from Eqs. (104) and (106), the kinetic terms follow:

$$\omega(\phi) = -\frac{2}{\kappa^2}\left[\frac{H_0}{(t_s - \phi)^2} - \frac{2H_1}{\phi^3} + g(\phi)\right], \qquad \sigma(\phi) = \frac{2}{\kappa^2}g(\chi). \tag{107}$$

The function $f(\phi, \chi)$ is

$$f(\phi, \chi) = \frac{H_0}{t_s - \phi} + \frac{H_1}{\phi^2} + \int d\phi g(\phi) - \int d\chi g(\chi), \tag{108}$$

while the scalar potential is

$$V(\phi, \chi) = \frac{1}{\kappa^2}\left[3f(\phi, \chi)^2 + \frac{H_0}{(t_s - \phi)^2} - \frac{2H_1}{\phi^3} + g(\phi) - g(\chi)\right]. \tag{109}$$

This scalar potential leads to the cosmological solution (17).

It is possible to further restrict $g(t)$ by studying the stability of the system considered. To this end, we define the functions

$$X_\phi = \dot{\phi}, \qquad X_\chi = \dot{\chi}, \qquad Y = \frac{f(\phi, \chi)}{H}. \tag{110}$$

Then, the Friedmann and scalar field equations can be written as

$$\frac{dX_\phi}{dN} = -\frac{1}{2H}\frac{\omega'(\phi)}{\omega(\phi)}(X_\phi^2 - 1) - 3(X_\phi - Y), \quad (111)$$

$$\frac{dX_\sigma}{dN} = -\frac{1}{2H}\frac{\sigma'(\chi)}{\sigma(\chi)}(X_\chi^2 - 1) - 3(X_\chi - Y),$$
$$\frac{dY}{dN} = \frac{\kappa^2}{2H^2}[\omega(\phi)X_\phi(YX_\phi - 1) + \sigma(\chi)X_\chi(YX_\chi - 1)], \quad (112)$$

where $\frac{d}{dN} = \frac{1}{H}\frac{1}{dt}$. At $X_\phi = X_\chi = Y = 1$, we consider the perturbations

$$X_\phi = 1 + \delta X_\phi, \qquad X_\chi = 1 + \delta X_\chi, \qquad Y = 1 + \delta Y, \quad (113)$$

then

$$\frac{d}{dN}\begin{pmatrix}\delta X_\phi\\ \delta X_\chi\\ \delta Y\end{pmatrix} = M\begin{pmatrix}\delta X_\phi\\ \delta X_\chi\\ \delta Y\end{pmatrix},$$
$$M = \begin{pmatrix} -\frac{\omega'(\phi)}{H\omega(\phi)} - 3 & 0 & 3\\ 0 & -\frac{\sigma'(\chi)}{H\sigma(\chi)} - 3 & 3\\ \kappa^2\frac{\omega(\phi)}{2H^2} & \kappa^2\frac{\sigma(\chi)}{2H^2} & \kappa^2\frac{\omega(\phi)+\sigma(\chi)}{2H^2}\end{pmatrix}. \quad (114)$$

The eigenvalue equation is given by

$$\left(\frac{\omega'(\phi)}{H\omega(\phi)} + 3 + \lambda\right)\left(\frac{\sigma'(\chi)}{H\sigma(\chi)} + 3 + \lambda\right)\left(\frac{\kappa^2}{2H^2}(\omega(\phi) + \sigma(\phi)) - \lambda\right) + \frac{3\kappa^2\omega(\phi)}{2H^2}\left(\frac{\sigma'(\chi)}{H\sigma(\chi)} + 3 + \lambda\right) + \frac{3\kappa^2\sigma(\chi)}{2H^2}\left(\frac{\omega'(\phi)}{H\omega(\phi)} + 3 + \lambda\right) = 0. \quad (115)$$

To avoid divergences in the eigenvalues, we choose the kinetic functions to satisfy

$$\omega(\phi) \neq 0, \qquad \sigma(\chi) \neq 0, \quad (116)$$

hence, the eigenvalues in Eq. (115) are finite. Summing up, under these conditions, the solution (106) has no infinite instability when the transition from the nonphantom to the phantom phase occurs.

As an example, we may choose $g(t) = \alpha/t^3$, where α is a constant that satisfies $\alpha > 2H_1$. Then, the $f(\phi, \chi)$ function (108) is given by

$$f(\phi, \chi) = \frac{H_0}{t_s - \phi} - \frac{(\alpha - 2H_1)}{2\phi^2} + \frac{\alpha}{2\chi^2}. \quad (117)$$

As a result, the kinetic terms (107) are expressed as

$$\omega(\phi) = -\frac{2}{\kappa^2}\left[\frac{H_0}{(t_s - \phi)^2} - \frac{2H_1}{\phi^3} + \frac{\alpha}{\phi^3}\right],$$
$$\sigma(\chi) = \frac{2}{\kappa^2}\frac{\alpha}{\chi^3}, \quad (118)$$

and the potential reads

$$V(\phi, \chi) = \frac{1}{\kappa^2}\left[3f(\phi, \chi)^2 + \frac{H_0}{(t_s - \phi)^2} + \frac{\alpha - 2H_1}{\phi^3} - \frac{\alpha}{\chi^3}\right]. \quad (119)$$

This potential reproduces the solution above, which unifies inflation and late-time acceleration in the context of scalar-tensor theories, involving two-scalar fields. Notice that the extra degree of freedom gives the possibility to select a different kinetic and scalar potential in such a manner that we get the same solution.

In the case in which the condition (116) is not imposed, the kinetic terms (104) may have zeros for $0 < t < t_s$, so that the perturbation analysis performed above ceases to be valid because some of the eigenvalues could diverge.

B. General case: *n* scalar fields

As a generalization of the action (97), we now consider the corresponding one for n scalar fields,

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2\kappa^2}R - \frac{1}{2}\sum_{i=1}^{n}\omega_i(\phi_i)\partial_\mu\phi_i\partial^\mu\phi_i - V(\phi_1, \phi_2, \cdots, \phi_n)\right]. \quad (120)$$

The associated Friedmann equations are

$$H^2 = \frac{\kappa^2}{3}\left[\sum_{i=1}^{n}\frac{1}{2}\omega_i(\phi_i)\dot{\phi}_i^2 + V(\phi_1, \cdots, \phi_n)\right],$$
$$\dot{H} = -\frac{\kappa^2}{2}\left[\sum_{i=1}^{n}\omega_i(\phi_i)\dot{\phi}_i^2\right]. \quad (121)$$

We can proceed analogously to the case of two-scalar fields so that the kinetic terms are written as

$$\sum_{i=1}^{n}\omega_i(t) = -\frac{2}{\kappa^2}f'(t), \quad (122)$$

hence,

$$\omega_i(\phi_i) = -\frac{2}{\kappa^2}\frac{df(\phi_1, \cdots, \phi_n)}{d\phi_i},$$
$$V(\phi_1, \cdots, \phi_n) = \frac{1}{\kappa^2}\left[3f(\phi_1, \cdots, \phi_n)^2 + \sum_{i=1}^{n}\frac{df(\phi_1, \cdots, \phi_n)}{d\phi_i}\right], \quad (123)$$

where $f(t, t, \cdots, t) \equiv f(t)$. Then the following solution is found:

$$\phi_i = t, \qquad H(t) = f(t). \tag{124}$$

From (122) we can choose, as done above, the kinetic terms to be

$$\omega_1(\phi_1) = -\frac{2}{\kappa^2}[f'(\phi_1) + g_2(\phi_1) + \cdots + g_n(\phi_1)],$$
$$\omega_2(\phi_2) = \frac{2}{\kappa^2} g_2(\phi_2), \cdots, \omega_n(\phi_n) = \frac{2}{\kappa^2} g_n(\phi_n). \tag{125}$$

Then, there are $n-1$ arbitrary functions that reproduce the solution (124) so reconstruction may be successfully done. They could be chosen so that dark matter is also represented by some of the scalar fields appearing in the action (120).

VII. RECONSTRUCTION OF INFLATION AND COSMIC ACCELERATION FROM TWO-SCALAR THEORY

In the present section, inflation and cosmic acceleration are reconstructed separately, by means of a two-scalar field model that reproduces some of the cosmological constraints at each epoch. We explore an inflationary model in which the scalar potential, given for a pair of scalar fields, exhibits an extra degree of freedom and can be chosen in such way that slow-roll conditions are satisfied. Also, the cosmic acceleration is reproduced with a pair of scalar fields plus an ordinary matter term, in which the values of the observed cosmological density parameter ($\Omega_{\rm DE} \simeq 0.7$) and of the EoS parameter ($w_{\rm DE} \simeq -1$) are reproduced in a quite natural way. For the case of a single scalar, the reconstruction for similarly distant epochs was given in Ref. [19] (for other reconstruction versions, see also [20]).

A. Inflation

In the previous sections, models describing inflation and late-time accelerated expansion have been constructed by using certain convenient scalar-tensor theories. In this section, we present an inflationary model with two-scalar fields, which can be constructed in such a way that the inflationary conditions are carefully accounted for. For this purpose, we use some of the techniques given in the previous section. The action during the inflationary epoch is written as

$$S = \int \sqrt{-g}\Big[\frac{R}{2\kappa^2} - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\partial_\mu\chi\partial^\mu\chi - V(\phi,\chi)\Big]. \tag{126}$$

We will show that a general solution can be constructed, where the scalar field potential is not completely specified because of the extra degree of freedom represented by the second scalar field added, in a way similar to the situation occurring in the previous section. Considering a spatially flat FRW metric, the Friedmann and scalar field equations are obtained by using the Einstein equations and varying the action with respect to both scalar fields:

$$H^2 = \frac{\kappa^2}{3}\Big[\frac{1}{2}\dot\phi^2 + \frac{1}{2}\dot\chi^2 + V(\phi,\chi)\Big],$$
$$\ddot\phi + 3H\dot\phi + \frac{\partial V(\phi,\chi)}{\partial\phi} = 0, \tag{127}$$
$$\ddot\chi + 3H\dot\chi + \frac{\partial V(\phi,\chi)}{\partial\chi} = 0.$$

We assume that the slow-roll conditions are satisfied, that is, $\ddot\phi \ll 3H\dot\phi$ ($\ddot\chi \ll 3H\dot\chi$), and $\dot\phi^2 \ll V(\phi,\chi)$ ($\dot\chi^2 \ll V(\phi,\chi)$), in order for inflation to occur. Then, Eqs. (127) take the form

$$H^2 \approx \frac{\kappa^2}{3}V(\phi,\chi), \qquad 3H\dot\phi + \frac{\partial V(\phi,\chi)}{\partial\phi} \approx 0, \qquad 3H\dot\chi + \frac{\partial V(\phi,\chi)}{\partial\chi} \approx 0 \tag{128}$$

and the slow-roll conditions read

$$\frac{1}{3\kappa^2}\frac{V_{,i}V^{,i}}{V^2} \ll 1, \qquad \frac{1}{3\kappa^2}\frac{\sqrt{V_{,ij}V^{,ij}}}{V} \ll 1. \tag{129}$$

Here, $V_{,i}$ denotes the partial derivative of V with respect to one of the scalar fields ($i = \phi, \chi$). As done previously, a scalar potential $V(\phi,\chi)$ can be constructed, although in this case the conditions for inflation need to be taken into account. From (128), the potential is given by

$$V(\phi,\chi) = \frac{3}{\kappa^2}H^2(\phi,\chi). \tag{130}$$

We can choose this potential such that

$$V(\phi,\chi) = \frac{3}{\kappa^2}[f^2(\phi,\chi) + g_1(\phi) - g_2(\chi)]. \tag{131}$$

The three components f, g_1, and g_2 are arbitrary functions, and $g(N) = g_1(N) = g_2(N)$ where, for convenience, we use the number of e-folds $N \equiv \ln\frac{a(t)}{a_i}$ instead of the cosmic time, and a_i denotes the initial value of the scale factor before inflation. Then, the following solution is found:

$$H(N) = f(N). \tag{132}$$

Hence, Eqs. (128) may be expressed as a set of differential equations with the number of e-folds as an independent variable,

$$3f^2(N)\frac{d\phi}{dN} + \frac{\partial V(\phi,\chi)}{\partial\phi} \approx 0, \qquad 3f^2(N)\frac{d\chi}{dN} + \frac{\partial V(\phi,\chi)}{\partial\chi} \approx 0. \tag{133}$$

To illustrate this construction, let us use a simple example. The following scalar potential, as a function of the number of e-folds N, is considered:

$$V(\phi, \chi) = \frac{3}{\kappa^2}[H_0^2 N^{2\alpha}], \tag{134}$$

where α and H_0 are free parameters. By specifying the arbitrary function $g(N)$, one can find a solution for the scalar fields. Let us choose, for the sake of simplicity, $g(N) = g_0 N^{2\alpha}$, where g_0 is a constant, and $f(\phi, \chi) = f(\phi)$ (i.e., as a function of the scalar field ϕ only). Then, using Eqs. (133), the solutions for the scalar fields are found to be

$$\begin{aligned}\phi(N) &= \phi_0 - \frac{1}{\kappa H_0}\sqrt{2\alpha(H_0^2 + g_0)}\ln N,\\ \chi(N) &= \chi_0 - \frac{1}{\kappa H_0}\sqrt{2 g_0 \alpha N},\end{aligned} \tag{135}$$

and the scalar potential can be written as

$$\begin{aligned}V(\phi, \chi) = \frac{3}{\kappa^2}\Bigg[&(H_0 + g_0)\exp\left(\kappa\frac{\sqrt{2\alpha}H_0}{\sqrt{H_0^2 + g_0}}(\phi_0 - \phi)\right)\\ &- g_0\left(\frac{\kappa^2 H_0^2(\chi_0 - \chi)^2}{2 g_0 \alpha}\right)^{2\alpha}\Bigg].\end{aligned} \tag{136}$$

We are now able to impose the slow-roll conditions by evaluating the slow-roll parameters

$$\begin{aligned}&\frac{1}{3\kappa^2}\frac{V_{,i}V^{,i}}{V^2} = \frac{2\alpha}{3H_0^2}\left(\frac{(H_0 + g_0)^2}{H_0^2 + g_0} + \frac{4 g_0}{N}\right) \ll 1,\\ &\frac{1}{3\kappa^2}\frac{\sqrt{V_{,ij}V^{,ij}}}{V} = \frac{2}{3}\alpha^2\sqrt{\frac{(H_0 + g_0)^2}{(H_0^2 + g_0)^2} + \frac{16 g_0(4\alpha - 1)^2}{4 g_0^2 \alpha^2}\frac{1}{N^2}}\\ &\qquad \ll 1.\end{aligned} \tag{137}$$

Hence, we may choose conveniently the free parameters so that the slow-roll conditions are satisfied, and, therefore, inflation takes place. From these expressions we see that the desired conditions will be obtained, in particular, when α is sufficiently small and/or H_0 and N are large enough. All these regimes help to fulfill the slow-roll conditions, in a quite natural way.

B. Cosmic acceleration with a pair of scalar fields

It is quite reasonable, and rather aesthetic, to think that the cosmic acceleration could be driven by the same mechanism as inflation. To this purpose, we apply the same model with two-scalar fields, with the aim of reproducing late-time acceleration in a universe filled with a fluid with EoS $p_m = w_m \rho_m$. The free parameters given by the model could be adjusted to fit the observational data, as shown below. We begin with the action representing this model,

$$\begin{aligned}S = \int d^4x\sqrt{-g}\Bigg[&\frac{R}{2\kappa^2} - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}(\partial\chi)^2\\ &- V(\phi, \chi) + L_m\Bigg].\end{aligned} \tag{138}$$

By assuming a spatially flat FRW metric, one obtains the Friedmann equations

$$\begin{aligned}H^2 &= \frac{\kappa^2}{3}\left[\frac{1}{2}(\dot{\phi})^2 + \frac{1}{2}(\dot{\chi})^2 + V(\phi, \chi) + \rho_m\right],\\ \dot{H} &= -\frac{\kappa}{2}\left[\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\dot{\chi}^2 + V(\phi, \chi) + \rho_m\right].\end{aligned} \tag{139}$$

Variation of the action (138) yields the scalar field equations

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0, \qquad \ddot{\chi} + 3H\dot{\chi} + V_{,\chi} = 0. \tag{140}$$

This set of independent equations may be supplemented by a fifth one, the conservation of matter-energy density, ρ_m,

$$\dot{\rho_m} + 3H\rho_m(1 + w_m) = 0. \tag{141}$$

As done in [19], we perform the substitutions

$$\begin{aligned}&\Omega_m = \frac{\rho_m}{3H^2/\kappa^2}, \qquad \Omega_{sf} = \frac{\frac{1}{2}(\dot{\phi})^2 + \frac{1}{2}(\dot{\chi})^2 + V(\phi, \chi)}{3H^2/\kappa^2},\\ &\epsilon = \frac{\dot{H}}{H^2}, \qquad w_{sf} \equiv \frac{p_{sf}}{\rho_{sf}} = \frac{\frac{1}{2}(\dot{\phi})^2 + \frac{1}{2}(\dot{\chi})^2 - V(\phi, \chi)}{\frac{1}{2}(\dot{\phi})^2 + \frac{1}{2}(\dot{\chi})^2 + V(\phi, \chi)}.\end{aligned} \tag{142}$$

For convenience, we consider both scalar fields together under the subscript sf, so that the corresponding density parameter is $\Omega_{sf} = \Omega_\phi + \Omega_\chi$. Then, using the transformations (142), the Friedmann equations and the scalar equation read

$$\begin{aligned}\Omega_m + \Omega_{sf} &= 1,\\ 2\epsilon + 3(1 + w_{sf})\Omega_{sf} + 3(1 + w_m)\Omega_m &= 0,\\ \Omega'_{sf} + 2\epsilon\Omega_{sf} + 3\Omega_{sf}(1 + w_{sf}) &= 0.\end{aligned} \tag{143}$$

Here, the prime denotes differentiation with respect to the number of e-folds $N \equiv \ln\frac{a(t)}{\text{const}}$. We can now combine Eqs. (142) to write the EoS parameter for the scalar fields as a function of Ω_{sf} and of the time derivative of the scalar fields, namely

$$w_{sf} = \frac{\kappa^2(\phi'^2 + \chi'^2) - 3\Omega_{sf}}{3\Omega_{sf}}. \tag{144}$$

Hence, it is possible to find an analytic solution for the Eqs. (143), for a given evolution of the scalar fields, as will be seen below. Before doing that, it is useful to write the effective EoS parameter, which is given by

$$w_{\rm eff} = -1 - \frac{2\epsilon}{3}, \qquad \rho_{\rm eff} = \rho_m + \rho_{sf}, \qquad p_{\rm eff} = p_m + p_{sf}, \tag{145}$$

and the deceleration parameter

$$q = -\frac{\ddot{a}}{aH^2} = -1 - \epsilon. \tag{146}$$

As usual, for $q > 0$ the Universe is in a decelerated phase, while $q < 0$ denotes an accelerated epoch, such that for $w_{\rm eff} < -1/3$ the expansion is accelerated. To solve the equations, we consider a universe which, at present, is filled with a pressureless component ($w_m = 0$) representing ordinary matter, and two-scalar fields which represent a dynamical dark energy and a dark matter species. To show this, we make the following assumption on the evolution of the scalar fields, which are given as functions of N:

$$\phi(N) = \phi_0 + \frac{\alpha}{\kappa^2} N, \qquad \chi(N) = \chi_0 + \frac{\beta}{\kappa^2} N. \tag{147}$$

Then, Eqs. (143) can be solved, and the scalar field density parameter takes the form

$$\Omega_{sf} = \Omega_\phi + \Omega_\chi = 1 - \frac{\lambda}{k e^{\lambda N} + 3}, \tag{148}$$

where k is an integration constant and $\lambda = 3 - (\alpha^2 + \beta^2)$. It is possible to introduce an arbitrary function $g(N)$ to express the energy density parameter for each scalar field in the following way:

$$\Omega_\phi = 1 - \frac{\lambda}{k e^{\lambda N} + 3} - g(N), \qquad \Omega_\chi = g(N). \tag{149}$$

The function $g(N)$ may be chosen in such a way that the scalar field χ represents a cold dark matter contribution at present ($w_\chi \simeq 0$), and the scalar field ϕ represents the dark energy responsible for the accelerated expansion of our Universe. On the other hand, using Eqs. (143), $\epsilon = \frac{H'}{H}$ is obtained as

$$\epsilon = -\frac{3}{2}\left\{1 - \frac{k\lambda(k e^{\lambda N} + \alpha^2\beta^2)}{[(\alpha^2 + \beta^2)e^{-\lambda N} + 3k](k e^{\lambda N} + 3)}\right\}. \tag{150}$$

Then, it is possible to calculate the effective parameter of EoS given by Eq. (133)

$$w_{\rm eff} = -1 - \frac{2}{3}\epsilon = -\frac{k\lambda(k e^{\lambda N} + \alpha^2\beta^2)}{[(\alpha^2 + \beta^2)e^{-\lambda N} + 3k](k e^{\lambda N} + 3)}. \tag{151}$$

We have four free parameters (N, k, α, β) that may be adjusted to fit the constraints derived from observations. With this purpose, we use the observational input $\Omega_m \simeq 0.03$, referred to as baryonic matter, and normalize the number of e-folds N, taking $N = 0$ at present, then the integration constant k may be written as a function of α and β as

$$\Omega_m(N = 0) = 0.03 \rightarrow k = \frac{2.01 - (\alpha^2 + \beta^2)}{0.03}. \tag{152}$$

The free parameter β may be fixed in such a way that the scalar field χ represents cold dark matter at present, i.e., $w_\chi \simeq 0$ and $\Omega_\chi \simeq 0.27$, and its EoS parameter is written as

$$w_\chi = \frac{\kappa^2 \chi'^2 - 3\Omega_\chi}{3\Omega_\chi} = \frac{\beta^2 - 3g(N)}{3g(N)}. \tag{153}$$

For convenience, we choose $g(N) = \frac{\beta^2}{3} e^{-N}$, then the energy density and EoS parameter are given by

$$w_\chi = \frac{1 - e^{-N}}{e^{-N}}, \qquad \Omega_\chi = \frac{\beta^2}{3} e^{-N}. \tag{154}$$

Hence, at present ($N = 0$), the expressions (154) can be compared with the observational values and the β parameter is given by

$$w_\chi(N = 0) = 0, \qquad \Omega_\chi(N = 0) \simeq 0.27 \rightarrow \beta^2 = 0.81. \tag{155}$$

Finally, the energy density of ϕ expressed by Eq. (149) takes the form

$$\Omega_\phi = 1 - \frac{\lambda}{k e^{\lambda N} + 3} - \frac{\beta^2}{3} e^{-N}. \tag{156}$$

The value for α could be taken so that $\Omega_\phi \simeq 0.7$ and $w_\phi \simeq -1$ at present. Hence, it has been shown that cosmic acceleration can be reproduced with a pair of scalar fields, where due to the presence of the extra scalar that can be identified with the dark matter component.

It is interesting to point out that one can unify these realistic descriptions of the inflationary and late-time acceleration eras within a single theory. However, the corresponding potential looks quite complicated. The easiest way would be to use step (θ-function) potentials in order to unify the whole description in the easiest way (as was pioneered in [21]).

We may also construct a model unifying early universe inflation and the present accelerated expansion era. To this end we can choose $f(\phi)$ in (30), which gives the Hubble parameter (31). Then, using (104), one can define $\omega(\phi)$ and $\sigma(\chi)$ with the help of an arbitrary function g. After defining then $f(\phi, \chi)$ with Eq. (101), we can construct the potential $V(\phi, \chi)$ using Eq. (102). Finally, we obtain the two-scalar-tensor theory (97) reproducing the Hubble rate (31), which describes both inflation and the accelerated expansion.

VIII. CONCLUSIONS

Modeling both the early inflation and late-time acceleration epochs within the context of a single field theory has, undoubtedly, much aesthetic appeal and seems a worthy goal, which we have attempted here. To summarize, we have developed, step by step, the reconstruction program for the expansion history of the Universe, by using a single or multiple (canonical and/or phantom) scalar fields. Already in the case of a single scalar, we have presented a number of examples which prove that it is actually possible to unify early-time inflation (at very high redshift) with late-time acceleration (at low redshift). The reconstruction technique has then been generalized to the case of a scalar field nonminimally coupled to the Ricci curvature, and to nonminimal (Brans-Dicke-type) scalars. Again, various explicit examples of unification of early-time inflation and late-time acceleration have been presented in those formulations. Because of the special role of de Sitter space, which often appears as an attractor in the inflationary epoch, as well as in the present cosmic acceleration era, special attention was paid to this specific space-time. Conformal transformations to the Einstein frame and stability conditions for the de Sitter space were discussed.

Moreover, the case of several minimally coupled scalar fields has been considered for the description of the realistic evolution of the Hubble parameter, and we have shown that it is qualitatively easier to achieve a realistic unification of late and early epochs in a model of this kind, in such a way as to satisfy the cosmological bounds coming from the observational data. This is due to the arbitrariness in the choice of the scalar potential and the scalar kinetic factor in the description of a universe with a given scale factor $a(t)$. Using the freedom of choosing these scalar functions, one can constrain the theory in an observationally acceptable manner. Specifically, slow-roll conditions and stability conditions may be satisfied in different ways for different scalar functions, while the scale factor remains the same. This can be used also to obtain the correct structure of perturbations, etc.

As a matter of fact, many questions remain to be discussed in greater detail, a more realistic matter content should be taken into account, and the universe expansion history should be described in a more precise and detailed manner. After all, we live in an era of increasingly more precise cosmological tests. Anyhow, the unified effective description of the cosmic expansion history presented here seems quite promising. Using it in more realistic contexts—in which, of course, technical details become necessarily more complicated—appears to be quite possible.

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