Instanton calculus and loop operator in supersymmetric gauge theory

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We compute the one-point function of the glueball loop operator in the maximally confining phase of supersymmetric gauge theory using instanton calculus. In the maximally confining phase the residual symmetry is the diagonal $U(1)$ subgroup and the localization formula implies that the chiral correlation functions are the sum of the contributions from each fixed point labeled by the Young diagram. The summation can be performed exactly by operator formalism of free fermions, which is also featured in the equivariant Gromov-Witten theory of $\mathbf{P}^1$. By taking the Laplace transformation of the glueball loop operator, we find an exact agreement with the previous results for the generating function (resolvent) of the glueball one-point functions.

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I. INTRODUCTION

Symmetry helps us to understand physics, in particular, the nonperturbative dynamics. Guided by holomorphy coming from $\mathcal{N} = 2$ supersymmetry and the asymptotic behavior in a semiclassical region, Seiberg and Witten wrote down the effective prepotential of $\mathcal{N} = 2$ supersymmetric gauge theory [1]. However, it is still desirable to derive their result following the standard process of quantum field theory—path integral. After a lot of work on direct integration over the moduli space of instantons, Nekrasov [2] applied the localization theorem for toric actions to compute the equivariant integral over the instanton moduli space. Subsequently, it was noted by [3] that a similar approach also applies to the instanton calculus of $\mathcal{N} = 1$ supersymmetric gauge theory where $\mathcal{N} = 2$ theory is perturbed by a superpotential $W(\Phi)$. Here the integration is localized to the summation over the (isolated) fixed points labeled by the Young diagrams, which are in some sense regarded as saddle points. One might expect the instanton calculus on the saddle points has more applications than kinematical constraints.

The computation of the holomorphic quantities in $\mathcal{N} = 1$ theory such as effective superpotential is related to the matrix model in [4]. This proposal is derived by relating them to the B-model topological string amplitudes on certain noncompact Calabi-Yau manifolds. The relation is a mirror to a gauge/string duality due to the geometric transition on the A-model side [5]. Thus the reproduction of the holomorphic quantities in $\mathcal{N} = 1$ theory from the microscopic instanton calculus can be regarded as a check of the gauge/string correspondence involving the matrix model. For a recent progress in microscopic approach to $\mathcal{N} = 1$ supersymmetric gauge theory, see [6–8].

Though the localization formulas are given explicitly for both the partition function and the chiral correlation functions, in general it is not straightforward to perform the summation over the Young diagrams exactly. In the maximally confining phase of $U(N)$ supersymmetric gauge theory,[1] where $SU(N)$ is confined and the residual symmetry at low energy is the diagonal $U(1) \subset U(N)$, the task of summation gets considerably tractable. In [9], we adopted the standard correspondence between Young diagrams and neutral states in the fermion Fock space and calculated explicitly the chiral one-point functions $\langle \mathrm{Tr}\varphi^J \rangle$ in the maximally confining phase. We found that the results can be summarized compactly in terms of the loop operator:

$$\langle \mathrm{Tr} e^{u\varphi} \rangle = I_0\left(4\sqrt{q}\frac{\sinh(u\hbar/2)}{\hbar}\right) \to I_0(2\sqrt{q}u), \qquad (\hbar \to 0), \tag{1.1}$$

with $I_n(x)$ being the modified Bessel function. The vacuum expectation value of the loop operator has two parameters $q = \Lambda^{2N}$ and $\hbar$. The series expansion in q is the instanton expansion of gauge theory, while that in $\hbar$ is interpreted as the genus expansion of the corresponding string theory. In the gauge/string theory correspondence, the partition function and the correlation functions allow the double perturbative expansion in the instanton number of gauge theory and the genus of the string theory. Usually we can sum up the expansion only in one of the two expansion parameters, but we have to compute the functions order by order in the other parameter. It is very amusing that the above result (1.1) gives an example where we can perform the summation of the double expansion completely in a closed form.

The chiral one-point functions $\langle \mathrm{Tr}\varphi^J \rangle$ in $\mathcal{N} = 1$ theory do not detect the superpotential $W(\Phi)$ which is used to deform the original $\mathcal{N} = 2$ theory [3,9]. In this paper we

[1]We reduce the computation to the instanton calculus of $U(1)$ gauge theory. However, the existence of underlying $U(N)$ theory should be assumed, since we will consider the expansion in the second Chern number whose meaning is lost in a genuine $U(1)$ theory.

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would like to proceed to the computation of the chiral glueball one-point functions $\langle \mathrm{Tr}\lambda^\alpha\lambda_\alpha\varphi^J\rangle$, which depend on the details of $W(\Phi)$. The chiral glueball one-point functions were obtained previously by several methods. For example, one can relate the holomorphic (F-term) quantities in supersymmetric gauge theory to amplitudes of topological string theory [10] and then compute with matrix model using open/closed string duality [4]. We can also relate the correlation functions to the generalized Konishi anomaly and perform a purely field theoretical computation [11]. The resulting glueball resolvent is given by

$$R(z) := \left\langle \mathrm{Tr}\lambda^\alpha\lambda_\alpha \frac{1}{z-\varphi}\right\rangle = -W'(z) + \sqrt{W'(z)^2 - 4f(z)}, \tag{1.2}$$

with an unknown polynomial $f(z)$ which is determined so that the asymptotic behavior of $R(z)$ in $z \to \infty$ is $\langle \mathrm{Tr}\lambda^\alpha\lambda_\alpha\rangle \cdot z^{-1} + O(z^{-2})$. In the maximally confining phase, the polynomial $f(z)$ is further tuned so that the square root is factorized into

$$\sqrt{W'(z)^2 - 4f(z)} = H(z)\sqrt{(z-a)^2 - 4q}. \tag{1.3}$$

In the matrix model the maximally confining phase is described by a one-cut solution, while on the moduli space of Seiberg-Witten theory it corresponds to the degenerating loci where the maximal number of mutually nonintersecting cycles of the Seiberg-Witten curve collapse. In this paper we would like to add one more method to derive the glueball resolvent $R(z)$. We find our instanton calculus gives an exact agreement to the result (1.2) with the factorization (1.3). The reproduction of previously known results should serve as a nontrivial consistency check of the instanton calculus. Besides, along the way of computation we also find an explicit expression of the polynomial $f(z)$ that causes the complete factorization (1.3). It is interesting that our $U(1)$ instanton calculus automatically gives a closed form for the glueball resolvent in the maximally confining phase.

The plan of the present paper is as follows. We will shortly review some necessary ingredients of the instanton calculus in Sec. II. Before proceeding to our computation of the chiral glueball one-point functions in Sec. IV, we will first compute two-point functions of the loop operators as a preparation in Sec. III. Finally in Sec. V we comment on the higher genus correction to our computation.

II. REVIEW OF INSTANTON CALCULUS

Chiral operators $\mathcal{O}$ in supersymmetric field theories are, by definition, those annihilated by the fermionic charges $\bar{Q}_{\dot\alpha}$ of one chirality. Two chiral operators are defined to be equivalent, if the difference is $\bar{Q}_{\dot\alpha}$-exact. The set of chiral operators is closed under the multiplication and form a ring, which we call chiral ring. From the supersymmetry algebra in four dimensions, $\{Q_\alpha, \bar{Q}_{\dot\alpha}\} = \sigma^\mu_{\alpha\dot\alpha}P_\mu$, we can see that correlation functions of chiral operators are "topological" in the sense that they are independent of the positions of operators. Especially, topological one-point functions characterize the phase structure of vacua. In the four dimensional $\mathcal{N}=1$ supersymmetric gauge theory with a single adjoint matter Φ, we have a vector multiplet (A_μ, λ_α) and a chiral multiplet $\Phi = (\varphi, \psi_\alpha)$, which are all in the adjoint representation. One can show that the generators of the chiral ring are of the form $\mathrm{Tr}\varphi^J$, $\mathrm{Tr}\lambda_\alpha\varphi^J$, and $\mathrm{Tr}\lambda^\alpha\lambda_\alpha\varphi^J$ [11]. The correlation functions of the chiral operators $\mathcal{O}$ are defined by

$$\langle\mathcal{O}\rangle_{\mathcal{N}=1} := \frac{1}{VZ_{\mathcal{N}=1}}\int_{\mathcal{M}}\left\{\int_{\mathbb{C}^2}\mathcal{O}\right\}\exp(-S_{\mathcal{N}=1}), \tag{2.1}$$

where the action for $\mathcal{N}=1$ supersymmetric gauge theory $S_{\mathcal{N}=1}$ is that of $\mathcal{N}=2$ theory $S_{\mathcal{N}=2}$ perturbed by a superpotential $W(\Phi)$:

$$S_{\mathcal{N}=1} := S_{\mathcal{N}=2} + \int dx^4 d\theta^2 W(\Phi). \tag{2.2}$$

The correlator $\langle\mathcal{O}\rangle_{\mathcal{N}=1}$ is normalized by the volume V of the noncommutative $\mathbb{C}^2$ and the partition function $Z_{\mathcal{N}=1} := \int_{\mathcal{M}}\exp(-S_{\mathcal{N}=1})$. The integral is over the moduli space $\mathcal{M}$ of $U(N)$ instantons described, for example, by the ADHM (Atiyah-Drinfeld-Hitchin-Manin) construction. The moduli space has the decomposition $\mathcal{M} := \sqcup_k\mathcal{M}_{N,k}$, that is, the correlation function is a sum over the contributions from each moduli space $\mathcal{M}_{N,k}$ with a specific instanton number k. The correlation function $\langle\mathcal{O}\rangle_{\mathcal{N}=1}$ for $\mathcal{O} = \mathrm{Tr}\varphi^J$ or $\mathrm{Tr}\lambda^\alpha\lambda_\alpha\varphi^J$ is of our prime interest in this paper. We shall omit the subscripts $\mathcal{N}=1$ or $\mathcal{N}=2$ of the correlators hereafter as long as it is clear from the context.

The strategy to calculate these correlation functions is to first replace all the operators by their equivariant extensions [3]:

$$\mathrm{Tr}\,\varphi^J \mapsto \alpha_{(2,2)}\wedge\mathrm{Tr}\mathcal{F}^J, \tag{2.3}$$

$$\mathrm{Tr}\,\lambda^\alpha\lambda_\alpha\varphi^J \mapsto -\frac{1}{(J+2)(J+1)}\alpha_{(0,2)}\wedge\mathrm{Tr}\mathcal{F}^{J+2}, \tag{2.4}$$

$$d^2\theta W(\Phi) \mapsto \alpha_{(2,0)}\wedge\mathrm{Tr}W(\mathcal{F}), \tag{2.5}$$

with α being the equivariantly closed forms

$$\alpha_{(0,0)} := 1, \tag{2.6}$$

$$\alpha_{(2,0)} := dz^1\wedge dz^2 + i\hbar z^1z^2, \tag{2.7}$$

$$\alpha_{(0,2)} := d\bar{z}^1\wedge d\bar{z}^2 - i\hbar\bar{z}^1\bar{z}^2, \tag{2.8}$$

$$\alpha_{(2,2)} := \alpha_{(2,0)}\wedge\alpha_{(0,2)}. \tag{2.9}$$

The curvature $\mathcal{F}$ of the universal bundle over $\mathbb{C}^2\times\mathcal{M}_{N,k}$

is expanded according to the direct product structure of the base space as follows[2]:

$$\mathcal{F} =: F + \Psi + \varphi = F_{\mu\nu}dx^\mu dx^\nu + \{\lambda_m dz^m + \psi_{\bar{m}} d\bar{z}^{\bar{m}}\} + \{(d_{\mathcal{M}}U^\dagger)(d_{\mathcal{M}}U) - U^\dagger \mathcal{L}_\xi U\}. \tag{2.10}$$

If we untwist the theory,[3] the components $F_{\mu\nu}$ and λ_m are identified with the field strength and the gaugino, which gives a vector multiplet in $\mathcal{N} = 1$ theory. The remaining pair $\psi_{\bar{m}}$ and φ gives a chiral multiplet. Then due to the localization theorem, our computation reduces to picking up the value at the fixed points. Since the fixed points are classified by the Young diagrams, the result is given in terms of summation over the Young diagrams. Especially in the maximally confining phase, the correlation functions (2.1) simplifies into

$$\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \sum_{k=0}^{\infty} \left[\frac{q}{\hbar^2}\right]^k \sum_{|Y|=k} \frac{\mathcal{O}_Y}{\prod_{\square \in Y} (h(\square))^2}, \tag{2.11}$$

where the partition function is given by

$$\mathcal{Z} = \sum_{k=0}^{\infty} \left[\frac{q}{\hbar^2}\right]^k \sum_{|Y|=k} \frac{1}{\prod_{\square \in Y} (h(\square))^2}. \tag{2.12}$$

We have introduced the parameter q of the instanton expansion. $\hbar$ is the equivariant parameter associated with the $U(1)$ action $T_\hbar$: $(z_1, z_2) \to (e^{i\hbar} z_1, e^{-i\hbar} z_2)$ on $\mathbb{C}^2$. Note that the partition function $\mathcal{Z}$ depends on these parameters only through the combination $q/\hbar^2$. We denote by $|Y|$ the total number of boxes of a Young diagram Y. The hook length at a box $\square \in Y$ is denoted by $h(\square)$ and the weight $\prod_{\square \in Y} (h(\square))^{-2}$ is called the Plancherel measure. At each fixed point Y we can estimate the chiral operator $\mathcal{O}$ to obtain $\mathcal{O}_Y$. Thus $\mathcal{O}_Y$ is a function on the space of Young diagrams and $\langle \mathcal{O} \rangle$ is nothing but the integration of $\mathcal{O}_Y$ with respect to the Plancherel measure.[4]

For the loop operator $\mathcal{O} = \mathrm{Tr} e^{t\varphi}$, which is a generating function of $\mathrm{Tr}\varphi^J$, the function $\mathcal{O}_Y$ is given by

$$\mathrm{Ch}_Y(\alpha) = e^{ta}\left(1 + \mathrm{sh}^2(\alpha) \sum_{\square \in Y} e^{\alpha(c(\square))}\right), \tag{2.13}$$

where $a = \langle \mathrm{Tr}\varphi \rangle$, $\alpha = t\hbar$, and $\mathrm{sh}(\alpha) := e^{\alpha/2} - e^{-\alpha/2}$ [9]. For a box $\square$ at the mth row and the nth column of the Young diagram, we define the content by $c(\square) := n - m$. In [3] the glueball operator $\mathcal{O} = \mathrm{Tr}\lambda^\alpha \lambda_\alpha \varphi^J$ is related to the connected two-point function by

$$\langle \mathrm{Tr}\lambda^\alpha \lambda_\alpha \varphi^J \rangle_{\mathcal{N}=1} = -\frac{2}{(J+2)(J+1)\hbar^2} \times \langle \mathrm{Tr}W(\varphi)\mathrm{Tr}\varphi^{J+2} \rangle_{\mathrm{conn}}. \tag{2.14}$$

The relation (2.14) may be derived as follows[5]: The definition (2.1) implies that the $\mathcal{N} = 1$ correlators are related to the $\mathcal{N} = 2$ correlators by

$$\langle \mathcal{O} \rangle_{\mathcal{N}=1} = \langle \mathcal{O} \rangle_{\mathcal{N}=2} + \left\langle \mathcal{O} \int_{\mathbb{C}^2} \alpha_{(2,0)} \wedge \mathrm{Tr}W(\mathcal{F}) \right\rangle_{\mathcal{N}=2} + \cdots. \tag{2.15}$$

Hence we have

$$\left\langle \int_{\mathbb{C}^2} \alpha_{(0,2)} \wedge \mathrm{Tr}\mathcal{F}^{J+2} \right\rangle_{\mathcal{N}=1} = \hbar \bar{z}^1 \bar{z}^2 \langle \mathrm{Tr}\varphi^{J+2} \rangle + \hbar^2 z^1 z^2 \bar{z}^1 \bar{z}^2 \langle \mathrm{Tr}\varphi^{J+2}\, \mathrm{Tr}W(\varphi) \rangle + \cdots, \tag{2.16}$$

$$V\langle 1 \rangle_{\mathcal{N}=1} = \hbar^2 z^1 z^2 \bar{z}^1 \bar{z}^2 (\langle 1 \rangle + \hbar z^1 z^2 \langle \mathrm{Tr}W(\varphi) \rangle + \cdots). \tag{2.17}$$

Note that in the expansion the first term vanishes after setting $\bar{z}^1$, $\bar{z}^2 \to 0$, while the higher-order terms vanish because of z^1, $z^2 \to 0$. Summing up all the terms in (2.14) into the exponential function, we find that the glueball loop operator is given as

$$\langle \mathrm{Tr}\lambda^\alpha \lambda_\alpha e^{u\varphi} \rangle_{\mathcal{N}=1} = -\frac{2}{\hbar^2} \left\langle \mathrm{Tr}W(\varphi)\ \mathrm{Tr}\frac{e^{u\varphi} - 1 - u\varphi}{u^2} \right\rangle_{\mathrm{conn}}. \tag{2.18}$$

The more conventional quantity is the glueball resolvent, which is the Laplace transformation of the glueball loop operator

$$R(z) := \left\langle \mathrm{Tr}\lambda^\alpha \lambda_\alpha \frac{1}{z - \varphi} \right\rangle = \int_0^\infty du\, e^{-zu} \langle \mathrm{Tr}\lambda^\alpha \lambda_\alpha e^{u\varphi} \rangle. \tag{2.19}$$

The explicit form of the glueball resolvent is known to be

$$R(z) = -W'(z) + \sqrt{W'(z)^2 - 4f(z)}, \tag{2.20}$$

from the relation to the matrix model [4] or the generalized Konishi anomaly [11]. For the superpotential $W(z)$ of degree $n + 1$, we need to include a polynomial $f(z)$ of degree $n - 1$. Since we are considering the maximally confining phase, we have to choose the coefficients of $f(z)$ so that $\sqrt{W'(z)^2 - 4f(z)}$ is factorized into a product of a polynomial and the square root of a quadratic function. Therefore, the resolvent $R(z)$ in the maximally confining phase is characterized by the following conditions:

[2] The matrix U appears in the ADHM construction as the zero modes of the Dirac operator.

[3] Note that we consider the case $\mathbb{C}^2 \simeq \mathbb{R}^4$ in this paper. In general the topological twist of $\mathcal{N} = 1$ theory is possible on the Kähler manifold.

[4] The Plancherel measure can be regarded as a discretization of the Vandermonde measure. This interpretation suggests a natural connection to the matrix model.

[5] Two proofs of (2.14) have been presented recently in [6,8].

(i) The resolvent $R(z)$ is a linear combination of 1 and the square root of a quadratic function with the coefficients being polynomials of z.
(ii) The behavior of $z \to \infty$ is $\langle \mathrm{Tr}\lambda^{\alpha}\lambda_{\alpha}\rangle \cdot z^{-1} + O(z^{-2})$.
(iii) The coefficient of 1 is $-W'(z)$.

We would like to reproduce all these features from the following instanton calculus.

III. SCALAR TWO-POINT FUNCTIONS

In this section we compute the two-point correlation function of the loop operator:

$$\langle \mathrm{Tr}e^{t\varphi}\,\mathrm{Tr}e^{u\varphi}\rangle = \frac{1}{Z}\sum_{k=0}^{\infty}\left[\frac{q}{\hbar^2}\right]^k \sum_{|Y|=k}\frac{\mathrm{Ch}_Y(\alpha)\mathrm{Ch}_Y(\beta)}{\prod_{\square\in Y}(h(\square))^2}, \quad (3.1)$$

where

$$\mathrm{Ch}_Y(\alpha) = 1 + \mathrm{sh}^2(\alpha)\sum_{\square\in Y} e^{\alpha(c(\square))}, \quad (3.2)$$

with $\alpha = t\hbar$ and $\beta = u\hbar$. (We set $a = 0$ for simplicity in this section, but we can easily reproduce the contribution by multiplying the final result by the factors e^{ta} and e^{ua}.) Taking care of the constant term separately, we find

$$\langle \mathrm{Tr}e^{t\varphi}\,\mathrm{Tr}e^{u\varphi}\rangle = \mathcal{W} + \langle \mathrm{Tr}e^{t\varphi}\rangle + \langle \mathrm{Tr}e^{u\varphi}\rangle - 1, \quad (3.3)$$

where

$$\mathcal{W} := \frac{1}{Z}\sum_{k=0}^{\infty}\left[\frac{q}{\hbar^2}\right]^k \mathrm{sh}^2(\alpha)\mathrm{sh}^2(\beta) \times \sum_{|Y|=k}\frac{\sum_{\square\in Y} e^{\alpha(c(\square))}\sum_{\square\in Y} e^{\beta(c(\square))}}{\prod_{\square\in Y}(h(\square))^2}. \quad (3.4)$$

A. Operator formalism

It is well known that there is a correspondence between the Young diagrams and the fermion Fock states with neutral charge. By making use of the correspondence, we can compute the summations over the set of Young diagrams in operator formalism. Let us introduce a pair of charged (Neveu-Schwarz) free fermions

$$\psi(z) = \sum_{r\in\mathbb{Z}+1/2}\psi_r z^{-r-(1/2)}, \qquad \psi^*(z) = \sum_{s\in\mathbb{Z}+1/2}\psi^*_s z^{-s-(1/2)}, \quad (3.5)$$

with the anticommutation relation

$$\{\psi_r, \psi^*_s\} = \delta_{r+s,0}, \qquad r, s \in \mathbb{Z} + \tfrac{1}{2}. \quad (3.6)$$

We define the Fock vacuum $|0\rangle$ by

$$\psi_r|0\rangle = \psi^*_s|0\rangle = 0, \qquad r, s > 0. \quad (3.7)$$

Recall that the Young diagram is specified by a partition $\lambda = (\lambda_i)$, where λ_i is the length of the ith row. Then the corresponding state is given by

$$|\lambda\rangle = \prod_{i=1}^{\infty}\psi_{i-\lambda_i-(1/2)}||0\rangle\rangle, \quad (3.8)$$

with $\psi^*_s||0\rangle\rangle = 0$, $\forall\ s$. One can show that

$$J^k_{-1}|0\rangle = \sum_{|\lambda|=k}\frac{k!}{\prod_{\square\in\lambda}h(\square)}|\lambda\rangle, \quad (3.9)$$

where J_{-1} is the constant mode in the standard $U(1)$ current $J(z) :=: \psi(z)\psi^*(z):$. From (3.9) we can confirm the following famous relation

$$\sum_{|Y|=k}\prod_{\square\in Y}h(\square)^{-2} = \frac{1}{k!}, \quad (3.10)$$

which is used for computing the summation over the Young diagrams appearing in the partition function Z. We find (2.12) has a simple form:

$$Z = \exp\left(\frac{q}{\hbar^2}\right). \quad (3.11)$$

This is the most fundamental example of the computation in operator formalism of the summation with the Plancherel measure [12].

To compute the correlation functions, it is convenient to introduce the operator [13,14]

$$\mathcal{E}^{(\alpha)}(z) := :\psi(e^{\alpha/2}z)\psi^*(e^{-\alpha/2}z): = \sum_{n\in\mathbb{Z}}\mathcal{E}^{(\alpha)}_n z^{-n-1}, \quad (3.12)$$

which is a "point splitting" deformation of $J(z) = \mathcal{E}^{(0)}(z)$. The modes of $\mathcal{E}^{(\alpha)}(z)$

$$\mathcal{E}^{(\alpha)}_n := \sum_{r\in\mathbb{Z}+1/2} e^{\alpha(r-n/2)}:\psi_{n-r}\psi^*_r:, \quad (3.13)$$

satisfy the commutation relation,[6]

$$[\mathcal{E}^{(\alpha)}_n, \mathcal{E}^{(\beta)}_m] = \mathrm{sh}(n\beta - m\alpha)\mathcal{E}^{(\alpha+\beta)}_{n+m} + \delta_{n+m,0}\frac{\mathrm{sh}n(\alpha+\beta)}{\mathrm{sh}(\alpha+\beta)}. \quad (3.14)$$

We will use the following important relation in the computation of correlation functions:

[6] The infinite-dimensional Lie algebra with the commutation relation (3.14) appeared first in [15], where it was called area-preserving torus diffeomorphism algebra. The representation theory was initiated in [16] and the algebra was identified with $W_{1+\infty}$ algebra. Recently the same algebra is used to reveal a connection of the counting of plane partitions and the Toda hierarchy [17].

$$\mathcal{E}_0^{(\alpha)}|\lambda\rangle = \sum_{i=1}^{\infty}(e^{\alpha(\lambda_i - i + 1/2)} - e^{\alpha(-i+1/2)})|\lambda\rangle = \mathrm{sh}(\alpha)\sum_{\square\in\lambda}\exp(\alpha c(\square))|\lambda\rangle. \tag{3.15}$$

The second term comes from $\mathcal{E}_0^{(\alpha)}||0\rangle\rangle = -(\mathrm{sh}(\alpha))^{-1}||0\rangle\rangle$, which can be calculated directly from the definition of $||0\rangle\rangle$ or from the consistency $\mathcal{E}_0^{(\alpha)}|0\rangle = 0$.

B. Computation

After the preparation of the operator formalism, let us proceed to the explicit calculation of $\mathcal{W}$ defined by (3.4). First of all, note that the formulas in the operator formalism imply

$$\langle 0|J_1^k \mathcal{E}_0^{(\alpha)}\mathcal{E}_0^{(\beta)}J_{-1}^k|0\rangle = (k!)^2\mathrm{sh}(\alpha)\mathrm{sh}(\beta) \times \sum_{|Y|=k}\frac{\sum_{\square\in Y}e^{\alpha(c(\square))}\sum_{\square\in Y}e^{\beta(c(\square))}}{\prod_{\square\in Y}(h(\square))^2}. \tag{3.16}$$

Thus the problem of evaluation of $\mathcal{W}$ reduces to the computation of $\langle 0|J_1^k\mathcal{E}_0^{(\alpha)}\mathcal{E}_0^{(\beta)}J_{-1}^k|0\rangle$. With the help of the commutation relation (3.14) with $J_1 = \mathcal{E}_1^{(0)}$, we obtain

$$\langle 0|J_1^k\mathcal{E}_0^{(\alpha)}\mathcal{E}_0^{(\beta)}J_{-1}^k|0\rangle = \sum_{l=0}^{k}\sum_{m=0}^{k-l}\frac{(k!)^2\mathrm{sh}^l(\alpha)\mathrm{sh}^m(\beta)}{l!m!(k-l-m)!(l+m)!} \times \langle 0|\mathcal{E}_l^{(\alpha)}\mathcal{E}_m^{(\beta)}J_{-1}^{l+m}|0\rangle, \tag{3.17}$$

where we have used

$$J_1^{k-l-m}J_{-1}^k|0\rangle = \frac{k!}{(l+m)!}J_{-1}^{l+m}|0\rangle. \tag{3.18}$$

Bringing J_{-1}'s to the most left in $\langle 0|\cdots|0\rangle$ by iteratively using the commutation relation

$$[\mathcal{E}_l^{(\alpha)}\mathcal{E}_m^{(\beta)}, J_{-1}] = \mathcal{E}_l^{(\alpha)}(\mathrm{sh}(\beta)\mathcal{E}_{m-1}^{(\beta)} + \delta_{m,1}) + (\mathrm{sh}(\alpha)\mathcal{E}_{l-1}^{(\alpha)} + \delta_{l,1})\mathcal{E}_m^{(\beta)}, \tag{3.19}$$

we are left with a constant term and terms such as $\langle 0|\mathcal{E}_n^{(\alpha)}\mathcal{E}_{-n}^{(\beta)}|0\rangle$. The coefficients of these terms can be calculated as follows. We first place the operator $\mathcal{E}_l^{(\alpha)}\mathcal{E}_m^{(\beta)}$ at the point (l, m) in the two-dimensional plane. (See Fig. 1.) Using the commutation relation (3.19) once means that we bring the operator downwards or leftwards by one unit in the plane. After $(l+m)$ times of the iterative moves, we finally arrive at the integer point $(n, -n)$. The coefficient of the $\langle 0|\mathcal{E}_n^{(\alpha)}\mathcal{E}_{-n}^{(\beta)}|0\rangle$ term (or the constant term) can be interpreted as the combinatorial factor of moving from (l, m) to $(n, -n)$ [or $(0, 0)$, respectively] by these iterative moves. In this way we find

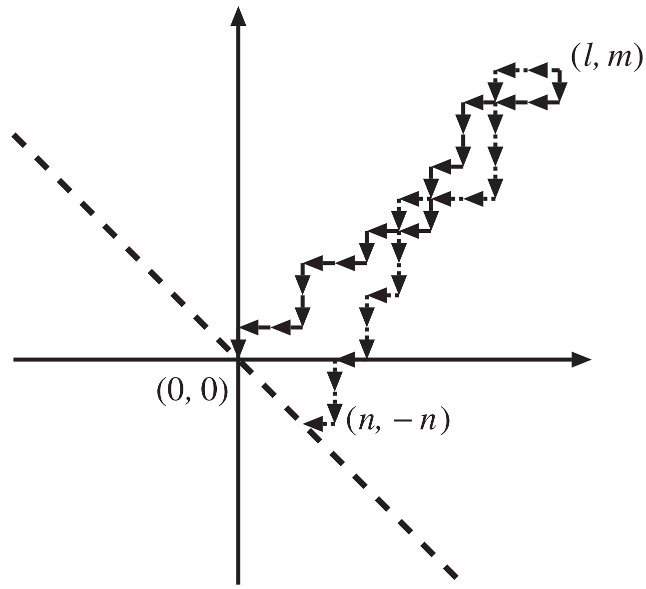

FIG. 1. Combinatorics of the coefficients.

$$\frac{\langle 0|\mathcal{E}_l^{(\alpha)}\mathcal{E}_m^{(\beta)}J_{-1}^{l+m}|0\rangle}{(l+m)!} = \sum_{n=0}^{l}\frac{[\mathrm{sh}(\alpha)]^{l-n}[\mathrm{sh}(\beta)]^{m+n}}{(l-n)!(m+n)!} \times \frac{\mathrm{sh}n(\alpha+\beta)}{\mathrm{sh}(\alpha+\beta)} + \frac{[\mathrm{sh}(\alpha)]^{l-1}[\mathrm{sh}(\beta)]^{m-1}}{l!m!}\Theta_{l>0}\Theta_{m>0}. \tag{3.20}$$

Note that the second term, which comes from the constant term in the commutation relation (3.19), contributes only when $l \neq 0$ and $m \neq 0$. Plugging back into (3.4), we call each contribution from the above two terms as $\mathcal{W}_1$ and $\mathcal{W}_2$, respectively.

Let us concentrate on $\mathcal{W}_2$ first:

$$\mathcal{W}_2 = \frac{1}{Z}\sum_{k=0}^{\infty}\sum_{l=0}^{k}\sum_{m=0}^{k-l}\left[\frac{q}{\hbar^2}\right]^k \times \frac{[\mathrm{sh}(\alpha)]^{2l}[\mathrm{sh}(\beta)]^{2m}}{(l!)^2(m!)^2(k-l-m)!}\Theta_{l>0}\Theta_{m>0}. \tag{3.21}$$

We can exchange the order of summation by bringing the k-summation into the most right

$$\sum_{k=0}^{\infty}\sum_{l=0}^{k}\sum_{m=0}^{k-l}\cdots = \sum_{l=0}^{\infty}\sum_{k=l}^{\infty}\sum_{m=0}^{k-l}\cdots = \sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=l+m}^{\infty}\cdots. \tag{3.22}$$

After the exchange of summation, we find $\mathcal{W}_2$ is computed to be

$$\mathcal{W}_2 = (I_0(A) - 1)(I_0(B) - 1), \tag{3.23}$$

with $A = 2\sqrt{q}\mathrm{sh}(\alpha)/\hbar$, $B = 2\sqrt{q}\mathrm{sh}(\beta)/\hbar$ and $I_n(x)$ is the nth modified Bessel function.

Now let us turn to $\mathcal{W}_1$:

$$\mathcal{W}_1 = \frac{1}{Z}\sum_{k=0}^{\infty}\sum_{l=0}^{k}\sum_{m=0}^{k-l}\sum_{n=0}^{l}\left[\frac{q}{\hbar^2}\right]^k \times \frac{[\mathrm{sh}(\alpha)]^{2l-n+1}[\mathrm{sh}(\beta)]^{2m+n+1}}{l!(l-n)!m!(m+n)!(k-l-m)!} \times \frac{\mathrm{sh}n(\alpha+\beta)}{\mathrm{sh}(\alpha+\beta)}. \tag{3.24}$$

Here the exchange of summation goes as

$$\sum_{k=0}^{\infty}\sum_{l=0}^{k}\sum_{m=0}^{k-l}\sum_{n=0}^{l}\cdots = \sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=l+m}^{\infty}\sum_{n=0}^{l}\cdots = \sum_{l=0}^{\infty}\sum_{n=0}^{l}\sum_{m=0}^{\infty}\sum_{k=l+m}^{\infty}\cdots = \sum_{n=0}^{\infty}\sum_{l=n}^{\infty}\sum_{m=0}^{\infty}\sum_{k=l+m}^{\infty}\cdots, \tag{3.25}$$

where we have used (3.22) in the first equality and then brought the n-summation to the most left. Performing the k-summation, we have

$$\mathcal{W}_1 = \sum_{n=0}^{\infty}\frac{\mathrm{sh}n(\alpha+\beta)}{\mathrm{sh}(\alpha+\beta)}\left(\sum_{l=n}^{\infty}\left[\frac{q}{\hbar^2}\right]^l\frac{[\mathrm{sh}(\alpha)]^{2l-n+1}}{l!(l-n)!}\right) \times\left(\sum_{m=0}^{\infty}\left[\frac{q}{\hbar^2}\right]^m\frac{[\mathrm{sh}(\beta)]^{2m+n+1}}{m!(m+n)!}\right), \tag{3.26}$$

which implies

$$\mathcal{W}_1 = \mathrm{sh}(\alpha)\mathrm{sh}(\beta)\sum_{n=0}^{\infty}I_n(A)I_n(B)\frac{\mathrm{sh}n(\alpha+\beta)}{\mathrm{sh}(\alpha+\beta)}. \tag{3.27}$$

Plugging back into (3.4) all our results including the one-point function $\langle \mathrm{Tr}e^{t\varphi}\rangle = I_0(A)$, we finally find that the two-point function is given by

$$\langle \mathrm{Tr}e^{t\varphi}\,\mathrm{Tr}e^{u\varphi}\rangle = I_0(A)I_0(B) + \mathrm{sh}(\alpha)\mathrm{sh}(\beta)\sum_{n=0}^{\infty}I_n(A)I_n(B) \times\frac{\mathrm{sh}n(\alpha+\beta)}{\mathrm{sh}(\alpha+\beta)}. \tag{3.28}$$

Note that the first term can be interpreted as the disconnected contribution. Hence,

$$\langle \mathrm{Tr}e^{t\varphi}\,\mathrm{Tr}e^{u\varphi}\rangle_{\mathrm{conn}} := \langle \mathrm{Tr}e^{t\varphi}\,\mathrm{Tr}e^{u\varphi}\rangle - \langle \mathrm{Tr}e^{t\varphi}\rangle\langle \mathrm{Tr}e^{u\varphi}\rangle = \mathrm{sh}(\alpha)\mathrm{sh}(\beta)\sum_{n=0}^{\infty}I_n(A)I_n(B)\frac{\mathrm{sh}n(\alpha+\beta)}{\mathrm{sh}(\alpha+\beta)}. \tag{3.29}$$

IV. GLUEBALL ONE-POINT FUNCTIONS

Having finished our computation of the scalar two-point function in the previous section, let us turn to the glueball one-point function. As (2.18) implies, all the information we need is encoded in the scalar two-point function (3.29) in the limit $\hbar \to 0$:

$$\lim_{\hbar\to 0}\frac{1}{\hbar^2}\langle \mathrm{Tr}e^{t\varphi}\,\mathrm{Tr}e^{u\varphi}\rangle_{\mathrm{conn}} = e^{ta}e^{ua}tu\sum_{n=0}^{\infty}nI_n(2\sqrt{q}t)I_n(2\sqrt{q}u), \tag{4.1}$$

where we have restored the classical value $a = \langle \mathrm{Tr}\varphi\rangle$ by simply multiplying the factors e^{ta} and e^{ua}. Using this result, we find only the exponential term in (2.18) gives a nontrivial contribution:

$$\langle \mathrm{Tr}\lambda^{\alpha}\lambda_{\alpha}e^{u\varphi}\rangle = \lim_{\hbar\to 0} -\frac{2}{\hbar^2u^2}\langle \mathrm{Tr}W(\varphi)\,\mathrm{Tr}e^{u\varphi}\rangle_{\mathrm{conn}}. \tag{4.2}$$

Then, all we have to do is to pick up necessary terms out of (4.1) and perform the Laplace transformation to obtain the glueball resolvent.

A. An example

Before proceeding to the general superpotential, let us consider a simple example

$$W(z) = \frac{z^4}{4}, \tag{4.3}$$

which allows an explicit computation and is instructive for general cases. Since the connected two-point function is given by

$$\lim_{\hbar\to 0}\frac{1}{\hbar^2}\langle \mathrm{Tr}e^{t\varphi}\,\mathrm{Tr}e^{u\varphi}\rangle_{\mathrm{conn}} = e^{ta}e^{ua}tu(I_1(2\sqrt{q}t)I_1(2\sqrt{q}u) + 2I_2(2\sqrt{q}t)I_2(2\sqrt{q}u) + 3I_3(2\sqrt{q}t)I_3(2\sqrt{q}u) + \cdots), \tag{4.4}$$

with the modified Bessel functions being

$$I_1(z) = \frac{(z/2)}{0!1!} + \frac{(z/2)^3}{1!2!} + \cdots, \qquad I_2(z) = \frac{(z/2)^2}{0!2!} + \cdots, \qquad I_3(z) = \frac{(z/2)^3}{0!3!} + \cdots, \tag{4.5}$$

we find

$$\lim_{\hbar\to 0}\frac{1}{\hbar^2}\left\langle \mathrm{Tr}\frac{\varphi^4}{4!}\,\mathrm{Tr}e^{u\varphi}\right\rangle_{\mathrm{conn}} = \frac{\sqrt{q}^3}{0!}\left[\frac{1}{0!3!}3uI_3(2\sqrt{q}u)e^{ua} + \frac{1}{1!2!}uI_1(2\sqrt{q}u)e^{ua}\right] + \frac{\sqrt{q}^2a}{1!}\left[\frac{1}{0!2!}2uI_2(2\sqrt{q}u)e^{ua}\right] + \frac{\sqrt{q}a^2}{2!}\left[\frac{1}{0!1!}uI_1(2\sqrt{q}u)e^{ua}\right], \tag{4.6}$$

by picking up the t^4 terms. Therefore, the one-point function of the glueball loop operator is given as

$$\langle \mathrm{Tr}\lambda^\alpha\lambda_\alpha e^{u\varphi}\rangle = -2\sqrt{q}^3\frac{3!}{0!3!}\left[\frac{3!}{0!3!}\frac{3}{u}I_3(2\sqrt{q}u)e^{ua} + \frac{3!}{1!2!}\frac{1}{u}I_1(2\sqrt{q}u)e^{ua}\right] - 2\sqrt{q}^2a\frac{3!}{1!2!}\left[\frac{2!}{0!2!}\frac{2}{u}I_2(2\sqrt{q}u)e^{ua}\right] - 2\sqrt{q}a^2\frac{3!}{2!1!}\left[\frac{1!}{0!1!}\frac{1}{u}I_1(2\sqrt{q}u)e^{ua}\right], \tag{4.7}$$

which is transformed into the glueball resolvent through the Laplace transformation. The Laplace transformation can be performed as

$$\int_0^\infty due^{-(z-a)u}\frac{n}{u}I_n(2\sqrt{q}u) = Z^n, \tag{4.8}$$

with Z (and $\bar{Z}$ which will appear later) defined by

$$Z = \frac{z-a-\sqrt{(z-a)^2-4q}}{2\sqrt{q}}, \qquad \bar{Z} = \frac{z-a+\sqrt{(z-a)^2-4q}}{2\sqrt{q}}. \tag{4.9}$$

Here we have used the Laplace transformation formula for the modified Bessel function

$$\int_0^\infty dte^{-st}I_n(\omega t) = \frac{(s-\sqrt{s^2-\omega^2})^n}{\sqrt{s^2-\omega^2}\omega^n}, \tag{4.10}$$

and the recursive relation of the modified Bessel function

$$\frac{2n}{x}I_n(x) = I_{n-1}(x) - I_{n+1}(x). \tag{4.11}$$

Using (4.8) we easily find

$$\left\langle \mathrm{Tr}\lambda^\alpha\lambda_\alpha\frac{1}{z-\varphi}\right\rangle = -2\sqrt{q}^3\frac{3!}{0!3!}\left[\frac{3!}{0!3!}Z^3 + \frac{3!}{1!2!}Z\right] - 2\sqrt{q}^2a\frac{3!}{1!2!}\left[\frac{2!}{0!2!}Z^2\right] - 2\sqrt{q}a^2\frac{3!}{2!1!}\left[\frac{1!}{0!1!}Z\right]. \tag{4.12}$$

It is remarkable that the Laplace transform of the modified Bessel function gives a linear combination of 1 and $\sqrt{(z-a)^2-4q}$ with the coefficient being the polynomials of z, which is required for the resolvent in the maximally confining phase. Furthermore, it behaves as $O(z^{-1})$ in the limit $z \to \infty$, as can be seen from

$$Z = \bar{Z}^{-1}. \tag{4.13}$$

Hence, in the final step of matching our computation to the expected result, all we have to do is to prove that the coefficient polynomial of 1 is $-z^3$. This can be done without explicit calculation. Let us first rewrite our final result (4.12) as

$$-V(z) + H(z)\sqrt{(z-a)^2-4q} = -2\sqrt{q}^3\frac{3!}{0!3!}\left[\frac{3!}{0!3!}Z^3 + \frac{3!}{1!2!}Z\right] - 2\sqrt{q}^2a\frac{3!}{1!2!}\left[\frac{2!}{0!2!}Z^2\right] - 2\sqrt{q}a^2\frac{3!}{2!1!}\left[\frac{1!}{0!1!}Z\right]. \tag{4.14}$$

Then, our task is to prove $V(z) = z^3$. To pick up $V(z)$ let us consider the "conjugate" of (4.14)

$$-V(z) - H(z)\sqrt{(z-a)^2-4q} = -2\sqrt{q}^3\frac{3!}{0!3!}\left[\frac{3!}{0!3!}\bar{Z}^3 + \frac{3!}{1!2!}\bar{Z}\right] - 2\sqrt{q}^2a\frac{3!}{1!2!}\left[\frac{2!}{0!2!}\bar{Z}^2\right] - 2\sqrt{q}a^2\frac{3!}{2!1!}\left[\frac{1!}{0!1!}\bar{Z}\right], \tag{4.15}$$

and add up with original (4.14). Then we can sum up the right-hand side into

$$-2V(z) = -2\sqrt{q}^3\frac{3!}{0!3!}[Z+\bar{Z}]^3 - 2\sqrt{q}^2a\frac{3!}{1!2!}\left([Z+\bar{Z}]^2 - \frac{2!}{1!1!}\right) - 2\sqrt{q}a^2\frac{3!}{2!1!}[Z+\bar{Z}], \tag{4.16}$$

because of (4.13). We can further rewrite the result into

$$-2V(z) = -2\sqrt{q}^3\left[Z+\bar{Z}+\frac{a}{\sqrt{q}}\right]^3 + 2F_4, \tag{4.17}$$

with F_4 defined as

$$F_4 = \sqrt{q}^3\left(\frac{3!}{1!2!}\frac{2!}{1!1!}\frac{a}{\sqrt{q}} + \frac{3!}{3!0!}\frac{0!}{0!0!}\left[\frac{a}{\sqrt{q}}\right]^3\right). \tag{4.18}$$

Using

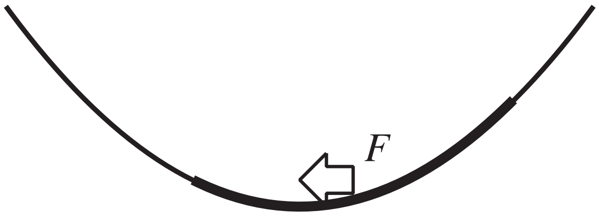

FIG. 2. Force felt by the cut.

$$Z + \bar{Z} + \frac{a}{\sqrt{q}} = \frac{z}{\sqrt{q}}, \tag{4.19}$$

we find (4.17) reduces to

$$V(z) = z^3 - F_4. \tag{4.20}$$

To finish our proof of $V(z) = z^3$ we need to choose $a = 0$ so that F_4 vanishes.

Note that since $V(z)$ is the first derivative of the potential (in the matrix model terminology), we can interpret the constant part F_4 as the force felt by the cut. (See Fig. 2.) Here the main contribution of F_4 is $F_4 \sim a^3$ for large a and the subleading term depends on the width of the cut $\sqrt{q}$.

B. Monomial superpotentials

The above argument can be easily generalized to any monomial superpotential. To compute the glueball one-point function for a superpotential

$$W(z) = \frac{z^{k+1}}{k+1}, \tag{4.21}$$

we have to pick up the t^{k+1}-terms in the right-hand side of (4.1):

$$\langle \mathrm{Tr}\lambda^\alpha \lambda_\alpha e^{u\varphi} \rangle = -2e^{ua} \sum_{l=0}^{k-1} \sqrt{q}^{k-l} a^l \binom{k}{l} \sum_{m=0}^{[(k-l-1)/2]} \binom{k-l}{m} \times \frac{k-l-2m}{u} I_{k-l-2m}(2\sqrt{q}u). \tag{4.22}$$

Using the Laplace transformation formula (4.8), we find the resolvent is given by

$$\left\langle \mathrm{Tr}\lambda^\alpha \lambda_\alpha \frac{1}{z-\varphi} \right\rangle = -2 \sum_{l=0}^{k-1} \sqrt{q}^{k-l} a^l \binom{k}{l} \times \sum_{m=0}^{[(k-l-1)/2]} \binom{k-l}{m} Z^{k-l-2m}. \tag{4.23}$$

Note again that the result takes the form of $-V(z) + H(z)\sqrt{z^2 - 4q}$. To see the result is the expected one, we have to show $V(z) = z^k$. This can be done by adding the conjugate. In this way, we will complete "half" of the binomial expansion in (4.23) into a full one except constant terms. Finally, we find $V(z) = z^k - F_{k+1}$ with the force F_{k+1} given by

$$F_{k+1} = \sum_{l=0}^{[k/2]} \sqrt{q}^{2l} a^{k-2l} \frac{k!}{(k-2l)!(l!)^2}. \tag{4.24}$$

If the potential is even, the force is odd and we can always make $F_{k+1} = 0$ by setting $a = 0$. This means that in the even potential we can stabilize the cut at the center. However, this does not work for the odd monomial potentials.

C. General superpotentials

If the superpotential is $W_2(z) = z^2/2$, the glueball resolvent is given as

$$\left\langle \mathrm{Tr}\lambda^\alpha \lambda_\alpha \frac{1}{z-\varphi} \right\rangle = -2\sqrt{q} \frac{1!}{0!1!} \left[\frac{1!}{0!1!} Z \right], \tag{4.25}$$

and the force felt by the cut is given as

$$F_2 = \sqrt{q} \left(\frac{1!}{1!0!} \frac{0!}{0!0!} \frac{a}{\sqrt{q}} \right), \tag{4.26}$$

while if the potential is $W_3(z) = z^3/3$, the glueball resolvent is

$$\left\langle \mathrm{Tr}\lambda^\alpha \lambda_\alpha \frac{1}{z-\varphi} \right\rangle = -2\sqrt{q}^2 \frac{2!}{0!2!} \left[\frac{2!}{0!2!} Z^2 \right] - 2\sqrt{q} a \frac{2!}{1!1!} \left[\frac{1!}{0!1!} Z \right], \tag{4.27}$$

and the force is

$$F_3 = \sqrt{q}^2 \left(\frac{2!}{0!2!} \frac{2!}{1!1!} + \frac{2!}{2!0!} \frac{0!}{0!0!} \left[\frac{a}{\sqrt{q}} \right]^2 \right). \tag{4.28}$$

Hence for the potential of the linear combination

$$V = \frac{m}{2}\varphi^2 + \frac{g}{3}\varphi^3, \tag{4.29}$$

we have to impose the force balance condition:

$$mF_2 + gF_3 = m\sqrt{q} \left(\frac{1!}{1!0!} \frac{0!}{0!0!} \frac{a}{\sqrt{q}} \right) + g\sqrt{q}^2 \left(\frac{2!}{0!2!} \frac{2!}{1!1!} + \frac{2!}{2!0!} \frac{0!}{0!0!} \left[\frac{a}{\sqrt{q}} \right]^2 \right) = 0. \tag{4.30}$$

As is clear from the above example the linearity holds for any superpotential. Therefore, as long as we are careful with the force balance condition, we can reproduce the results of all the superpotentials in the maximally confining phase.

V. HIGHER GENUS CORRECTION

So far we have computed the classical limit of the glueball one-point function by instanton calculus and found a complete agreement with the matrix model result.

In the computation we have to introduce an equivariant parameter $\hbar$ for the toric action on $\mathbb{C}^2$, which can be related to the (local) $SO(4)$ rotations of the Ω background [2].

Note that the $\hbar$-expansion of the scalar one-point and two-point functions computed in this paper agrees with the genus expansion of the Gromov-Witten theory of $\mathbf{P}^1$ developed in [13,14]. The correspondence is described as follows: Okounkov and Pandharipande computed all genus correlation functions of the Kähler class ω and its descendents $\tau_p(\omega)$ (the so-called stable sector) with respect to the two ramification data at the north and the south poles of $\mathbf{P}^1$, which are labeled by the partitions. To obtain the gauge theory correlation functions we identify the operator $\mathrm{Tr}\varphi^{2j}$ as the cohomology class $\tau_p(\omega)$. Then, the k-instanton sector is recovered by taking both the ramification data to be (1^k).

On the other hand, the same glueball one-point function was evaluated in [4] by relating the holomorphic quantities in supersymmetric gauge theory to amplitudes of topological string theory [10] and computing with the matrix model using open/closed string duality. Here the loop expansion parameter of the matrix model was interpreted as the genus expansion parameter of topological string theory, which, on the gauge theory side, was identified with a constant graviphoton background. A comparison between instanton calculus and matrix model at higher genus through topological string amplitudes was made in [18,19].

Although it was pointed out in [20] that the Ω background and the graviphoton background are different,[7] here we would like to compute the first order correction to the one-point function on both backgrounds and compare with each other, because these two backgrounds look similar and share the interpretation of genus expansion. We shall start with computing the $\hbar$-correction of the glueball one-point function from the instanton calculus in Sec. V A and proceed to recapitulating the matrix model computation in Sec. V B. Comparing the result of the glueball one-point function from the instanton calculus (5.4) with the result of the one-point function in the matrix model (5.11) or (5.14), we find a discrepancy. Though we cannot find any way to relate the two results, we still expect there to be a relation between these two genus expansions, for example, by change of variables.

A. $\hbar$-corrections of the glueball one-point function

Let us study the $\hbar$-corrections of the glueball one-point function. We shall choose the simplest Gaussian superpotential here.

Since the connected two-point function is given as

$$\left\langle \mathrm{Tr}\frac{\varphi^2}{2}\,\mathrm{Tr}e^{u\varphi}\right\rangle = e^{ua}\hbar\sqrt{q}\,\mathrm{sh}(u\hbar)I_1\left(2\sqrt{q}\frac{\mathrm{sh}(u\hbar)}{\hbar}\right), \quad (5.1)$$

[7] The relation between these two backgrounds was also discussed in [21].

we find the $\hbar$-corrections of the glueball one-point function is

$$\begin{aligned}\langle \mathrm{Tr}\lambda^\alpha\lambda_\alpha e^{u\varphi}\rangle &= -2\sqrt{q}\left(1+\frac{(u\hbar)^2}{24}\right) \\ &\quad\times\frac{1}{u}I_1\left(2\sqrt{q}u\left(1+\frac{(u\hbar)^2}{24}\right)\right)e^{u\varphi} \\ &= -2\sqrt{q}\left\{\frac{1}{u}I_1(2\sqrt{q}u)e^{u\varphi}\right. \\ &\quad\left.+\frac{(u\hbar)^2}{24}2\sqrt{q}I_0(2\sqrt{q}u)e^{u\varphi}\right\},\end{aligned} \quad (5.2)$$

where we have used identities of the modified Bessel function

$$\begin{aligned}I_{n-1}(x)-I_{n+1}(x)&=\frac{2n}{x}I_n(x),\\ I_{n-1}(x)+I_{n+1}(x)&=2\frac{d}{dx}I_n(x).\end{aligned} \quad (5.3)$$

Therefore, the glueball resolvent is

$$\begin{aligned}\left\langle \mathrm{Tr}\lambda^\alpha\lambda_\alpha\frac{1}{z-\varphi}\right\rangle &= -2\sqrt{q}\left\{\frac{z-a-\sqrt{(z-a)^2-4q}}{2\sqrt{q}}\right. \\ &\quad\left.+\frac{\hbar^2}{24}2\sqrt{q}\frac{2(z-a)^2+4q}{[(z-a)^2-4q]^{5/2}}\right\}.\end{aligned} \quad (5.4)$$

Note that the following formula holds for the Laplace transformation

$$\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s), \quad (5.5)$$

if we denote the Laplace transformation as $\mathcal{L}\{f(t)\} = F(s)$.

B. Matrix model computation

The computation on the matrix model side can be performed with several methods. The first method is due to loop equation: The resolvent in the matrix model is determined by the loop equation.

We shall consider the matrix model with potential $W(M)$:

$$Z = \int \mathcal{D}M \exp(-N\,\mathrm{Tr}W(M)). \quad (5.6)$$

The loop equation for the resolvent $R(z) = \langle N^{-1}\mathrm{Tr}(z-M)^{-1}\rangle$,

$$\int\frac{dy}{2\pi i}\frac{W'(y)}{z-y}R(y) = R(z)^2 + \frac{1}{N^2}\frac{\delta}{\delta W(z)}R(z), \quad (5.7)$$

(see (13.52) in [22]) gives

$$\int\frac{dy}{2\pi i}\frac{W'(y)}{z-y}R_1(y) = 2R_0(z)R_1(z) + \frac{\delta}{\delta W(z)}R_0(z), \quad (5.8)$$

at one loop. For the Gaussian model $W'(y) = \mu y$, the one-loop contribution is only nonvanishing for $\langle \mathrm{Tr}\varphi^n \rangle$ with $n \geq 4$. Hence, the resolvent for $z \to \infty$ should behave as $R_1(z) \sim 1/z^5$. This means there is no pole at $y = \infty$ and we can perform the y integration easily by picking up the pole at $y = z$:

$$(2R_0(z) - W'(z))R_1(z) = -\frac{\delta}{\delta W(z)} R_0(z). \tag{5.9}$$

The right-hand side is a two-point loop correlator and given by [23]

$$\frac{\delta}{\delta W(z)} R_0(z) = R_0(z, z) = \frac{1}{\mu(z^2 - 4/\mu)^2}. \tag{5.10}$$

Therefore, we find

$$R_1(z) = \frac{1}{\mu^2(z^2 - 4/\mu)^{5/2}}. \tag{5.11}$$

For the case of the Gaussian matrix model, an exact manipulation is possible. The exact solution for the Gaussian matrix model

$$\left\langle \frac{1}{N} \mathrm{Tr} e^{uM} \right\rangle = \frac{1}{Z} \int \mathcal{D}M \frac{1}{N} \mathrm{Tr} e^{uM} \exp\left(-\frac{N\mu}{2} \mathrm{Tr} M^2 \right), \tag{5.12}$$

is given by [24]

$$\left\langle \frac{1}{N} \mathrm{Tr} e^{uM} \right\rangle = \frac{\sqrt{\mu}}{u} I_1(2u/\sqrt{\mu}) + \frac{u^2}{12N^2\mu} I_2(2u/\sqrt{\mu}), \tag{5.13}$$

up to genus one. After the Laplace transformation we find

$$\left\langle \frac{1}{N} \mathrm{Tr} \frac{1}{z - M} \right\rangle = \frac{\mu}{2}(z - \sqrt{z^2 - 4/\mu}) + \frac{1}{N^2\mu^2} \frac{1}{(z^2 - 4/\mu)^{5/2}}. \tag{5.14}$$

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