

# Quantum limits of measurements induced by multiplicative conservation laws: Extension of the Wigner-Araki-Yanase theorem

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The Wigner-Araki-Yanase (WAY) theorem shows that additive conservation laws limit the accuracy of measurements. Recently, various quantitative expressions have been found for quantum limits on measurements induced by additive conservation laws, and have been applied to the study of fundamental limits on quantum-information processing. Here, we investigate generalizations of the WAY theorem to multiplicative conservation laws. The WAY theorem is extended to show that an observable not commuting with the modulus of, or equivalently the square of, a multiplicatively conserved quantity cannot be precisely measured. We also obtain a lower bound for the mean-square noise of a measurement in the presence of a multiplicatively conserved quantity. To overcome this noise it is necessary to make large the coefficient of variation (the so-called relative fluctuation), instead of the variance as is the case for additive conservation laws, of the conserved quantity in the apparatus.

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## I. INTRODUCTION

In recent investigations [1–9] it has been established that conservation laws put a precision limit or compensating resource requirements on quantum-information processing. Quantitative analysis suggests that conservation laws lead to undesirable entanglement of the object system with the control system, such as atom qubits controlled by the Jaynes-Cummings interaction with an electromagnetic field, causing decoherence of the object system, even if the environment induced decoherence is completely suppressed. Such limitations would disappear if the control system were considered to be macroscopic, so that this effect is of a quantum nature of the control system.

This sort of quantum limit has, however, long been known for measurements. Wigner [10] first claimed in 1952 that an observable which does not commute with an additively conserved quantity cannot be measured precisely. In 1960, Araki and Yanase [11] rigorously proved the impossibility of non-destructive [44] and precise measurement of a discrete observable not commuting with a bounded additively conserved quantity such as angular momentum; the result has been called the Wigner-Araki-Yanase (WAY) theorem. Subsequently, Yanase [12] found a bound for the accuracy of spin measurement and concluded that in order to increase the accuracy one needs to use a very large measuring apparatus; see also Wigner [13] and Ghirardi, Miglietta, Rimini, and Weber [14,15].

In order to extend the WAY theorem to continuous observables, one of the present authors [16,17] introduced a quantitative approach using commutation relations of noise operators. He showed the impossibility of nondestructive and

precise measurement of an observable not commuting with additively conserved quantity in an appropriate limit sense, and yet showed the possibility of arbitrarily precise, nondestructive measurement of any observable having a  $c$ -number commutator with additively conserved quantity, such as position measurement under the momentum conservation law; see also Stein and Shimony [18]. As a quantitative generalization of the WAY theorem, a lower bound for the sum of the mean-square noise and the mean-square disturbance was obtained in Ref. [4] and used to give a precision limit to realizing universal quantum gates in Ref. [1].

There has been a debate as to whether the WAY theorem can be extended to destructive measurements. Ohira and Pearle [19] constructed a model by which they claimed the possibility of precise measurement of a spin component under the spin conservation law in any direction. However, one of the present authors [20] pointed out that the claim is a circular argument, since the model only transfers the problem of measurement of object spin in one direction to the measurement of the spin of the probe in the same direction. In the same paper, it was shown by establishing a lower bound for the mean-square noise of the measurement that there is no measurement model for the precise measurement of an observable not commuting with a bounded conserved quantity if the meter observable is required to be measured nondestructively.

This lower bound was subsequently shown [5] to be a consequence of the noise-disturbance uncertainty principle [21,22], to solve a long-standing question as to how the WAY theorem relates to the uncertainty principle, and was used to derive the limit of the achievable gate fidelity of realizations of the Hadamard gate under the angular momentum conservation law; see also Refs. [7–9].

In this paper, we investigate an entirely new generalization of the WAY theorem and show that the WAY theorem can be extended in a relatively similar formulation to multiplicative conservation laws qualitatively and quantitatively

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[45]. The inevitable mean-square noise induced by the presence of a multiplicatively conserved quantity is evaluated to show that to overcome the limit we need to make large the coefficient of variation (or the so-called relative fluctuation) of the conserved quantity in the apparatus. We also show that the extension of the WAY theorem to the multiplicative case includes the additive case as a corollary. Furthermore, we obtain a limitation on such a measurement of an observable that the measuring interaction has an invariant state.

In Sec. II the concept of generalized measurements is reviewed, and the measurement models considered by Araki and Yanase [11] are characterized as those measurement models with both zero mean-square noise and zero mean-square disturbance. In Sec. III an extension of the WAY theorem to multiplicatively conserved quantities is proved and an application to quantum statistical mechanics is discussed. In Sec. IV a lower bound for the mean-square noise induced by the presence of a multiplicatively conserved quantity is obtained. Furthermore, an extension of the WAY theorem is also derived to destructive measurements under multiplicative conservation laws.

## II. MEASURING PROCESSES

A mathematical proof of the Wigner-Araki-Yanase theorem was first given by Araki and Yanase [11]. In their formulation, a measuring process is modelled as follows. Let  $A$  be an observable of a quantum system  $\mathbf{S}$  represented by a Hilbert space  $\mathcal{H}$ ; in this paper, all Hilbert spaces are assumed separable, or of at most countably infinite dimension. It is assumed that  $A$  has eigenvalues  $\mu$  with corresponding complete orthonormal eigenvectors  $|\phi_{\mu\rho}\rangle$  in  $\mathcal{H}$ , i.e.,

$$A|\phi_{\mu\rho}\rangle = \mu|\phi_{\mu\rho}\rangle \quad (1)$$

and  $\langle\phi_{\mu\rho}|\phi_{\mu'\rho'}\rangle = \delta_{\mu\mu'}\delta_{\rho\rho'}$ . Here the index  $\rho$  represents the degeneracy parameter of the eigenvalues of  $A$ . Following von Neumann [23], it is assumed that the measurement of  $A$  in a state  $|\phi\rangle \in \mathcal{H}$  is carried out by an interaction with a probe system  $\mathbf{P}$  described by a Hilbert space  $\mathcal{K}$  in its initial state  $|\xi\rangle \in \mathcal{K}$ . The time evolution of the composite system  $\mathbf{S}+\mathbf{P}$  during the interaction is represented by a unitary operator  $U$  on  $\mathcal{H}\otimes\mathcal{K}$ . It is assumed that the relation

$$U|\phi_{\mu\rho}\otimes\xi\rangle = \sum_{\rho'} |\phi_{\mu\rho'}\rangle \otimes X_{\mu\rho\rho'} \quad (2)$$

holds with the distinguishability condition (see footnote 3 of Ref. [11])

$$\langle X_{\mu\rho\rho'} | X_{\nu\sigma\sigma'} \rangle = 0 \quad (3)$$

if  $\mu \neq \nu$ . After the interaction, a meter observable  $M$  on  $\mathcal{K}$  is measured to obtain the outcome of the measurement, where  $M$  is given by

$$M|X_{\mu\rho\rho'}\rangle = \mu|X_{\mu\rho\rho'}\rangle \quad (4)$$

for all  $\mu, \rho, \rho'$ , where  $\mu$  varies over a countable set of real numbers, and  $\rho$  and  $\rho'$  vary over a countable index set depending on the value of  $\mu$ .

The above model describes a class of physically realizable measurements [23], but is not sufficiently general to include

all of the physically realizable measurements. In the modern approach [22,24–26], an exhaustive class of measurements is formulated as follows. Let  $\mathbf{A}(\mathbf{x})$  be a measuring apparatus with macroscopic output variable  $\mathbf{x}$  to measure observable  $A$  possibly with some error. The probe system  $\mathbf{P}$  with a Hilbert space  $\mathcal{K}$ , a part of the apparatus, is initially prepared in a state  $|\xi\rangle \in \mathcal{K}$ , and interacts with the system  $\mathbf{S}$  during a finite but short time interval [46] in which the composite system  $\mathbf{S}+\mathbf{P}$  undergoes the time evolution described by a unitary operator  $U$  on  $\mathcal{H}\otimes\mathcal{K}$ . After the interaction, a meter observable  $M$  on  $\mathcal{K}$  is measured to obtain the macroscopic output  $\mathbf{x}$ . According to the Born statistical formula, if the system  $\mathbf{S}$  is initially in a state  $|\phi\rangle \in \mathcal{H}$ , the probability distribution of the output  $\mathbf{x}$  is given by

$$\Pr\{\mathbf{x} \in \Delta\} = \|[I_1 \otimes E^M(\Delta)]U|\psi \otimes \xi\rangle\|^2, \quad (5)$$

where  $E^M(\Delta)$  is the spectral projection of  $M$  corresponding to a Borel set or an interval  $\Delta$ ; throughout this paper, index 1 refers to the system  $\mathbf{S}$  and 2 to the probe  $\mathbf{P}$ , and accordingly  $I_1$  and  $I_2$  refer to the identity operators of  $\mathbf{S}$  and  $\mathbf{P}$ , respectively. The mapping  $E^M: \Delta \mapsto E^M(\Delta)$  is called the spectral measure of  $M$  [27].

It is well known that every measurement is associated with a probability operator-valued measure (POVM) [28] that describes the output probability distribution. The POVM  $\Pi$  of the apparatus  $\mathbf{A}(\mathbf{x})$  is given by

$$\Pi(\Delta) = \langle \xi | U^\dagger [I_1 \otimes E^M(\Delta)] U | \xi \rangle \quad (6)$$

for all Borel set  $\Delta$ , where  $\langle \xi | \cdots | \xi \rangle$  stands for the partial inner product on  $\mathcal{K}$ ; the mapping  $\Pi: \Delta \mapsto \Pi(\Delta)$  is a positive operator-valued measure satisfying the normalization condition  $\Pi(\mathbb{R}) = I_1$ . Then, from Eq. (5) the output probability distribution satisfies

$$\Pr\{\mathbf{x} \in \Delta\} = \|\Pi(\Delta)^{1/2}|\psi\rangle\|^2. \quad (7)$$

We say that the apparatus  $\mathbf{A}(\mathbf{x})$  described by the measuring process  $(\mathcal{K}, |\xi\rangle, U, M)$  precisely measures observable  $A$  if the output probability distribution satisfies the Born statistical formula for observable  $A$ , i.e.,

$$\Pr\{\mathbf{x} \in \Delta\} = \|E^A(\Delta)|\psi\rangle\|^2 \quad (8)$$

for any state  $|\psi\rangle \in \mathcal{H}$ . In terms of POVM, from (6) this condition is equivalent to the condition that the POVM of the apparatus coincides with spectral measure of the observable  $A$ , i.e.,  $\Pi = E^A$ . This condition is also equivalent to the relation

$$U^\dagger [I_1 \otimes E^M(\Delta)] U |\psi \otimes \xi\rangle = E^A(\Delta) \otimes I_2 |\psi \otimes \xi\rangle \quad (9)$$

for any Borel set  $\Delta$  and state  $|\psi\rangle \in \mathcal{H}$ , since two projection operators  $U^\dagger [I_1 \otimes E^M(\Delta)] U$  and  $E^A(\Delta) \otimes I_2$  are identical on the space  $\mathcal{H} \otimes |\xi\rangle$  if and only if their expectation values are identical for all states in that space. The last condition is also equivalent to the condition that in the Heisenberg picture the meter observable  $M$  precisely evolves to the observable  $A$  to be measured, i.e.,

$$U^\dagger (I_1 \otimes M) U |\psi \otimes \xi\rangle = A \otimes I_2 |\psi \otimes \xi\rangle \quad (10)$$

for any states  $|\psi\rangle$  in the domain of  $A$ , since two operators  $U^\dagger (I_1 \otimes M) U$  and  $A$  are identical on the space  $\mathcal{H} \otimes |\xi\rangle$  if and

only if their spectral measures are identical on that space. To quantify the difference between the both sides, we introduce the root-mean-square noise  $\epsilon(A, |\psi\rangle)$  of the measurement of  $A$  in the state  $|\psi\rangle$  defined by

$$\epsilon(A, |\psi\rangle) = \|N|\psi \otimes \xi\rangle\|, \quad (11)$$

where the noise operator  $N$  is defined by

$$N = U^\dagger(I_1 \otimes M)U - A \otimes I_2. \quad (12)$$

Then, the precise measurement is equivalently characterized by the condition

$$\epsilon(A, |\psi\rangle) = 0 \quad (13)$$

for any states  $|\psi\rangle$  in the domain of  $A$  [22].

We say that the apparatus  $\mathbf{A}(\mathbf{x})$  does not disturb an observable  $B$  on  $\mathcal{H}$  if the time evolution  $U$  does not change the probability distribution of observable  $B$ , i.e.,

$$\|E^B(\Delta)|\psi\rangle\|^2 = \|[E^B(\Delta) \otimes I_2]U|\psi \otimes \xi\rangle\|^2 \quad (14)$$

for all states  $|\psi\rangle \in \mathcal{H}$ . This condition is also equivalent to the relation

$$U^\dagger[E^B(\Delta) \otimes I_2]U|\psi \otimes \xi\rangle = E^B(\Delta) \otimes I_2|\psi \otimes \xi\rangle \quad (15)$$

for any Borel set  $\Delta$  and state  $|\psi\rangle \in \mathcal{H}$  by a similar reasoning as above. The last condition is also equivalent to the condition that in the Heisenberg picture the time evolution during the measuring interaction does not change the observable  $B$ , i.e.,

$$U^\dagger(B \otimes I_2)U|\psi \otimes \xi\rangle = B \otimes I_2|\psi \otimes \xi\rangle \quad (16)$$

$|\psi\rangle$  in the domain of  $B$  [22], or the condition that  $B \otimes I_2$  commutes with  $U$ , i.e.,

$$[B \otimes I_2, U]|\psi \otimes \xi\rangle = 0 \quad (17)$$

for any states  $|\psi\rangle$  in the domain of  $B$  [22].

To quantify the difference between both sides of Eq. (16) the root-mean-square disturbance  $\eta(B, |\psi\rangle)$  of the observable  $B$  in a state  $|\psi\rangle$  is naturally defined by

$$\eta(B, |\psi\rangle) = \|D|\psi \otimes \xi\rangle\|, \quad (18)$$

where the disturbance operator  $D$  is defined by

$$D = U^\dagger(B \otimes I_2)U - B \otimes I_2. \quad (19)$$

According to Eq. (16), the apparatus  $\mathbf{A}(\mathbf{x})$  does not disturb the observable  $B$  if and only if

$$\eta(B, |\psi\rangle) = 0$$

for all states  $|\psi\rangle$  in the domain of  $B$  [22].

We say that an apparatus  $\mathbf{A}(\mathbf{x})$  nondestructively and precisely measures an observable  $A$  if  $\mathbf{A}(\mathbf{x})$  precisely measures  $A$  without disturbing  $A$ , i.e.,

$$\epsilon(A, \psi) = \eta(A, \psi) = 0 \quad (20)$$

for any state  $|\psi\rangle$  in the domain of  $A$ .

Now, we shall show that the measurement models considered by Araki and Yanase [11] mentioned above are characterized by the above condition (20). To formulate the state-

ment, an apparatus  $\mathbf{A}(\mathbf{x})$  described by  $(\mathcal{K}, |\xi\rangle, U, M)$  is said to be of the Araki-Yanase type if there is a complete orthonormal basis  $\{|\phi_{\mu\rho}\rangle\}$  in  $\mathcal{H}$  and a family  $\{X_{\mu\rho\rho'}\}$  of vectors in  $\mathcal{K}$  satisfying Eqs. (2)–(4).

**Theorem 1.** An apparatus  $\mathbf{A}(\mathbf{x})$  described by  $(\mathcal{K}, |\xi\rangle, U, M)$  nondestructively and precisely measures an observable  $A$  on  $\mathcal{H}$  if and only if  $\mathbf{A}(\mathbf{x})$  is of the Araki-Yanase type. In this case, the observable  $A$  is uniquely determined by relation (1).

*Proof.* First, suppose that  $\mathbf{A}(\mathbf{x})$  is of the Araki-Yanase type and let  $A$  be defined by (1). Then, from conditions (2) and (4), we have

$$U^\dagger(I_1 \otimes M)U|\phi_{\mu\rho} \otimes \xi\rangle = A \otimes I_2|\phi_{\mu\rho} \otimes \xi\rangle \quad (21)$$

for any eigenvector  $|\phi_{\mu\rho}\rangle$  of  $A$ . From the completeness of  $|\phi_{\mu\rho}\rangle$  in  $\mathcal{H}$ , the preciseness condition holds, i.e., Eq. (10) holds for all states  $|\psi\rangle$  in the domain of  $B$ . Similarly, the condition for nondestructive measurements is shown to be satisfied. Next, suppose that  $\mathbf{A}(\mathbf{x})$  nondestructively and precisely measures an observable  $A$  on  $\mathcal{H}$ . First, we shall show that the observable  $A$  has a purely discrete spectrum by appealing to a general theorem stating that if the apparatus  $\mathbf{A}(\mathbf{x})$  satisfies the repeatability hypothesis, then the observable  $A$  has a purely discrete spectrum (Theorem 6.6 of Ref. [24]). In order to formulate the repeatability hypothesis, we define the joint probability distribution of the repeated measurement using the apparatus  $\mathbf{A}(\mathbf{x})$  by

$$\text{Pr}\{\mathbf{x} \in \Delta, \mathbf{y} \in \Gamma\} = \|[E^A(\Gamma) \otimes E^M(\Delta)]U|\psi \otimes \xi\rangle\|^2. \quad (22)$$

Then, the apparatus  $\mathbf{A}(\mathbf{x})$  is said to satisfy the repeatability hypothesis if

$$\text{Pr}\{\mathbf{x} \in \Delta, \mathbf{y} \in \Gamma\} = \text{Pr}\{\mathbf{x} \in \Delta \cap \Gamma\}, \quad (23)$$

or equivalently

$$[E^A(\Gamma) \otimes E^M(\Delta)]U|\psi \otimes \xi\rangle = [I_1 \otimes E^M(\Delta \cap \Gamma)]U|\psi \otimes \xi\rangle. \quad (24)$$

Thus, it suffices to show the last relation. Equation (15) with  $A=B$  implies

$$[E^A(\Gamma) \otimes I_2]U|\psi \otimes \xi\rangle = U[E^A(\Gamma) \otimes I_2]|\psi \otimes \xi\rangle, \quad (25)$$

and similarly, Eq. (10) implies

$$[I_1 \otimes E^M(\Gamma)]U|\psi \otimes \xi\rangle = U[E^A(\Gamma) \otimes I_2]|\psi \otimes \xi\rangle. \quad (26)$$

Combining the above two equations, we have

$$[E^A(\Gamma) \otimes I_2]U|\psi \otimes \xi\rangle = [I_1 \otimes E^M(\Gamma)]U|\psi \otimes \xi\rangle. \quad (27)$$

Multiplying the both sides by  $I_1 \otimes E^M(\Delta)$  from the left-hand side, we obtain Eq. (24). Thus, the apparatus satisfies the repeatability hypothesis, and hence the observable  $A$  has a purely discrete spectrum. Thus,  $A$  has complete orthonormal eigenvectors  $|\phi_{\mu\rho}\rangle$  satisfying (1). Applying Eqs. (10) and (16) with  $B=A$  to the state  $|\psi\rangle = |\phi_{\mu\rho}\rangle$ , we obtain that  $U|\phi_{\mu\rho} \otimes X\rangle$  is an eigenvector belonging to the eigenvalue  $\mu$  of both  $A \otimes I_2$  and  $I_1 \otimes M$ . Since the eigenspace of  $A \otimes I_2$  with eigenvalue  $\mu$  is spanned by all  $|\phi_{\mu\rho} \otimes X\rangle$  with arbitrary  $\rho$  and arbitrary  $|X\rangle \in \mathcal{K}$ , vector  $U|\phi_{\mu\rho} \otimes X\rangle$  is generally written as Eq. (2). Since this is an eigenvector of  $I_1 \otimes M$ , the vector

$|X_{\mu\rho\rho'}\rangle$  is in the eigenspace of  $I_1 \otimes M$  with eigenvalue  $\mu$ , and hence satisfies condition (3). Thus, we have shown that  $\mathbf{A}(\mathbf{x})$  is of the Araki-Yanase type. Since the precisely measured observable  $A$  is uniquely determined by the output probability distribution, the uniqueness of  $A$  follows obviously. ■

In the next section, we discuss the limitation on nondestructive and precise measurements under a multiplicative conservation law, while in Sec. IV we discuss the limitation to arbitrary precise measurements with nondestructively measurable meters.

### III. LIMITATION ON PRECISE AND NONDESTRUCTIVE MEASUREMENTS INDUCED BY MULTIPLICATIVE CONSERVATION LAWS

Let  $\mathbf{A}(\mathbf{x})$  be an apparatus described by  $(\mathcal{K}, |\xi\rangle, U, M)$  for a Hilbert space  $\mathcal{H}$ . Let  $L_1$  and  $L_2$  be observables on  $\mathcal{H}$  and  $\mathcal{K}$ , respectively. The observable  $L=L_1 \otimes I_2 + I_1 \otimes L_2$  is called an additively conserved quantity of  $\mathbf{A}(\mathbf{x})$  if  $U$  satisfies

$$[L_1 \otimes I_2 + I_1 \otimes L_2, U] = 0. \quad (28)$$

The additive conservation law is generally associated with a continuous symmetry, and often holds for such quantities as energy, angular momentum, and spin. With reference to the discussions in the preceding section, Araki and Yanase [11] proved the following.

**Wigner-Araki-Yanase theorem:** An apparatus  $\mathbf{A}(\mathbf{x})$  with an additively conserved quantity  $L=L_1 \otimes I_2 + I_1 \otimes L_2$  nondestructively and precisely measures an observable  $A$ , then the observable  $A$  must commute with the conserved quantity, i.e.,  $[A, L_1]=0$ , provided that  $L_1$  is bounded.

An observable  $L=L_1 \otimes L_2$  with observables  $L_1$  on  $\mathcal{H}$  and  $L_2$  on  $\mathcal{K}$  is called a multiplicatively conserved quantity of  $\mathbf{A}(\mathbf{x})$  if  $U$  satisfies

$$[L_1 \otimes L_2, U] = 0. \quad (29)$$

Multiplicative conservation laws are related to discrete symmetries such as parity, charge conjugation, and time reversal. Moreover, they also formally include all the additive conservation laws by exponentiating the additively conserved quantities. We shall show that a similar limitation to that for the additive case arises for measurements under multiplicative conservation laws.

Let  $L_1$  be an observable on  $\mathcal{H}$ . An observable  $A$  is said to be nondestructively and precisely measurable under the multiplicative conservation law with  $L_1$ , if there is an apparatus  $\mathbf{A}(\mathbf{x})$  described by  $(\mathcal{K}, |\xi\rangle, U, M)$  such that  $\mathbf{A}(\mathbf{x})$  precisely measures  $A$  without disturbing  $A$  and that  $\mathbf{A}(\mathbf{x})$  has a multiplicatively conserved quantity  $L_1 \otimes L_2$  for some invertible observable  $L_2$  on  $\mathcal{K}$ . In the following,  $|L|$  stands for the modulus of the observable  $L$  defined by  $|L|^2=L^2$  and  $|L| \geq 0$ .

**Theorem 2.** Every nondestructively and precisely measurable observable  $A$  under the multiplicative conservation law with  $L_1$  commutes with  $|L_1|$ , i.e.,  $[A, |L_1|]=0$ , provided that  $L_1$  is bounded and that  $L_1$  has a bounded inverse or 0 is an isolated eigenvalue (the above conditions are automatically satisfied if  $\mathcal{H}$  is finite dimensional).

*Proof.* Suppose that  $L_1$  is bounded and that  $L_1$  has a

bounded inverse or 0 is an isolated eigenvalue of  $L_1$ . Suppose that  $A$  can be precisely and nondestructively measured by an apparatus  $\mathbf{A}(\mathbf{x})$  described by  $(\mathcal{K}, |\xi\rangle, U, M)$  with conserved quantity  $L_1 \otimes L_2$  with an invertible  $L_2$ . Let  $P_1$  be the projection operator to the kernel of  $L_1$ . Then, the projection operator to the kernel of  $L_1 \otimes L_2$  on  $\mathcal{H} \otimes \mathcal{K}$  is  $P_1 \otimes I_2$ , since  $L_2$  is invertible. From the conservation law (29), we have  $[P_1 \otimes I_2, U]=0$  and  $[P_1^\perp \otimes I_2, U]=0$ . Then, observing the equality  $\langle \phi_{\mu\rho} \otimes \xi | U^\dagger (P_1^\perp \otimes I_2) U | \phi_{\nu\sigma} \otimes \xi \rangle = \langle \phi_{\mu\rho} \otimes \xi | P_1^\perp \otimes I_2 | \phi_{\nu\sigma} \otimes \xi \rangle$  and Eq. (2), we have

$$\sum_{\rho', \sigma'} \langle \phi_{\mu\rho'} | P_1^\perp | \phi_{\nu\sigma'} \rangle \langle X_{\mu\rho\rho'} | X_{\nu\sigma\sigma'} \rangle = \langle \phi_{\mu\rho} | P_1^\perp | \phi_{\nu\sigma} \rangle \langle \xi | \xi \rangle$$

for all  $\mu, \nu$ . From condition (3), if  $\mu \neq \nu$ , we have

$$\langle \phi_{\mu\rho} | P_1^\perp | \phi_{\nu\sigma} \rangle = 0, \quad (30)$$

and hence  $[A, P_1^\perp]=0$ . Let  $\mathcal{H}' = P_1^\perp \mathcal{H}$ . Then, from the above,  $\mathcal{H}' \otimes \mathcal{K}$  is invariant under  $U$ , and  $\mathcal{H}'$  is invariant under  $A$  and  $L_1$ . Let  $A'$  and  $L'_1$  be the restrictions of  $A$  and  $L_1$  to  $\mathcal{H}'$ , respectively. Let  $U'$  be the restriction of  $U$  to  $\mathcal{H}' \otimes \mathcal{K}$ . Then, the measuring process  $(\mathcal{K}, |\xi\rangle, U', M)$  precisely and nondestructively measures  $A'$  with multiplicatively conserved quantity  $L'_1 \otimes L_2$ . Since  $L'_1 \otimes L_2$  is invertible, the observable  $\ln|L'_1 \otimes L_2|$  is well defined and satisfies

$$\ln|L'_1 \otimes L_2| = \ln|L'_1| \otimes I_2 + I'_1 \otimes \ln|L_2|,$$

where  $I'_1$  is the identity on  $\mathcal{H}'$ . Then, we have  $[U, \ln|L'_1 \otimes L_2|]=0$ , so that  $\ln|L'_1| \otimes I_2 + I'_1 \otimes \ln|L_2|$  is an additively conserved quantity for  $(\mathcal{K}, |\xi\rangle, U', M)$ . It follows from assumptions on  $L_1$  that  $\ln|L'_1|$  is bounded. Thus, from the Wigner-Araki-Yanase theorem [11] it follows that  $[A', \ln|L'_1|]=0$ . Thus, we have  $[A', |L'_1|]=0$  from  $|L'_1| = \exp \ln|L'_1|$ . Therefore, we have  $[A, |L_1|] = [A, |L_1| P_1^\perp] = [A P_1^\perp, |L_1| P_1^\perp] = 0$ . ■

In comparing the WAY theorem for additive conservation laws with the present extension to multiplicative conservation laws, the following two features should be noticed. First, the measured observable  $A$  is required to commute with  $|L_1|$  not with  $L_1$ . Second, for the theorem to be valid, we need an additional assumption that  $L_2$  is invertible. Indeed, these are necessary assumptions in the multiplicative cases as the following examples show. Let  $\mathcal{H}$  and  $\mathcal{K}$  be two-dimensional Hilbert spaces, and for  $i=1, 2$  let  $\sigma_x^{(i)}, \sigma_y^{(i)}, \sigma_z^{(i)}$  be Pauli matrices for system  $\mathcal{H}$  if  $i=1$ , and  $\mathcal{K}$  if  $i=2$ , where  $\{|a_i\rangle\}_{i=1,2}$  and  $\{|\xi_j\rangle\}_{j=1,2}$  are eigenvectors of  $\sigma_z^{(1)}$  and  $\sigma_z^{(2)}$ , respectively. Let us consider a measurement of  $A = \sigma_x^{(1)}$ . With the measuring interaction  $U$ , which is a controlled-NOT gate defined by  $U = |a_1\rangle\langle a_1| \otimes I_1 + |a_2\rangle\langle a_2| \otimes \sigma_x^{(2)}$ , it is easy to check that the relation

$$U|a_i\rangle \otimes |\xi_1\rangle = |a_i\rangle \otimes |\xi_i\rangle \quad (31)$$

holds for  $i=1, 2$ , and hence the apparatus  $\mathbf{A}(\mathbf{x})$  described by  $(\mathcal{K}, |\xi_1\rangle, U, M)$  nondestructively and precisely measures  $A$ .

*Example 1.* Let  $L_1 = \sigma_x^{(1)}$  and  $L_2 = \sigma_x^{(2)}$ ; notice that  $\sigma_x^{(2)}$  is invertible. It is easy to check that  $L_1 \otimes L_2$  satisfies the multiplicative conservation law with respect to  $U$  in (31). However, the observable  $A = \sigma_x^{(1)}$  does not commute with  $L_1 = \sigma_x^{(1)}$ , while  $A$  commutes with  $|L_1| = I_1$ . This shows that the

commutativity with  $A$  applies to  $|L_1|$  and not to  $L_1$  itself. Note that it is natural to impose the invertibility restriction on  $L_2$ , since we have the trivial counterexample  $L_2=0$ .

*Example 2.* Let  $L_1$  be an arbitrary observable on  $\mathcal{H}$  and let  $L_2$  be a noninvertible observable on  $\mathcal{K}$  defined by  $L_2=|\tilde{\xi}\rangle\langle\tilde{\xi}|$  with  $|\tilde{\xi}\rangle=\frac{1}{\sqrt{2}}(|\xi_1\rangle+|\xi_2\rangle)$ . Then, it is easy to see  $[U, L_1 \otimes L_2]=0$ . However there exists an observable  $L_1$  with which  $|L_1|$  does not commute with  $A$ , e.g.,  $L_1=|L_1|=|\tilde{a}\rangle\langle\tilde{a}|$  with  $|\tilde{a}\rangle=\frac{1}{\sqrt{2}}(|a_1\rangle+|a_2\rangle)$ . This shows that the invertibility of  $L_2$  is a necessary assumption for Theorem 2 to hold.

We shall now show that the usual WAY theorem for additive conservation laws is obtained as a corollary of Theorem 2. Let  $L$  be an additively conserved quantity for an apparatus  $\mathbf{A}(\mathbf{x})$  as defined in (28) with  $L_1$  bounded. Then,  $\mathbf{A}(\mathbf{x})$  has a multiplicatively conserved quantity  $\exp(L_1) \otimes \exp(L_2)$  satisfying (29). In this case,  $\exp(L_2)$  is invertible and  $\exp(L_1)$  is bounded and has a bounded inverse, so that Theorem 2 concludes  $[A, |\exp(L_1)|]=0$ . Since  $\exp(L_1)$  is positive, we have  $[A, \exp(L_1)]=0$  and hence  $[A, L_1]=0$ . It is important to point out that Theorem 2 for multiplicative conservation laws is not directly obtained from the usual WAY theorem by taking the logarithm of the multiplicatively conserved quantity. Since  $L_1$  in condition (29) is not assumed to be invertible,  $\ln|L_1 \otimes L_2|$  is not necessarily an additively conserved quantity.

Another application of Theorem 2 leads to a limitation on a measurement such that the measuring interaction has an invariant state.

**Theorem 3.** Suppose that an apparatus  $\mathbf{A}(\mathbf{x})=(\mathcal{K}, |\xi\rangle, U, M)$  nondestructively and precisely measures an observable  $A$  on  $\mathcal{H}$ . If the measuring interaction  $U$  leaves a product state  $\rho_1 \otimes \rho_2$  invariant, where  $\rho_1$  is a density operator on  $\mathcal{H}$  with finite rank and  $\rho_2$  is an invertible density operator (such as the Gibbs state) on  $\mathcal{K}$ , we have

$$[A, \rho_1]=0. \quad (32)$$

The above theorem follows easily from Theorem 2 with  $L_1=\rho_1$  and  $L_2=\rho_2$ .

#### IV. LOWER BOUND FOR THE MEAN-SQUARE NOISE AND LIMITATION ON ARBITRARY PRECISE MEASUREMENTS

Now we consider quantitative limitations to measurements under multiplicative conservation laws. For this purpose, a technique previously developed in Ref. [20] is used, and we obtain a bound for the mean-square noise  $\epsilon(A, |\psi\rangle)$  under a multiplicative conservation law.

Suppose that an apparatus  $\mathbf{A}(\mathbf{x})$  described by  $(\mathcal{K}, |\xi\rangle, U, M)$  has a multiplicatively conserved quantity  $L_1 \otimes L_2$ . From the Heisenberg-Robertson uncertainty relation for standard deviations of the noise operator  $N$  and the modulus of the multiplicatively conserved quantity,  $|L_1 \otimes L_2|=|L_1| \otimes |L_2|$ , in the state  $|\psi \otimes \xi\rangle$ , we have

$$\sigma(N)^2 \sigma(|L_1 \otimes L_2|)^2 \geq \frac{1}{4} |[N, |L_1| \otimes |L_2|]|^2, \quad (33)$$

where  $\sigma(X)$  and  $\langle X \rangle$  denote the standard deviation and the mean of  $X$  in state  $|\psi \otimes \xi\rangle$ , respectively. Since

$[U, |L_1| \otimes |L_2|]=0$ , we have the commutation relation

$$[N, |L_1| \otimes |L_2|]=[A, |L_1|] \otimes |L_2| - U^\dagger(|L_1| \otimes [M, |L_2|])U. \quad (34)$$

Observing the relations  $\epsilon(A, |\psi\rangle)^2 \geq \epsilon(A, |\psi\rangle)^2 - \langle N \rangle^2 = \sigma(N)^2$  and  $\langle \psi | |L_1|^2 | \psi \rangle \langle \xi | |L_2|^2 | \xi \rangle = \langle |L_1 \otimes L_2|^2 \rangle \geq \sigma(|L_1 \otimes L_2|)^2$ , we obtain

$$4\epsilon(A, |\psi\rangle)^2 \langle \psi | |L_1|^2 | \psi \rangle \langle \xi | |L_2|^2 | \xi \rangle \geq \langle [A, |L_1|] \otimes |L_2| - U^\dagger(|L_1| \otimes [M, |L_2|])U \rangle^2. \quad (35)$$

Let  $\text{Ker}(L_j)$  for  $j=1,2$  denote the kernel of  $L_j$  in  $\mathcal{H}$  or  $\mathcal{K}$ , respectively. Let  $|\psi\rangle \notin \text{Ker}(L_1)$  and  $|\xi\rangle \notin \text{Ker}(L_2)$ . Then, we have  $\langle \psi | |L_1|^2 | \psi \rangle \langle \xi | |L_2|^2 | \xi \rangle \neq 0$  and hence we obtain a bound for the mean-square noise as

$$\epsilon(A, |\psi\rangle)^2 \geq \frac{\langle [A, |L_1|] \otimes |L_2| - U^\dagger(|L_1| \otimes [M, |L_2|])U \rangle^2}{4\langle \psi | |L_1|^2 | \psi \rangle \langle \xi | |L_2|^2 | \xi \rangle}. \quad (36)$$

Now, we require Yanase's condition [12,20]

$$[M, |L_2|]=0. \quad (37)$$

This condition eliminates a circular argument to show the measurability of  $A$  under a conservation law by reducing it to the measurability of  $M$  whose measurability is still unresolved under the same conservation law. In order to ensure the measurability of  $A$ , even if we allow the measurement of  $A$  to be destructive, it is natural to assume that there is a measuring process in which the meter measurement can be done non-destructively to achieve the stability of the measurement outcome to be recorded. Then, this is possible in the presence of a multiplicative conservation law with  $L_2$  only if the relation  $[M, |L_2|]=0$  holds. Thus, it is natural to require the existence of a measuring process in which Yanase's condition holds for the meter observable.

Under Yanase's condition we have

$$\epsilon(A, |\psi\rangle)^2 \geq \frac{\langle \psi | [A, |L_1|] | \psi \rangle}{4\langle \psi | |L_1|^2 | \psi \rangle} R(|L_2|), \quad (38)$$

where  $R(|L_2|)$  is the ratio of the squared mean of  $|L_2|$  to the mean of  $L_2^2=|L_2|^2$  in state  $|\xi\rangle$ , i.e.,

$$R(|L_2|) \equiv \frac{\langle \xi | |L_2| | \xi \rangle^2}{\langle \xi | |L_2|^2 | \xi \rangle} \leq 1, \quad (39)$$

where the last inequality holds for any state  $|\xi\rangle$  since  $\langle \xi | |L_2|^2 | \xi \rangle - \langle \xi | |L_2| | \xi \rangle^2 = \sigma(|L_2|)^2 \geq 0$ . The ratio  $R(|L_2|)$  is directly related to the coefficient of variation (relative fluctuation)  $C_V(|L_2|)$  of  $|L_2|$ , the ratio of the standard deviation of  $|L_2|$  to the mean of  $|L_2|$  in state  $|\xi\rangle$ , i.e.,

$$R(|L_2|) = \frac{\langle \xi | |L_2| | \xi \rangle^2}{\sigma(|L_2|)^2 + \langle \xi | |L_2| | \xi \rangle^2} = \frac{1}{1 + C_V(|L_2|)^2}. \quad (40)$$

Let  $L_1$  be a bounded observable on  $\mathcal{H}$ . An observable  $A$  on  $\mathcal{H}$  is said to be precisely measurable under the multiplicative conservation law with  $L_1$  if there is an apparatus  $\mathbf{A}(\mathbf{x})$  described by  $(\mathcal{K}, |\xi\rangle, U, M)$  such that  $\mathbf{A}(\mathbf{x})$  precisely measures  $A$ , that  $\mathbf{A}(\mathbf{x})$  has a multiplicatively conserved quantity

$L_1 \otimes L_2$  for some invertible and bounded observable  $L_2$  on  $\mathcal{K}$ , and that  $\mathbf{A}(\mathbf{x})$  satisfies Yanase's condition. Then, we obtain the following generalization of the WAY theorem.

**Theorem 4.** Every precisely measurable observable under the multiplicative conservation law with  $L_1$  commutes with  $|L_1|$ .

*Proof.* Let  $\mathbf{A}(\mathbf{x})$  be an apparatus, described by  $(\mathcal{K}, |\xi\rangle, U, M)$ , to carry out a precise measurement of  $A$  having a multiplicatively conserved quantity  $L_1 \otimes L_2$ , where  $L_1$  is bounded and  $L_2$  is bounded and invertible, and satisfying Yanase's condition. We can assume without any loss of generality that  $A$  is bounded; otherwise, replace  $A$  by  $\tan^{-1} A$  and  $M$  by  $\tan^{-1} M$ , for instance. Then, we have  $\epsilon(A, |\psi\rangle) = 0$  for any state  $|\psi\rangle \in \mathcal{H}$ . Let  $|\psi\rangle \in \mathcal{H}$  and  $|\xi\rangle \in \mathcal{K}$ . Then, Eq. (35) concludes  $\langle \psi | [A, |L_1|] | \psi \rangle \langle \xi | |L_2| | \xi \rangle = 0$  under Yanase's condition. Since  $L_2$  is invertible, we have  $\langle \xi | |L_2| | \xi \rangle > 0$ , and  $\langle \psi | [A, |L_1|] | \psi \rangle = 0$  holds. Therefore, we conclude  $[A, |L_1|] = 0$ . ■

Note that the condition  $[A, |L_1|] = 0$  is equivalent with  $[A, L_1^2] = 0$ . Thus, we have shown that an observable not commuting with the modulus of, or equivalently the square of, a multiplicatively conserved quantity cannot be precisely measured.

To figure out the apparatus state  $|\xi\rangle$  which makes the measurement of  $A$  as precise as possible, let us consider the state that minimizes  $R(|L_2|)$ ; for simplicity we assume  $L_2$  to be of finite rank. If  $|L_2|$  is constant, then  $R(|L_2|)$  is always 1. Suppose that  $|L_2|$  is not constant, and let  $l_m$  and  $l_M$  be the minimum and maximum eigenvalues of  $|L_2|$ , respectively. Then,  $R(|L_2|)$  takes the minimum value  $\frac{4l_m l_M}{(l_M - l_m)^2} < 1$ . Indeed, it is straightforward to prove the following statement in discrete probability theory: Let  $L$  be a random variable with values  $(l_1, l_2, \dots, l_d)$  with  $d \geq 2$  and  $0 < l_1 < l_2 < \dots < l_d$ . The ratio

$$R(L) = \frac{\left(\sum_{i=1}^d l_i p_i\right)^2}{\sum_{i=1}^d l_i^2 p_i} \tag{41}$$

with the probability distribution  $(p_1, p_2, \dots, p_d)$  has the minimum  $\frac{4l_1 l_d}{(l_d - l_1)^2}$  with the unique probability distribution  $(\frac{l_d}{l_1 + l_d}, 0, \dots, 0, \frac{l_1}{l_1 + l_d})$ , while the variance of  $L$  is maximized by  $(\frac{1}{2}, 0, \dots, 0, \frac{1}{2})$ . The above minimum is attained by any state  $|\xi_{\min}\rangle$  with the following properties: The probabilities to obtain the outputs  $l_m$  and  $l_M$  are  $\frac{l_M}{l_m + l_M}$  and  $\frac{l_m}{l_m + l_M}$ , respectively, while the probabilities to obtain the other eigenvalues are zero. Therefore, any such state  $|\xi_{\min}\rangle$  can be written as  $|\xi_{\min}\rangle = \sqrt{\frac{l_M}{l_m + l_M}} |m\rangle + \sqrt{\frac{l_m}{l_m + l_M}} |M\rangle$  with  $|m\rangle$  and  $|M\rangle$  being eigenstates of  $|L_2|$  with eigenvalues  $l_m$  and  $l_M$ , respectively.

**V. CONCLUDING REMARKS**

In this paper natural generalizations of the WAY theorem to multiplicatively conserved quantities have been established. We have characterized nondestructive and precise measurements of an observable by the root-mean-square noise and disturbance. In the finite dimensional case, we have proved that every nondestructively and precisely mea-

surable observable under the multiplicative conservation law with arbitrary  $L_1$  and invertible  $L_2$  commutes with  $|L_1|$ ; here, we confine ourselves to the finite dimensional case for simplicity. By taking exponentials, every additive conservation law can be regarded as a multiplicative conservation law, such that both  $L_1$  and  $L_2$  are invertible. Thus, the original WAY theorem is recovered as a simple corollary. An example shows that there is a nondestructively and precisely measurable observable not commuting with  $L_1$  under the multiplicative conservation law with arbitrary  $L_1$  and invertible  $L_2$ . Thus, the noncommutativity applies to  $|L_1|$  instead of  $L_1$ . Another example shows that the invertibility of  $L_2$  cannot be dropped from the assumptions of the above statement. An interesting application of the above statement is given to invariant states of measuring interactions.

We have also investigated destructive measurements to drop the assumption of measurements to be nondestructive from the above statement. In this case, a general lower bound for the root-mean-square noise is established in Eq. (38) under Yanase's condition that ensures that the meter observable can be precisely and nondestructively measured with the same conserved quantity. This condition is necessary for eliminating a circular argument. The above lower bound shows that in order to overcome the limitation we need to make large the coefficient of variation of the conserved quantity in the apparatus. Then, we have concluded that every precisely measurable observable under the multiplicative conservation law with bounded  $L_1$  and bounded and invertible  $L_2$  commutes with  $|L_1|$ .

An interesting problem in experimental settings is to obtain a tighter lower limit for the root-mean-square noise in the presence of a multiplicatively conserved quantity; our result is the most general one but more specific results can be tighter than the most general. Theorems in the present paper are stepping stones towards developing a WAY-type theorem for conserved quantities that has both additive as well as multiplicative components, e.g., a Hamiltonian of a multiparticle system with an interaction term.

A broad area of applications of the present investigation is general implementation limitations on quantum computers. For additive conservation laws, there has been an extensive literature [1–9] on the conservation-law-induced quantum limits on the performance of elementary quantum gates; see also [29–35] for the model-dependent approach to the limitations of quantum gate operations realized by the atom-field interaction, which has turned out to be consistent with the model-independent approach based on conservation laws. Our method will be also expected to contribute to the problem of programmable quantum processors [36–38] and related subjects [39–41] in future investigations.

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- [44] Here, a measurement is “nondestructive” if the measured observable is not disturbed by the interaction between the system and the apparatus. Note also that “a quantum nondemolition (QND) measurement” is reserved to mean a sequence of precise measurements such that the result of each measurement is completely predictable from the results of the preceding measurement (cf. p. 363 of Ref. [42]). Thus, a sequence of nondestructive and precise measurements of a constant of motion is a QND measurement.
- [45] A preliminary consideration on this topic has already been presented in Ref. [43].
- [46] Here, we consider instantaneous measurements [26] in which the measuring interaction takes place in a time interval so short that the system time evolution is negligible. This is the case if the coupling between the system and probe is considered to be very strong.