Transversality Conditions for Infinite Horizon Optimality: Higher Order Differential Problems†

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Abstract

Michel [1] and Ekeland and Scheinkman [2] presented the transversality condition for the first order differential problems: $\max_{\mathbf{x}} \int_0^\infty v\left(\mathbf{x}(t), \dot{\mathbf{x}}(t), t\right) dt$, which may have unbounded objective functions. Kamihigashi [3] showed a generalization of their transversality condition that does not assume concavity. Using the variational approach, this paper deals with higher order differential problems: $\max_{\mathbf{x}} \int_0^\infty v\left(\mathbf{x}(t), \dot{\mathbf{x}}(t), \ddot{\mathbf{x}}(t), \cdots, \mathbf{x}^{(n)}(t), t\right) dt$. We derive two conditions: the Euler's condition and the transversality condition, for such problems in a simple manner. They are imperative to solve the variational problems. Furthermore, two assumptions are necessary to induce the two conditions. We construct a counterexample in which the transversality condition is not satisfied without the two assumptions.

Keywords: Transversality condition; Dynamic optimization; Infinite horizon; Unbounded; Higher order differential problems

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1. Introduction

We study the transversality conditions for the following reduced model

$$(1)\qquad \begin{cases} \max_{\mathbf{x}} \int_0^\infty v\left(\mathbf{x}(t),\dot{\mathbf{x}}(t),\ddot{\mathbf{x}}(t),\cdots,\mathbf{x}^{(n)}(t),t\right)dt \text{ subject to} \\ \mathbf{x}(0)=\mathbf{x}_0,\ \forall t\geq 0,\ \left(\mathbf{x}(t),\dot{\mathbf{x}}(t),\ddot{\mathbf{x}}(t),\cdots,\mathbf{x}^{(n)}(t)\right)\in X(t)\subset\left(\mathbb{R}^N\right)^n, \end{cases}$$

where $N\in\mathbb{N}$, v is a real-valued n-th order continuously differentiable function, and $\mathbf{x}(t)$ is n-th order continuously differentiable.[1] Notice that the objective functional of (1) is not necessarily finite. So far, the most general form transversality conditions for the $n=1$ case is presented in Kamihigashi [3, Theorem 3.2], which generalizes the results of Michel [1] and Ekeland and Scheinkman [2]. Kamihigashi considers the transversality condition for the first order differential problems: $\max_{\mathbf{x}}\int_0^\infty v\left(\mathbf{x}(t),\dot{\mathbf{x}}(t),t\right)dt$, using the Lebesgue integral. In this paper, we extend their results to higher order differential problems using the classical Riemannian integral. In order to attain the maximality in these problems, the following two conditions, the Euler's condition and the transversality condition, are needed. To realize the feasibility for the two conditions, further two assumptions are provided. The application of such higher order differential problems can be found in economics. Specially, second order differential problems appear in the discussion concerning the acceleration principle. When the investment depends on the variation of the income, consumption then depends on the second order difference of the capital. Under such a case, the utility will also depend on the second order difference of

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[1] Normally, v is defined on $\left(\mathbb{R}^N\right)^n\times\mathbb{R}$. The domain of v is denoted by $X(t)$, in included in $\left(\mathbb{R}^N\right)^n$, for all t.

the capital.

Obviously, if the objective function v is a piece-wise n-th order continuously differentiable function, we should need to consider the problem under the Lebesgue integral. Kamihigashi [3] considers such a case only for the first order differential problems. His Assumption 3.1 corresponds to our two assumptions when $n=1$.

We first use the variational approach to present a complete representation of the two conditions for higher order differntial problems. Using the variational approach, Chiang [4] also considers the transversality condition for the infinite horizon first order differential problems ($n=1$). The two assumptions should be needed in this approach. To show this fact, we provide a counterexample. However, the two assumptions are naturally satisfied when a discounting factor is incorporated into the model. Therefore, the transversality condition generalizes the result obtained in the presence of discounting. The optimal solutions to this class are discussed in Cai and Nitta [5-7].

2. A Complete Characterization of the Transversality Conditions

Suppose that the optimal path to (1) exists and is given by $\mathbf{x}^*$, optimal in the sense of overtaking criterion. We perturb it with n-th order continuously differentiable curves $\mathbf{p}(t)$,

(2) $$\mathbf{x}(t)=\mathbf{x}^*(t)+\varepsilon\cdot\mathbf{p}(t),$$

We define

(3)

$$V(\varepsilon,T)=\inf_{T\le T'}\int_0^{T'}\Big(v\big(\mathbf{x}^*(t)+\varepsilon\cdot\mathbf{p}(t),\dot{\mathbf{x}}^*(t)+\varepsilon\dot{\mathbf{p}}(t),\cdots,\mathbf{x}^{*(n)}(t)+\varepsilon\cdot\mathbf{p}^{(n)}(t),t\big)-v\big(\mathbf{x}^*(t),\dot{\mathbf{x}}^*(t),\cdots,\mathbf{x}^{*(n)}(t),t\big)\Big)dt.$$

We use Brock's [8] notion of weak maximality as our optimality criterion and assume that there exists an optimal path that satisfy the weak maximality criterion, which is defined as: an attainable path $\left(\mathbf{x}^*(t)\right)$ is optimal if no other attainable path overtakes it[2]:

(4)

$$\lim_{T\to\infty}\inf_{T\le T'}\int_0^{T'}\Big(v\big(\mathbf{x}^*(t)+\varepsilon\cdot\mathbf{p}(t),\dot{\mathbf{x}}^*(t)+\varepsilon\dot{\mathbf{p}}(t),\cdots,\mathbf{x}^{*(n)}(t)+\varepsilon\cdot\mathbf{p}^{(n)}(t),t\big)-v\big(\mathbf{x}^*(t),\dot{\mathbf{x}}^*(t),\cdots,\mathbf{x}^{*(n)}(t),t\big)\Big)dt\le 0.$$

Let $V(\varepsilon)=\lim_{T\to\infty}V(\varepsilon,T)$. Differentiating it with respect to ε, we have

(5)

$$\lim_{\varepsilon\to 0}\frac{V(\varepsilon)}{\varepsilon}=\lim_{\varepsilon\to 0}\lim_{T\to\infty}\inf_{T\le T'}\int_0^{T'}\frac{\Big(v\big(\mathbf{x}^*(t)+\varepsilon\cdot\mathbf{p}(t),\dot{\mathbf{x}}^*(t)+\varepsilon\dot{\mathbf{p}}(t),\cdots,\mathbf{x}^{*(n)}(t)+\varepsilon\cdot\mathbf{p}^{(n)}(t),t\big)-v\big(\mathbf{x}^*(t),\dot{\mathbf{x}}^*(t),\cdots,\mathbf{x}^{*(n)}(t),t\big)\Big)}{\varepsilon}dt.$$

Let $\lim_{\varepsilon\to 0}\frac{V(\varepsilon)}{\varepsilon}\equiv\Omega$. In general, $\frac{d}{d\varepsilon}\lim_{T\to\infty}f(\varepsilon,T)=\lim_{T\to\infty}\frac{d}{d\varepsilon}f(\varepsilon,T)$ only if $\lim_{T\to\infty}\frac{d}{d\varepsilon}f(\varepsilon,T)$ converges uniformly for ε (Lang [9]). We assume

Assumption 1. *Assume* Ω *converges uniformly for* ε *when* $T\to\infty$.

If Ω satisfies Assumption 1, we can then rewrite (5) as

(6)

$$\Omega=\lim_{T\to\infty}\lim_{\varepsilon\to^+0}\inf_{T\le T'}\int_0^{T'}\frac{\Big(v\big(\mathbf{x}^*(t)+\varepsilon\cdot\mathbf{p}(t),\dot{\mathbf{x}}^*(t)+\varepsilon\dot{\mathbf{p}}(t),\cdots,\mathbf{x}^{*(n)}(t)+\varepsilon\cdot\mathbf{p}^{(n)}(t),t\big)-v\big(\mathbf{x}^*(t),\dot{\mathbf{x}}^*(t),\cdots,\mathbf{x}^{*(n)}(t),t\big)\Big)}{\varepsilon}dt.$$

Next, we impose another assumption:

[2] Brock (1970) shows that once such a path exists once two assumptions are satisfied.

Assumption 2. *We assume for any* $T > 0$,

$\inf_{T \le T'} \int_0^{T'} \frac{\left(v\left(\mathbf{x}^*(t) + \varepsilon \cdot \mathbf{p}(t), \dot{\mathbf{x}}^*(t) + \varepsilon \cdot \dot{\mathbf{p}}(t), \cdots, t\right) - v\left(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), \cdots, t\right)\right)}{\varepsilon} dt$ *converges uniformly for* ε.

The assumption means: Let $A(T, \varepsilon) = \int_0^{T} \frac{\left(v\left(\mathbf{x}^*(t) + \varepsilon \cdot \mathbf{p}(t), \dot{\mathbf{x}}^*(t) + \varepsilon \cdot \dot{\mathbf{p}}(t), \cdots, t\right) - v\left(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), \cdots, t\right)\right)}{\varepsilon} dt.$ Then there exists a sequence $A\left(T_n', \varepsilon\right)$ for each $\varepsilon > 0$, so that $\lim_{n \to \infty} A\left(T_n', \varepsilon\right) = \inf_{T \le T'} A\left(T', \varepsilon\right)$, uniformly for ε, that is, the sequence is uniformly convergence for ε.

Assumption 1 and 2 corresponds to Assumption 3.1 in Kamihigashi [3], which uses the Lebesgue integral. When Assumption 1 and 2 are satisfied, then $\lim_{\varepsilon \to {}^+0}$ and $\inf_{T \le T'}$ can be interchanged, and equality (6) can then restated as

(7)

$$\Omega = \lim_{T \to \infty} \inf_{T \le T'} \lim_{\varepsilon \to {}^+0} \int_0^{T'} \frac{\left(v\left(\mathbf{x}^*(t) + \varepsilon \cdot \mathbf{p}(t), \dot{\mathbf{x}}^*(t) + \varepsilon \dot{\mathbf{p}}(t), \cdots, \mathbf{x}^{*(n)}(t) + \varepsilon \cdot \mathbf{p}^{(n)}(t), t\right) - v\left(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), \cdots, \mathbf{x}^{*(n)}(t), t\right)\right)}{\varepsilon} dt.$$

Because T' is finite uniformly for ε,

if $\int_0^{T'} \frac{\left(v\left(\mathbf{x}^*(t) + \varepsilon \cdot \mathbf{p}(t), \dot{\mathbf{x}}^*(t) + \varepsilon \dot{\mathbf{p}}(t), \cdots, \mathbf{x}^{*(n)}(t) + \varepsilon \cdot \mathbf{p}^{(n)}(t), t\right) - v\left(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), \cdots, \mathbf{x}^{*(n)}(t), t\right)\right)}{\varepsilon} dt$

exists, (7) is then rewritten as

(8)

$$\Omega = \lim_{T \to \infty} \inf_{T \le T'} \int_0^{T'} \lim_{\varepsilon \to {}^+0} \frac{\left(v\left(\mathbf{x}^*(t) + \varepsilon \cdot \mathbf{p}(t), \dot{\mathbf{x}}^*(t) + \varepsilon \dot{\mathbf{p}}(t), \cdots, \mathbf{x}^{(n)*}(t) + \varepsilon \cdot \mathbf{p}^{(n)}(t), t\right) - v\left(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), \cdots, \mathbf{x}^{(n)*}(t), t\right)\right)}{\varepsilon} dt.$$

Since v is differentiable, we see that

$$\lim_{\varepsilon\to^+0}\frac{\left(v\left(\mathbf{x}^*(t)+\varepsilon\cdot\mathbf{p}(t),\dot{\mathbf{x}}^*(t)+\varepsilon\dot{\mathbf{p}}(t),\cdots,\mathbf{x}^{(n)*}(t)+\varepsilon\cdot\mathbf{p}^{(n)}(t),t\right)-v\left(\mathbf{x}^*(t),\dot{\mathbf{x}}^*(t),\cdots,\mathbf{x}^{(n)*}(t),t\right)\right)}{\varepsilon}$$

$$=\frac{dv\left(v\left(\mathbf{x}^*(t)+\varepsilon\cdot\mathbf{p}(t),\dot{\mathbf{x}}^*(t)+\varepsilon\dot{\mathbf{p}}(t),\cdots,\mathbf{x}^{(n)*}(t)+\varepsilon\cdot\mathbf{p}^{(n)}(t),t\right)-v\left(\mathbf{x}^*(t),\dot{\mathbf{x}}^*(t),\cdots,\mathbf{x}^{(n)*}(t),t\right)\right)}{d\varepsilon}$$

$$=v_{\mathbf{x}(t)}\mathbf{p}(t)+v_{\dot{\mathbf{x}}(t)}\dot{\mathbf{p}}(t)+\cdots+v_{\mathbf{x}^{(n)}(t)}\mathbf{p}^{(n)}(t),$$

Hence, $\Omega=\lim_{T\to\infty}\inf_{T\le T'}\int_0^{T'}\left(v_{\mathbf{x}(t)}\mathbf{p}(t)+v_{\dot{\mathbf{x}}(t)}\dot{\mathbf{p}}(t)+\cdots+v_{\mathbf{x}^{(n)}(t)}\mathbf{p}^{(n)}(t)\right)dt.$

Using partial integral, we obtain

(9) $$\int_0^T v_{\mathbf{x}^{(k)}}p^{(k)}dt=p^{(k-1)}v_{\dot{\mathbf{x}}^{(k)}}-\int_0^T p^{(k-1)}(\dot{v}_{\dot{\mathbf{x}}^{(k)}})dt\,,$$

and

$$\int_0^{T'}\left(v_{\mathbf{x}(t)}\mathbf{p}(t)+v_{\dot{\mathbf{x}}(t)}\dot{\mathbf{p}}(t)+\cdots+v_{\mathbf{x}^{(n)}(t)}\mathbf{p}^{(n)}(t)\right)dt$$

$$=\int_0^T\left(v_{\mathbf{x}}-(v_{\dot{\mathbf{x}}})'+\cdots+(-1)^n(v_{\mathbf{x}^{(n)}})^{(n)}\right)p(t)dt+$$

$$\left[p(t)\left(v_{\dot{\mathbf{x}}}-(v_{\ddot{\mathbf{x}}})'+\cdots+(-1)^{n-1}(v_{\mathbf{x}^{(n)}})^{(n-1)}\right)+\dot{p}(t)\left(v_{\ddot{\mathbf{x}}}-(v_{\dddot{\mathbf{x}}})'+\cdots+(-1)^{n-2}(v_{\mathbf{x}^{(n)}})^{(n-2)}\right)+\cdots+p^{(n-1)}(t)v_{\mathbf{x}^{(n)}}\right]_0^T.$$

Note that when $\varepsilon\to{}^-0$, the argument is the same. Therefore, for arbitrary n-th order continuously differentiable curve $\mathbf{p}(t)$, we have

(10)

$$0\ge\lim_{T\to\infty}\inf\int_0^T\frac{d}{d\varepsilon}v(t,\mathbf{x}+\varepsilon p,\cdots,\mathbf{x}^{(n)}+\varepsilon p^{(n)})dt$$

$$\ge\lim_{T\to\infty}\inf\int_0^T\left(v_{\mathbf{x}}-(v_{\dot{\mathbf{x}}})'+\cdots+(-1)^n(v_{\mathbf{x}^{(n)}})^{(n)}\right)p(t)dt$$

$$+\lim_{T\to\infty}\inf\left[p(t)\left(v_{\dot{\mathbf{x}}}-(v_{\ddot{\mathbf{x}}})'+\cdots+(-1)^{n-1}(v_{\mathbf{x}^{(n)}})^{(n-1)}\right)+\dot{p}(t)\left(v_{\ddot{\mathbf{x}}}-(v_{\dddot{\mathbf{x}}})'+\cdots+(-1)^{n-2}(v_{\mathbf{x}^{(n)}})^{(n-2)}\right)+\cdots+p^{(n-1)}(t)v_{\mathbf{x}^{(n)}}\right]_0^T.$$

Hence, the Euler's condition is

(11)

$$v_{\mathbf{x}}\left(\mathbf{x}^*(t),\dot{\mathbf{x}}^*(t),\cdots,\mathbf{x}^{*(n)}(t),t\right)-\left(v_{\dot{\mathbf{x}}}\left(\mathbf{x}^*(t),\left(\mathbf{x}^*(t)\right)',\cdots,\mathbf{x}^{*(n)}(t),t\right)\right)'+\cdots+(-1)^n\left(v_{\mathbf{x}^{(n)}}\left(\mathbf{x}^*(t),\left(\mathbf{x}^*(t)\right)',\cdots,\mathbf{x}^{*(n)}(t),t\right)\right)^{(n)}=0,$$

which is an extenstion of the standard Euler's condition $v_{\mathbf{x}}-(\dot{v}_{\dot{\mathbf{x}}})=0$, and the transversality condition is

(12)

$$\liminf_{T\to\infty}\Big[\,p\Big(v_{\dot{\mathbf{x}}}-(\dot{v}_{\ddot{\mathbf{x}}})'+\cdots+(-1)^{n-1}(v_{\mathbf{x}^{(n)}})^{(n-1)}\Big)+\dot{p}\Big(v_{\ddot{\mathbf{x}}}-(\dot{v}_{\dddot{\mathbf{x}}})'+\cdots+(-1)^{n-2}(v_{\mathbf{x}^{(n)}})^{(n-2)}\Big)+\cdots+p^{(n-1)}v_{\mathbf{x}^{(n)}}\Big]_0^T\le 0.$$

We fix $0<\bar{\alpha}<1$ and $\alpha:\mathbb{R}^+\to\mathbb{R}^+, C^\infty$, $\alpha(0)=0,\cdots,\alpha^{(n-1)}(0)=0,\alpha(t)=\bar{\alpha},t\ge 1.$

Next, we consider a special curve $\mathbf{p}(t)$. Let $p(t)=\alpha x^*(t)$, then (12) is modified to

(13)

$$\liminf_{T\to\infty}\Big[\alpha x^*\Big(v_{\dot{\mathbf{x}}}-(\dot{v}_{\ddot{\mathbf{x}}})+\cdots+(-1)^{n-1}(v_{\mathbf{x}^{(n)}})^{(n-1)}\Big)+(\alpha x^*)'\Big(v_{\ddot{\mathbf{x}}}-(v_{\dddot{\mathbf{x}}})'+\cdots+(-1)^{n-2}(v_{\mathbf{x}^{(n)}})^{(n-2)}\Big)+\cdots+(\alpha x^*)^{(n-1)}v_{\mathbf{x}^{(n)}}\Big]_0^T$$

$$=\bar{\alpha}\liminf_{T\to\infty}\Big(x^*\Big(v_{\dot{\mathbf{x}}}-(\dot{v}_{\ddot{\mathbf{x}}})+\cdots+(-1)^{n-1}(v_{\mathbf{x}^{(n)}})^{(n-1)}\Big)+(x^*)'\Big(v_{\ddot{\mathbf{x}}}-(v_{\dddot{\mathbf{x}}})'+\cdots+(-1)^{n-2}(v_{\mathbf{x}^{(n)}})^{(n-2)}\Big)+\cdots+(x^*)^{(n-1)}v_{\mathbf{x}^{(n)}}\Big)\Big|_T$$

$$\le 0.$$

Because $\bar{\alpha}>0$, we then have

(14)

$$\liminf_{T\to\infty}\Big(x^*\Big(v_{\dot{\mathbf{x}}}-(\dot{v}_{\ddot{\mathbf{x}}})+\cdots+(-1)^{n-1}(v_{\mathbf{x}^{(n)}})^{(n-1)}\Big)+(x^*)'\Big(v_{\dot{\mathbf{x}}}-(\dot{v}_{\ddot{\mathbf{x}}})+\cdots+(-1)^{n-2}(v_{\mathbf{x}^{(n)}})^{(n-2)}\Big)+\cdots+(x^*)^{(n-1)}v_{\mathbf{x}^{(n)}}\Big)\Big|_T\le 0,$$

which is an extension of Kamihigashi [3]'s transversality condition.

4. A Counterexample

Next we show that the two assumptions we identified are imperative in the sense that (12) is invalid if one of them is violated. We consider the following simple counterexample ($n=2$):

(15) $\quad v\big(x(t),\dot{x}(t),\ddot{x}(t)\big)=\big(x(t)-a\big)^2+b\dot{x}(t)+c\ddot{x}(t)$,

where $a>0$, $b>0$, $c>0$, and the initial value x_0 is given, with $x_0=a$.

From (11), we see that the Euler's condition is

(16) $\quad v_x-\left(v_{\dot{x}}\right)'+\left(v_{\ddot{x}}\right)''=0$,

that is,

(16') $\quad 2\big(x(t)-a\big)-b'+c''=0$

Thus, we have $x(t)=a$.

Choosing a p so that $p(0)=0$ and $p(t)>0$, $\dot{p}(0)=0$, there exists $T_0>0$, so that $\dot{p}(t)=0$, $t\geq T_0$, that is, $p(t)$ is a constant $p_\infty>0$ for $t\geq T_0$.

From (12), we see that

(17) $$\liminf_{T\to\infty}\left[p\left(\nu_{\dot{x}}-\left(\nu_{\ddot{x}}\right)'\right)+\dot{p}\left(\nu_{\ddot{x}}\right)\right]_0^T\leq 0,$$

The left hand side (LHS) of (17) can be further rewritten as

$$\text{LHS of (17)}=\liminf_{T\to\infty}\left[p\left(b-c'\right)+\dot{p}c\right]_0^T$$

$$=\liminf_{T\to\infty}\left(\left(p(T)-p(0)\right)(b-c')+\dot{p}(0)c\right)$$
$$=\liminf_{T\to\infty}\left(p_\infty(b-c')\right)=p_\infty b>0.$$

We have then derived a contradiction to (12). Next we show that Assumption 1 is violated, which causes this contradiction.

We first consider $\nu\left(x+\varepsilon p,\dot{x}+\varepsilon\dot{p},\ddot{x}+\varepsilon\ddot{p}\right)-\nu\left(x,\dot{x},\ddot{x}\right)$. Substituting $x(t)=a$ into it, we have

(18) $$\begin{aligned}&\nu\left(x+\varepsilon p,\dot{x}+\varepsilon\dot{p},\ddot{x}+\varepsilon\ddot{p}\right)-\nu\left(x,\dot{x},\ddot{x}\right)\\&=\left(x+\varepsilon p-a\right)^2+b\left(\dot{x}+\varepsilon\dot{p}\right)+c\left(\ddot{x}+\varepsilon\ddot{p}\right)-\left(\left(x-a\right)^2+b\left(\dot{x}\right)+c\left(\ddot{x}\right)\right)\\&=\left(\varepsilon p\right)^2+\varepsilon\left(b\dot{p}+c\ddot{p}\right).\end{aligned}$$

Hence,

$$\inf_{T \le T'} \int_0^{T'} \frac{(\varepsilon p)^2 + \varepsilon(b\dot{p} + c\ddot{p})}{\varepsilon} dt$$

$$= \inf_{T \le T'} \int_0^{T'} \left(\varepsilon p^2 + (b\dot{p} + c\ddot{p})\right) dt$$

$$= \inf_{T \le T'} \left(\varepsilon \int_0^{T'} p^2 dt + \left[bp + c\dot{p} \right]_0^{T'} \right)$$

$$= \inf_{T \le T'} \left(\varepsilon \int_0^{T'} p^2 dt + b\left(p(T') - p(0)\right) + c\left(\dot{p}(T') - \dot{p}(0)\right) \right)$$

$$= \inf_{T \le T'} \left(\varepsilon \int_0^{T'} p^2 dt + bp_\infty \right)$$

$$= \inf_{T \le T'} \left(\varepsilon \int_0^{T'} p^2 dt \right) + bp_\infty$$

(20) $$= \varepsilon \int_0^{T} p^2 dt + bp_\infty$$

Ω is the limit of (20) when $T \to \infty$, $\varepsilon \to 0$. However, because $\lim_{\varepsilon \to \infty} \lim_{T \to \infty} \varepsilon \int_0^T p^2 dt = \infty$, whereas $\lim_{T \to \infty} \lim_{\varepsilon \to 0} \varepsilon \int_0^T p^2 dt = 0$, we see that Ω does not converge uniformly for ε when $T \to \infty$. Hence, Assumption 1 is not satisfied and (14) (when $n = 2$, $\liminf_{T \to \infty} \left(x^* \left(v_{\dot{x}} - (\dot{v}_{\ddot{x}}) \right) + (x^*)' v_{\ddot{x}} \right)\Big|_T \le 0$) is also not satisfied.

5. Conclusions

This paper gives the two assumptions that are needed to consider infinite horizon optimization problems in which the objective functions are unbounded. It generalizes the results of Michel [1], Ekeland and Scheinkman [2], and Kamihigashi [3] ($n = 1$) to higher order differential problems. Specifically, when $n = 1$, our transversality condition is exactly the same as Kamihigashi (Theorem 3.2). Moreover, Assumption 3.1 of

Kamihigashi corresponds to our Assumption 1 and 2. Our Assumption 1 and 2 obviously hold when a discounting factor is incorporated into the model. This paper also generalizes the transversality conditions examined in the presence of discounting.

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